§ 6.1, 6.2 Prof. 1 grader

Recall: recursive formula for det A:

Theorem 6.2.10 **Laplace expansion (or cofactor expansion)**

We can compute the determinant of an $n \times n$ matrix A by Laplace expansion down any column or along any row.

Expansion down the jth column:

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}).$$

Expansion along the ith row:

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}).$$

$\det(A) \neq 0 \iff A$ is invertible.

In Exercises 11 through 22, use the determinant to find out for which values of the constant k the given matrix A is invertible.

18. $\begin{bmatrix} 0 & 1 & k \\ 3 & 2k & 5 \\ 9 & 7 & 5 \end{bmatrix}$

find det(A), set equal to 0

$0 \cdot \det(\ ) - 1 \cdot \det\begin{pmatrix} 3 & 5 \\ 9 & 5 \end{pmatrix} + k \cdot \det\begin{pmatrix} 3 & 2k \\ 9 & 7 \end{pmatrix}$

$= 30 + 21k - 18k^2 = 0$

$\rightarrow k = -5/6$ or $k = 2$

$\Rightarrow$ Matrix is invertible if $k \neq -5/6$, $k \neq 2$

Find the determinants of the matrices A in Exercises 31 through 42.

31. $\begin{bmatrix} 1 & 9 & 8 & 7 \\ 0 & 2 & 9 & 6 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ **32.** $\begin{bmatrix} 2 & 5 & 7 & 11 \\ 0 & 3 & 5 & 13 \\ 0 & 0 & 5 & 11 \\ 0 & 0 & 0 & 7 \end{bmatrix}$

Recursive formula on bottom row:

$\det A = (-1)^{1+4} \cdot 0 \cdot \det(\ ) + (-1)^{1+5} \cdot 0 \cdot \det + (-1)^{1+6} \cdot 0 \cdot (\ ) +$

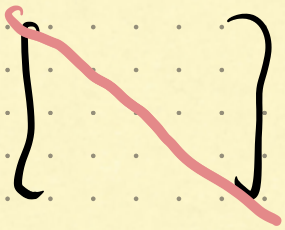
$+ (-1)^{4+4} \cdot 4 \cdot \underbrace{\det\begin{pmatrix} 1 & 9 & 8 \\ 0 & 2 & 9 \\ 0 & 0 & 3 \end{pmatrix}}_{\text{expand along bottom row}}$

$$\det\begin{bmatrix}1&9&8\\0&2&9\\0&0&3\end{bmatrix} = (-1)^{3+3}\cdot 3\cdot \det\begin{pmatrix}1&9\\0&2\end{pmatrix} = +3\cdot(1\cdot 2-0)$$

$$\Rightarrow \det A = 4\cdot 3\cdot 2 = 24$$

FACT: if a square matrix A has all zeros below diag, or all zeros above diagonal, then $\det A =$ prod diag entries

"A is triangular"



diagonal

eg.

$$\det\begin{bmatrix}1&2&3&4&5\\0&6&7&8&9\\0&0&10&11&12\\0&0&0&13&14\\0&0&0&0&15\end{bmatrix} = 1\cdot 6\cdot 10\cdot 13\cdot 15.$$

§6.2 Props of det

FACT $\det(A\cdot B) = \det(A)\cdot\det(B)$

eg $\det\left(\begin{bmatrix}1&0\\3&2\end{bmatrix}\cdot\begin{bmatrix}4&5\\0&7\end{bmatrix}\right) = \det\left(\begin{bmatrix}1&0\\3&2\end{bmatrix}\right)\cdot\det\left(\begin{bmatrix}4&5\\0&7\end{bmatrix}\right)$

$= (1\cdot 2)\cdot(4\cdot 7) = 56$

It follows: $\det(A^{-1}) = 1/\det(A)$

Pf $\det(A\cdot A^{-1}) = \det(I_n)$

$\det(A)\cdot\det(A^{-1}) = \det(I_n) = \det\begin{bmatrix}1&&0\\&\ddots&\\0&&1\end{bmatrix} = 1$

$\rightsquigarrow \det(A) = 1/\det(A^{-1})$, $\det(A^{-1}) = 1/\det(A)$

FACT $\det(A) = \det(A^T)$

Q let $\vec{u}_1, \dots, \vec{u}_n$ be an ONB for $\mathbb{R}^n$.

let $A = [\vec{u}_1, \dots, \vec{u}_n]$

What can we say about $\det(A)$?

Ans A is an orthogonal matrix

$$\Rightarrow A^T A = I_n$$

$$\Rightarrow \det(A^T A) = \det(I_n) = 1$$

$$\det(A^T A) = \det(A^T) \cdot \det(A)$$

$$\Rightarrow \det(A) \cdot \det(A) = 1 \quad \Rightarrow \det(A) = \pm 1$$

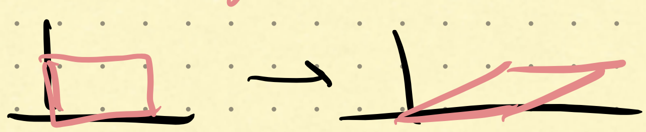
In summary, det of an orthogonal matrix is ± 1.

Row ops:

- Add cR_i to R_j: $\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \xrightarrow{R_1 \to R_1 + 4R_2} \begin{bmatrix} 1 & 11 \\ 0 & 2 \end{bmatrix}$

  $\det = 2$ $\qquad$ $\det = 2$

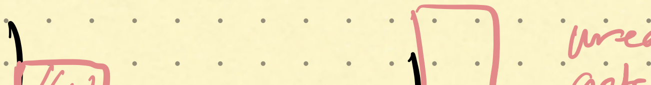
  This operation does not change determinant!



same area

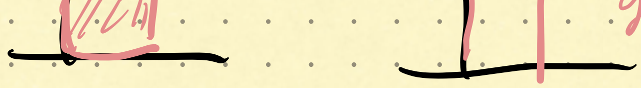
- Multiply R_i by c

  $\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \xrightarrow{6 \cdot R_1} \begin{bmatrix} 6 & 18 \\ 0 & 2 \end{bmatrix}$



area gets

det = 2 det = 12

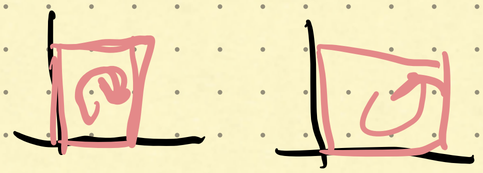


this operation multiplies det by c

- Swap R_i and R_j:

$$\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix}$$

det = 2 det = 0·3 − 2 = −2



orientation gets reversed.

This op. multiplies det by −1

Suppose

$$\det \begin{bmatrix} A & B & C & D \\ E & F & G & H \\ I & J & K & L \\ M & N & O & P \end{bmatrix} = 5$$

Find the following:

a) $\det \begin{bmatrix} 2A & 2B & 2C & 2D \\ E & F & G & H \\ M+E & N+F & O+G & P+H \\ I & J & K & L \end{bmatrix}$

b) $\det \begin{bmatrix} M & N & O & P \\ A & B & C & D \\ E & F & G & H \\ I & J & K & L \end{bmatrix}$

c) $\det \left(2 \begin{bmatrix} A & B & C & D \\ E & F & G & H \\ I & J & K & L \\ M & N & O & P \end{bmatrix} \right)$

d) $\det \begin{bmatrix} A & B+A & 2C & D \\ E & F+E & 2G & H \\ I & J+I & 2K & L \\ M & N+M & 2O & P \end{bmatrix}$

(Hint: think about doing row operations on the transpose!)

$$\begin{bmatrix} A & B & \cdots \\ E & F & \cdots \\ I & J & \cdots \\ M & N & \cdots \end{bmatrix} \longrightarrow \begin{bmatrix} 2A & 2B & \cdots \\ E & F & \cdots \\ I & J & \cdots \\ M & N & \cdots \end{bmatrix} \rightarrow \begin{bmatrix} 2A & 2B & \cdots \\ E & F & \cdots \\ M & N & \cdots \\ I & J & \cdots \end{bmatrix}$$

det · 2 det · (−1)

add R2 to R3

$$\begin{bmatrix} 2A & 2B & \cdots \\ E & F & \cdots \\ M+E & N+F & \cdots \\ I & J & \cdots \end{bmatrix}$$

(this does not affect det)

__Ans__: 5 · 2 · (−1) = −10