Midterm 2 review

October 31, 2022

1. Let $\vec{u}_1, \vec{u}_2, \vec{u}_3$ be an orthonormal basis for $\mathbb{R}^3$. Find the angle between $2\vec{u}_1 + 3\vec{u}_2$ and $4\vec{u}_2 + \vec{u}_3$.

2. Consider a QR-factorization,

$$\begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \\ 0 & 6 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

   (a) Find $\vec{v}_1 \cdot \vec{v}_2$

   (b) Find $||\vec{v}_3||$

   (c) Find the angle between $\vec{v}_1$ and $\vec{v}_3$.

3. Let $V \subseteq \mathbb{R}^4$ be the subspace defined by the equations $x_3 = x_1 + x_2$ and $x_4 = x_2 + x_3$. Find the matrix P_V of the orthogonal projection onto V.

45. Find the derivative of the function

$$f(x) = \det \begin{bmatrix} 1 & 1 & 2 & 3 & 4 \\ 9 & 0 & 2 & 3 & 4 \\ 9 & 0 & 0 & 3 & 4 \\ x & 1 & 2 & 9 & 1 \\ 7 & 0 & 0 & 0 & 4 \end{bmatrix}.$$

42. Consider an $n \times m$ matrix

$$A = QR,$$

where Q is an $n \times m$ matrix with orthonormal columns and R is an upper triangular $m \times m$ matrix with positive diagonal entries $r_{11}, \ldots, r_{mm}$. Express $\det(A^T A)$ in terms of the scalars r_{ii}. What can you say about the sign of $\det(A^T A)$?

47. If $A = QR$ is a QR factorization, what is the relationship between $A^T A$ and $R^T R$?

48. Consider an invertible $n \times n$ matrix A. Can you write A as $A = LQ$, where L is a *lower* triangular matrix and Q is orthogonal? *Hint*: Consider the QR factorization of A^T.

(Hint: recall $(AB)^T = B^T A^T$)

4. (a) Find an example of a 3×3-matrix M such that $\operatorname{rank}(M) < \operatorname{rank}(M^2)$, or show that this is not possible

   (b) Find an example of a 3×3-matrix M such that $\operatorname{rank}(M^2) < \operatorname{rank}(M)$, or show that this is not possible