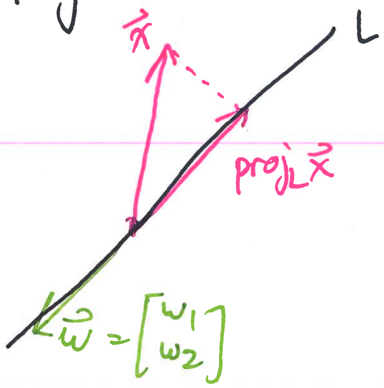
9/12/22

Agenda: 2.2 exc.
2.3 matrix mult.

Last time: • Stretch by k_1 in the x-direction and ~~by~~ by k_2 in the y-direction: $\begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}$

• Rotate by θ: $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

• Projection onto a line L: $\text{proj}_L(\vec{x}) = \dfrac{\vec{w}\cdot\vec{x}}{\vec{w}\cdot\vec{w}}\,\vec{w}$ for any vector $\vec{w}$ parallel to L



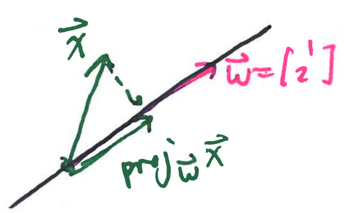
Matrix: $\dfrac{1}{\vec{w}\cdot\vec{w}}\begin{bmatrix} w_1\vec{w} & w_2\vec{w} \end{bmatrix}$

$$= \underbrace{\frac{1}{\vec{w}\cdot\vec{w}}}_{\vec{w}^T\vec{w}} \underbrace{\underbrace{\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}}_{\vec{w}} \underbrace{\begin{bmatrix} w_1 & w_2 \end{bmatrix}}_{\vec{w}^T}}_{\vec{w}\vec{w}^T} = \frac{1}{\vec{w}^T\vec{w}}(\vec{w}\vec{w}^T)$$

$$\text{proj}_L(\vec{x}) = \frac{1}{\vec{w}^T\vec{w}}(\vec{w}\vec{w}^T)\,\vec{x}$$

This formula holds in 3D, 4D, etc.

exc 2.2 #30: find a 2×2 matrix A s.t. $A\vec{x}$ is parallel to $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ for all $\vec{x}$



A one answer is to take A to be the matrix for $\text{proj}_{\vec{w}}$

$\rightsquigarrow A = \cancel{\dfrac{1}{\vec{w}\cdot\vec{w}}[1\ 2][}\ \dfrac{1}{\vec{w}^T\vec{w}}\vec{w}\vec{w}$

$\vec{w} \cdot \vec{w} = \vec{w}^T \vec{w}$

$\uparrow$ dot product $\qquad \uparrow$ matrix mult.

$\vec{w}^T \cdot \vec{w}$ weird.

$$= \frac{1}{1\cdot 1 + 2\cdot 2} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 1\cdot 1 & 2\cdot 1 \\ 2\cdot 1 & 2\cdot 2 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1/5 & 2/5 \\ 2/5 & 4/5 \end{bmatrix}$$

<u>Another answer</u>: We want $A\vec{x} = (\text{some constant}) \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ for all $\vec{x} \in \mathbb{R}^2$

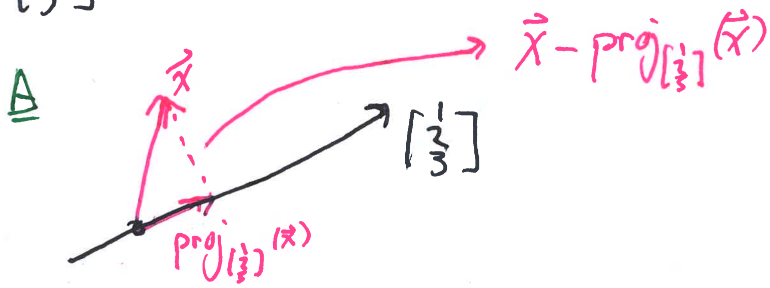
$$A = \begin{bmatrix} 67 \begin{bmatrix} 1 \\ 2 \end{bmatrix} & 22.1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 67 & 22.1 \\ 134 & 44.2 \end{bmatrix}$$

Then $A \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 67 \\ 134 \end{bmatrix} + x_2 \begin{bmatrix} 22.1 \\ 44.2 \end{bmatrix} = x_1 \cdot 67 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \cdot 22.1 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$= \underbrace{(67x_1 + 22.1x_2)}_{\text{some const.}} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

In general, $A = \begin{bmatrix} c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} & c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{bmatrix}$ works.

#3: Find a 3×3 matrix A s.t. $A\vec{x}$ is perpendicular to $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ for all $\vec{x} \in \mathbb{R}^3$



Matrix for $\text{proj}_{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}$:

$$M_L = \frac{1}{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

$$= \frac{1}{1\cdot 1 + 2\cdot 2 + 3\cdot 3} \begin{bmatrix} 1\cdot 1 & 1\cdot 2 & 1\cdot 3 \\ 2\cdot 1 & 2\cdot 2 & 2\cdot 3 \\ 3\cdot 1 & 3\cdot 2 & 3\cdot 3 \end{bmatrix}$$

Answer: $Id_3 - M_L$, because $(Id_3 - M_L)\vec{x} = Id\,\vec{x} - M_L\vec{x}$

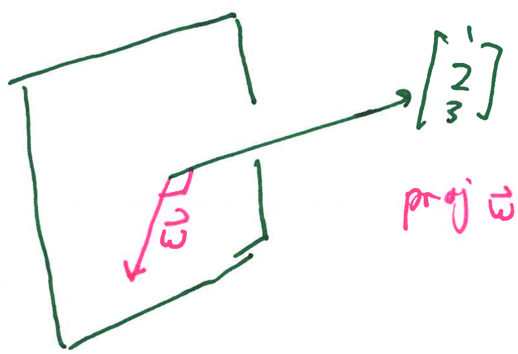
$$= \vec{x} - \mathrm{proj}_{\left[\begin{smallmatrix}1\\2\\3\end{smallmatrix}\right]}\vec{x}$$

A2 $\begin{bmatrix}-1\\-1\\1\end{bmatrix}$ is perpendicular to $\begin{bmatrix}1\\2\\3\end{bmatrix}$:

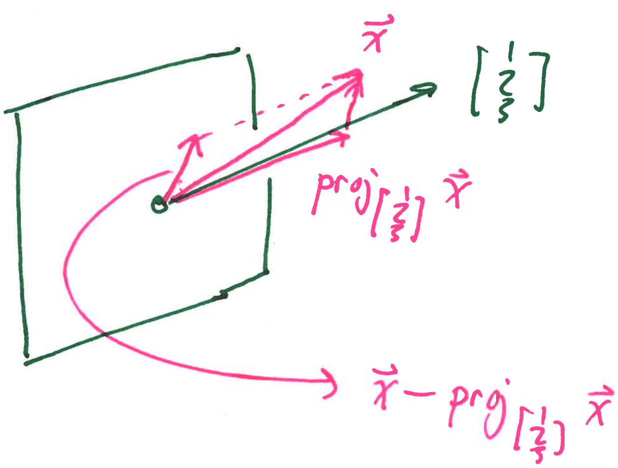
$$\begin{bmatrix}-1\\-1\\1\end{bmatrix} \cdot \begin{bmatrix}1\\2\\3\end{bmatrix} = -1\cdot 1 + -1\cdot 2 + 1\cdot 3 = 0$$

So the matrix for $\mathrm{proj}_{\left[\begin{smallmatrix}-1\\-1\\1\end{smallmatrix}\right]}$ works.

$\begin{bmatrix}-3\\0\\1\end{bmatrix}$ is also perpendicular to $\begin{bmatrix}1\\2\\3\end{bmatrix}$.



$\mathrm{proj}_{\vec{w}}$ works



§2.3

Let $T_1 : \mathbb{R}^a \longrightarrow \mathbb{R}^b$, $T_2 : \mathbb{R}^b \longrightarrow \mathbb{R}^c$ be linear.

The function sending any vector $\vec{v} \in \mathbb{R}^a$ to

$\overbrace{T_2(\underbrace{T_1(\vec{v})}_{\mathbb{R}^b})}^{\mathbb{R}^c}$ is a linear function, from $\mathbb{R}^a$

to $\mathbb{R}^c$

<u>Notation</u>: $(T_2 \circ T_1)(\vec{v}) = T_2(T_1(\vec{v}))$

So $T_2 \circ T_1 : \mathbb{R}^a \longrightarrow \mathbb{R}^c$ is linear.

What is its matrix?

Let M_1 be the matrix of T_1, M_2 be the matrix of T_2.

Then $(T_2 \circ T_1)(\vec{v}) = T_2(T_1(\vec{v})) = T_2(M_1 \vec{v}) = M_2 \cdot (M_1 \vec{v})$

$$= (M_2 \cdot M_1)\vec{v}$$

Composition of linear functions $\Longleftrightarrow$ Multiplication of matrices