Dets and row ops

Another way to think about it: performing a row op on a matrix is the same thing as multiplying by a certain matrix

e.g.
$$\underbrace{\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}}_{I_3 \text{ with bottom two rows swapped}}\overbrace{\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}}^{M}=\underbrace{\begin{bmatrix}1&2&3\\7&8&9\\4&5&6\end{bmatrix}}_{M \text{ with bottom two rows swapped.}}$$

Def Apply a single row transformation to the $n\times n$ id matrix, then the resulting matrix is called an "elementary matrix"

$$\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix},\quad \begin{bmatrix}1&&&\\&1&&\\&&1&\\&&&1\end{bmatrix}\xrightarrow[\text{to } R_4]{\text{add } 3\cdot R_2}\underbrace{\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&3&0&1\end{bmatrix}}_{\text{elementary matrix}}$$

FACT: applying a row op on an $m\times n$ matrix M is the same as multiplying M on the left by the corresp elementary matrix

e.g.
$$\underbrace{\begin{bmatrix}1&2&0\\0&1&0\\0&0&1\end{bmatrix}}_{\text{add } 2\cdot R_2 \text{ to } R_1 \text{ for } I_3}\cdot\overbrace{\begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix}}^{M}=\overbrace{\begin{bmatrix}a+2d&b+2e&c+2f\\d&e&f\\g&h&i\end{bmatrix}}^{M'}$$

$$\det(M')=\det\left(\begin{bmatrix}1&2&0\\0&1&0\\0&0&1\end{bmatrix}\cdot M\right)=\underbrace{\det\left(\begin{bmatrix}1&2&0\\0&1&0\\0&0&1\end{bmatrix}\right)}_{=1\cdot\det\begin{pmatrix}1&2\\0&1\end{pmatrix}=1}\cdot\det M$$

$$=1\cdot\det M$$

$\det(M') = \det(M)$

Q. how does multiplying a row by 3 change the det?

eg. mult. the middle row of a 3x3 matrix by 3

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}}_{M} = \underbrace{\begin{bmatrix} a & b & c \\ 3d & 3e & 3f \\ g & h & i \end{bmatrix}}_{M'}$$

$$\det(M') = \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \det(M)$$

$$= 1 \cdot 3 \cdot 1 \cdot \det(M) \rightsquigarrow \det(M') = 3 \cdot \det(M)$$

How do different row ops affect the det?

- Add c(Row i) to Row j: multiplying by the elementary matrix

$$\begin{array}{cc} \begin{bmatrix} 1 & & c & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} & \leftarrow j^{th} \text{ row} \\ \uparrow & \\ i^{th} \text{ column} & \end{array}$$

Triangular matrix!

So its det will always be prod of diag entries

$= 1 \cdot 1 \cdots 1 = 1$

$\Rightarrow$ this operation does not change the det!

- Mutiply row i by c: multiplying by the elementary matrix

$$\begin{bmatrix} 1 & & & 0 \\ & \ddots & & \\ & & c & \\ 0 & & & \ddots & 1 \end{bmatrix}.$$

det of this matrix is c

$\Rightarrow$ This op. multiplies the det of our original matrix by c.

Suppose

$$\det\begin{bmatrix}A&B&C&D\\E&F&G&H\\I&J&K&L\\M&N&O&P\end{bmatrix}=5 \quad M$$

$$M^T=\begin{bmatrix}A&E&\cdots\\B&F&\\C&G&\\D&H&\end{bmatrix}$$

Find the following:

a) $\det\begin{bmatrix}2A&2B&2C&2D\\E&F&G&H\\M+E&N+F&O+G&P+H\\I&J&K&L\end{bmatrix}$

b) $\det\begin{bmatrix}M&N&O&P\\A&B&C&D\\E&F&G&H\\I&J&K&L\end{bmatrix}$

c) $\det\left(2\begin{bmatrix}A&B&C&D\\E&F&G&H\\I&J&K&L\\M&N&O&P\end{bmatrix}\right)$

d) $\det\begin{bmatrix}A&B+A&2C&D\\E&F+E&2G&H\\I&J+I&2K&L\\M&N+M&2O&P\end{bmatrix} \longrightarrow M'$

(Hint: think about doing row operations on the transpose!)

M': Take M^T, add Row 1 to Row 2, multiply row 3 by 2. Take transpose of that. This gives you M'

b) $\begin{bmatrix}A&\cdots\\E&\cdots\\I&\cdots\\M&\cdots\end{bmatrix}\longrightarrow\begin{bmatrix}M&\cdots\\A&\cdots\\E&\cdots\\I&\cdots\end{bmatrix}$ 3 swaps!

swap R4 and R3 $\begin{bmatrix}A&\cdots\\E&\cdots\\M&\cdots\\I&\cdots\end{bmatrix}\xrightarrow[\text{swap R2, R3}]{}\begin{bmatrix}A&\cdots\\M&\cdots\\E&\cdots\\I&\cdots\end{bmatrix}\xrightarrow[\text{swap R1, R2}]{}\begin{bmatrix}M&\cdots\\A&\cdots\\E&\cdots\\I&\cdots\end{bmatrix}$

Answer: $(-1)^3\cdot 5=-5$

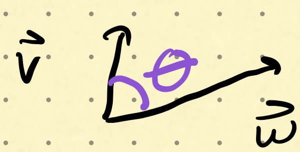
c) $\begin{bmatrix}A&\cdots\\E&\cdots\\I&\cdots\\M&\cdots\end{bmatrix}\longrightarrow 2\begin{bmatrix}A&\cdots\\E&\cdots\\I&\cdots\\M&\cdots\end{bmatrix}$ → mult each Row by 2

in other words: $2\begin{bmatrix}A&\cdots\\E&\cdots\\I&\cdots\\M&\cdots\end{bmatrix}=\begin{bmatrix}2&&&\\&2&&\\&&2&\\&&&2\end{bmatrix}\begin{bmatrix}A&\cdots\\E&\cdots\\I&\cdots\\M&\cdots\end{bmatrix}$

$$\det\left(2\begin{bmatrix}A&\cdots\\E&\\I&\\M&\end{bmatrix}\right)=2^4\cdot\det\left(\begin{bmatrix}A&\cdots\\E&\cdots\\I&\cdots\\M&\cdots\end{bmatrix}\right)=2^4\cdot 5$$

d) The way column ops affect det is actually the same as row ops, we added col 1 to col 2 and we multiplied col 3 by 2 so the det is $2\cdot\det\begin{bmatrix}A&B&C&D\\E&&\cdots&\\I&&&\\M&&&\end{bmatrix}=10$

Exam Review: Det products



$$\boxed{\vec v\cdot\vec w=\|\vec v\|\cdot\|\vec w\|\cdot\cos\theta}$$

$\vec{v} \cdot \vec{v} = \|\vec{v}\| \cdot \|\vec{v}\| \cdot \underbrace{\cos 0}_{=1} = \|\vec{v}\|^2$

<u>exc</u> Suppose $\vec{u}_1, \vec{u}_2, \vec{u}_3$ is an ONB for $\mathbb{R}^3$.

Find the angle between $\underbrace{2\vec{u}_1 + 3\vec{u}_2}_{\vec{v}}$ and $\underbrace{4\vec{u}_2 - \vec{u}_3}_{\vec{w}}$

<u>Ans</u>: $\underbrace{(2\vec{u}_1 + 3\vec{u}_2) \cdot (4\vec{u}_2 - \vec{u}_3)} = \|2\vec{u}_1 + 3\vec{u}_2\| \cdot \|4\vec{u}_2 - \vec{u}_3\| \cdot \cos(\theta)$

$2\underbrace{\vec{u}_1 \cdot \cancel{\vec{u}_2}}_{0} - 2\underbrace{\vec{u}_1 \cdot \cancel{\vec{u}_3}}_{0} + 12\underbrace{\vec{u}_2 \cdot \cancel{\vec{u}_2}}_{1} - 3\underbrace{\vec{u}_2 \cdot \cancel{\vec{u}_3}}_{0} = 12$