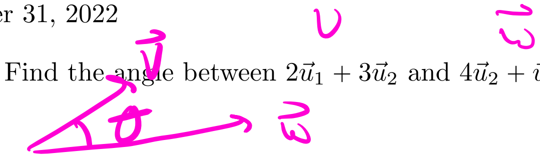
Midterm 2 review

October 31, 2022

1. Let $\vec{u}_1, \vec{u}_2, \vec{u}_3$ be an orthonormal basis for $\mathbb{R}^3$. Find the angle between $2\vec{u}_1 + 3\vec{u}_2$ and $4\vec{u}_2 + \vec{u}_3$.



$$\boxed{\vec{v}\cdot\vec{w} = \|\vec{v}\|\,\|\vec{w}\|\cos\theta}$$

$$\vec{v}\cdot\vec{w} = (2\vec{u}_1 + 3\vec{u}_2)\cdot(4\vec{u}_2 + \vec{u}_3) = \underbrace{\cancel{2\vec{u}_1\cdot 4\vec{u}_2}}_{0} + \underbrace{3\vec{u}_2\cdot 4\vec{u}_2}_{12} + \cancel{2\vec{u}_1\cdot\vec{u}_3} + \cancel{3\vec{u}_2\cdot\vec{u}_3}\ 0$$

$$= 12$$

$$\boxed{\|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}}} = \sqrt{2\cdot 2 + 3\cdot 3} = \sqrt{13}$$

$$\|\vec{w}\| = \sqrt{\vec{w}\cdot\vec{w}} = \sqrt{4\cdot 4 + 1\cdot 1} = \sqrt{17}$$

$$12 = \sqrt{13}\cdot\sqrt{17}\cdot\cos\theta$$

$$\boxed{\cos^{-1}\left(\frac{12}{\sqrt{13}\sqrt{17}}\right) = \theta}$$

2. Consider a QR-factorization,

$$\begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \\ | & | & | \end{bmatrix} \boxed{\begin{bmatrix} 2 & 1 & 2 \\ 0 & 6 & 1 \\ 0 & 0 & 3 \end{bmatrix}} \rightarrow r_{ij} = \vec{u}_i\cdot\vec{v}_j$$

$$= \left[\underset{\vec{v}_1}{2\vec{u}_1},\ \underset{\vec{v}_2}{\vec{u}_1 + 6\vec{u}_2},\ \underset{\vec{v}_3}{\boxed{2\vec{u}_1 + \vec{u}_2 + 3\vec{u}_3}}\right]$$

$\Rightarrow \vec{u}_1\cdot\vec{v}_2 = 1$

$\vec{v}_1 = 2\vec{u}_1$

$\Rightarrow \vec{v}_1\cdot\vec{v}_2 = 2\cdot 1$

(a) Find $\vec{v}_1\cdot\vec{v}_2$

(b) Find $\|\vec{v}_3\|$

(c) Find the angle between $\vec{v}_1$ and $\vec{v}_3$.

a) $\vec{v}_1\cdot\vec{v}_2 = 2\vec{u}_1\cdot(\vec{u}_1 + 6\vec{u}_2)$

$$= 2\underbrace{\vec{u}_1\cdot\vec{u}_1}_{1} + \underbrace{2\vec{u}_1\cdot 6\vec{u}_2}_{0} = 2$$

b) $\sqrt{(2\vec{u}_1 + \vec{u}_2 + 3\vec{u}_3)\cdot(2\vec{u}_1 + \vec{u}_2 + 3\vec{u}_3)} = \sqrt{2\cdot 2 + 1\cdot 1 + 3\cdot 3} = \sqrt{14}$

c) $v_1\cdot v_3 = \underbrace{2\vec{u}_1\cdot(2\vec{u}_1}+ \vec{u}_2 + 3\vec{u}_3) = 4$

$\|\vec{v}_3\| = \sqrt{14}$

$\|\vec{v}_1\| = \sqrt{2\vec{u}_1\cdot 2\vec{u}_1} = 2$

$$\rightsquigarrow \cos\theta = \frac{4}{\sqrt{14}\cdot 2}$$

$$\Rightarrow \theta = \cos^{-1}\left(2/\sqrt{14}\right)$$

$$\begin{bmatrix} -1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$x_1 = -x_2 + x_3$

$x_4 = \underline{x_2 + x_3}$

3. Let $V \subseteq \mathbb{R}^4$ be the subspace defined by the equations $\boxed{x_3 = x_1 + x_2 \text{ and } x_4 = x_2 + x_3.}$ Find the matrix P_V of the orthogonal projection onto V.

Formulas: • if $\vec{u}_1, \dots, \vec{u}_n$ is an ONB for V, and $Q = [\vec{u}_1, \dots, \vec{u}_n]$ then matrix for proj$_V$ is QQ^T

• if $\vec{v}_1, \dots, \vec{v}_n$ is any basis for V, and $A = [\vec{v}_1, \dots, \vec{v}_n]$, Then the matrix for proj$_V$ is $A(A^TA)^{-1} \cdot A^T$

Want to find a basis for V!

$$\begin{aligned} x_1 + x_2 - x_3 &= 0 \\ x_2 + x_3 - x_4 &= 0 \end{aligned} \qquad \ker\left(\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 \end{bmatrix}\right) = V.$$

Find basis of ker of matrix

RREF: $\begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$.

$x_1 = 2x_3 - x_4$

$x_2 = -x_3 + x_4$

$x_3 = x_3$

$x_4 = x_4$

basis: $\begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

answer: $\begin{bmatrix} 2 & -1 \\ -1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \left(\begin{bmatrix} 2 & -1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \right)^{-1} \cdot \begin{bmatrix} 2 & -1 \\ -1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

45. Find the derivative of the function

$$f(x) = \det \begin{bmatrix} 1 & 1 & 2 & 3 & 4 \\ 9 & 0 & 2 & 3 & 4 \\ 9 & 0 & 0 & 3 & 4 \\ x & 1 & 2 & 9 & 1 \\ 7 & 0 & 0 & 0 & 4 \end{bmatrix}.$$

$$f(x) = \underline{(-1)^{4+1}} \cdot x \cdot \det \underline{\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix}} + (-1)^{4+2} \cdot 1 \cdot \text{const}$$

$$+ (-1)^{4+3} \cdot 2 \cdot \text{const}$$

$$+ \text{const}$$

$$+ \text{const}.$$

triangular matrix

$$\Rightarrow f'(x) = (-1)^5 \cdot 1 \cdot 2 \cdot 3 \cdot 4$$

$$= -24.$$

42. Consider an $n \times m$ matrix

$$A = QR,$$

where Q is an $n \times m$ matrix with orthonormal columns and R is an upper triangular $m \times m$ matrix with positive diagonal entries $r_{11}, \ldots, r_{mm}$. Express $\det(A^T A)$ in terms of the scalars r_{ii}. What can you say about the sign of $\det(A^T A)$?

47. If $A = QR$ is a QR factorization, what is the relationship between $A^T A$ and $R^T R$?

48. Consider an invertible $n \times n$ matrix A. Can you write A as $A = LQ$, where L is a *lower* triangular matrix and Q is orthogonal? *Hint*: Consider the QR factorization of A^T.

(Hint: recall $(AB)^T = B^T A^T$)

4. (a) Find an example of a 3×3-matrix M such that $\text{rank}(M) < \text{rank}(M^2)$, or show that this is not possible

   (b) Find an example of a 3×3-matrix M such that $\text{rank}(M^2) < \text{rank}(M)$, or show that this is not possible