10/4 notes: §3.4 Lin transforms w.r.t a basis

Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a lin. transform. Let $\mathfrak{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ be a basis of $\mathbb{R}^n$. The matrix of T w.r.t. $\mathfrak{B}$ is the matrix

$$\left[[T(\vec{v}_1)]_{\mathfrak{B}}, \dots, [T(\vec{v}_n)]_{\mathfrak{B}} \right] = M.$$

Then $M[\vec{x}]_{\mathfrak{B}} = [T(\vec{x})]_{\mathfrak{B}}$ (not obvious) for all $\vec{x} \in \mathbb{R}^n$

Q If the matrix of T w.r.t the standard basis $\vec{e}_1, \dots, \vec{e}_n$ is A, how do we find M?

↳ "the" matrix of T. The usual one.

Here's one way: Recall: let $V = [\vec{v}_1 \cdots \vec{v}_n]$. Then

$V[\vec{x}]_{\mathfrak{B}} = \vec{x}$, $\quad [\vec{x}]_{\mathfrak{B}} = V^{-1}\vec{x}$

V? or V^{-1}? We want this matrix to send $\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ to $\vec{v}_1$

From the eq. $M[\vec{x}]_{\mathfrak{B}} = [T(\vec{x})]_{\mathfrak{B}}$ we get

$$MV^{-1}\vec{x} = V^{-1}T(\vec{x}) = V^{-1}A\vec{x} \text{ for all } \vec{x}$$

$$\Rightarrow MV^{-1} = V^{-1}A \Rightarrow \boxed{\begin{aligned} &M = V^{-1}AV \\ \Rightarrow\ &VMV^{-1} = A \end{aligned}}$$

In Exercises 25 through 30, find the matrix B of the linear transformation $T(\vec{x}) = A\vec{x}$ with respect to the basis $\mathfrak{B} = (\vec{v}_1, \dots, \vec{v}_m)$.

↳ not $[T(\vec{x})]_{\mathfrak{B}} = A[\vec{x}]_{\mathfrak{B}}$

25. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$; $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

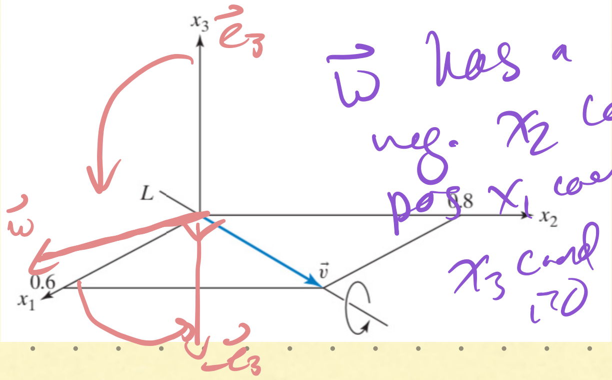
Given A = matrix for T in std coords $= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$V = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}. \rightsquigarrow B = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

73. Let L be the line in $\mathbb{R}^3$ spanned by the vector

$$\vec{v} = \begin{bmatrix} 0.6 \\ 0.8 \\ 0 \end{bmatrix}.$$

Let T from $\mathbb{R}^3$ to $\mathbb{R}^3$ be the rotation about this line through an angle of $\pi/2$, in the direction indicated in the accompanying sketch. Find the matrix A such that $T(\vec{x}) = A\vec{x}$.



Hint: try to find a basis in which the matrix for T is easier to find.

$T(\vec{v}) = \vec{v}$

In terms of the basis $\mathcal{B} = \{\vec{v}, \vec{w}, \vec{e}_3\}$, the matrix for T is:

$T(\vec{v}) = \vec{v}$

$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_{\mathcal{B}}\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_{\mathcal{B}}$

$T(\vec{w}) = -\vec{e}_3 \rightsquigarrow T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_{\mathcal{B}}\right) = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}_{\mathcal{B}}, \quad T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_{\mathcal{B}}\right) = \vec{w} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_{\mathcal{B}}$

Matrix for T w.r.t. $\mathcal{B}$ is $\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}}_{\text{this "M"}}$

want "A"

$M = V^{-1} A V$

$VMV^{-1} = A$

Answer:

$A = VMV^{-1} = \begin{bmatrix} \vec{v} & \vec{w} & \vec{e}_3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \vec{v} & \vec{w} & \vec{e}_3 \end{bmatrix}^{-1}$

???

$\vec{w} = ?$ $\quad \vec{w} \cdot \vec{v} = 0 \rightsquigarrow \vec{w} \cdot \begin{bmatrix} 0.6 \\ 0.8 \\ 0 \end{bmatrix} = 0$

$\vec{w} = \begin{bmatrix} 0.8 \\ -0.6 \\ 0 \end{bmatrix}$ $\quad$ check: $\vec{w} \cdot \vec{v} = 0$

$$A = \begin{bmatrix} 0.6 & 0.8 & 0 \\ 0.8 & -0.6 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0.6 & 0.8 & 0 \\ 0.8 & -0.6 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1}$$