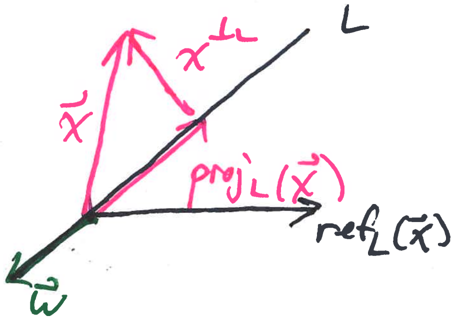
9/13/22 2.2 practice

2.3



Transform	Matrix
$proj_L$	$\frac{1}{\vec{w}\cdot\vec{w}}\vec{w}\,\vec{w}^T$
~~refl~~ ref_L	$\frac{2}{\vec{w}\cdot\vec{w}}\vec{w}\,\vec{w}^T - Id$

$ref_L(\vec{x}) = \cancel{2}\, proj_L(\vec{x}) - x^{\perp L}$

$\vec{x} = proj_L(\vec{x}) + x^{\perp L}$

$\Rightarrow ref_L(\vec{x}) = 2\,proj_L(\vec{x}) - \vec{x}$

Identity matrix "I"

$$\frac{1}{5}\begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2/5 & 4/5 \\ 4/5 & 8/5 \end{bmatrix} = 2\cdot \underbrace{\frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}}_{proj_L}.$$

$\Rightarrow$ matrix on the right was ref_L.

$$2\,proj_L(\vec{x}) - \vec{x} = 2\left(\frac{1}{\vec{w}\cdot\vec{w}}\vec{w}\,\vec{w}^T\right)\vec{x} - Id\cdot\vec{x} = \left(\frac{2}{\vec{w}\cdot\vec{w}}\vec{w}\,\vec{w}^T - Id\right)\vec{x}$$

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§2.3

(function composition) $\Longleftrightarrow$ Multiplication of matrices

Problem 1 One of the following matrices corresponds to projection onto a line and one of them corresponds to reflection about a line. Which is which?

$$\frac{1}{5}\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix} \qquad \frac{1}{5}\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix} \qquad \frac{1}{5}\begin{bmatrix}4 & -3\\ -3 & 4\end{bmatrix} \qquad \frac{1}{5}\begin{bmatrix}-3 & 4\\ 4 & 3\end{bmatrix}$$

Is this $2 \cdot proj - Id$?

If $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$, $proj_L$ has the matrix

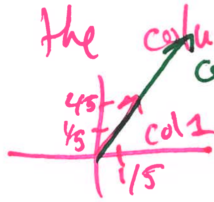
$$\frac{1}{w_1^2 + w_2^2}\begin{bmatrix} w_1^2 & w_1 w_2 \\ w_1 w_2 & w_2^2 \end{bmatrix}$$

Top right is not a projection matrix

If you choose $\vec{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$: $\frac{1}{1^2 + 2^2}\begin{bmatrix} 1^2 & 1\cdot 2 \\ 1\cdot 2 & 2^2 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

In a projection matrix, the columns will all be parallel

$\frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1/5 & 2/5 \\ 2/5 & 4/5 \end{bmatrix}$



col 2 = (some const) $\cdot$ col 1

To check reflection:

$ref_L = 2 proj_L - Id$

$$\frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/5 & 2/5 \\ 2/5 & 1/5 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6/5 & 2/5 \\ 2/5 & 6/5 \end{bmatrix}$$

if this matrix is $2 proj_L$, then one column has to be a const. times the other.

If col 1 = c $\cdot$ col 2, then

$\begin{bmatrix} 6/5 \\ 2/5 \end{bmatrix} = c \cdot \begin{bmatrix} 2/5 \\ 6/5 \end{bmatrix}$ c must be 3 but $2/5 \neq 3 \cdot 6/5$

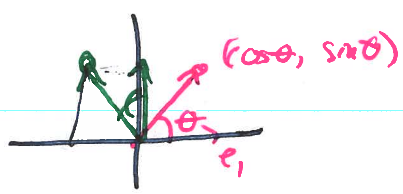
§2.3

Problem 2 Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the function which projects any vector onto the line containing $\begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}$, and then rotates by an angle of $\pi/3$ (counter clockwise). Find the matrix corresponding to T.

Matrix for projecting onto $\begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}$: $\dfrac{1}{\sqrt{3}^2+1^2} \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix} \begin{bmatrix} \sqrt{3} & 1 \end{bmatrix}$

$$= \frac{1}{4} \begin{bmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$$

Matrix for rotating $\pi/3$: $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

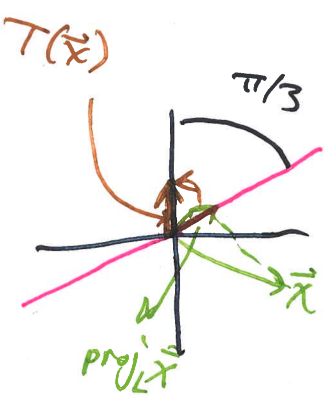


Matrix for T? Multiply them!

$$T(\vec{x}) = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \frac{1}{4} \begin{bmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}}_{\text{The matrix for } T.} \vec{x}$$

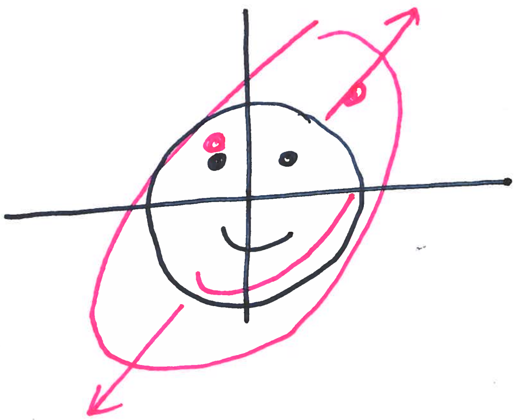
$$\rightsquigarrow \begin{bmatrix} 0 & 0 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix} 0 & 0 \\ \sqrt{3}/2 & 1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{3}/2\, x_1 + \frac{1}{2} x_2 \end{bmatrix}$$



$\Rightarrow T(\vec{x})$ is always parallel to y-axis!

Q



T stretches by a factor of 2 in the diagonal direction.

Q Find the matrix corresponding to T

$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = ?$ $\qquad$ $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = ?$

Problem 2 calculation

$$\begin{bmatrix} \overset{1/2}{\cancel{\cos \pi/3}} & \overset{-\sqrt{3}/2}{\cancel{-\sin \pi/3}} \\ \overset{\sqrt{3}/2}{\sin \pi/3} & \underset{1/2}{\cancel{\cos \pi/3}} \end{bmatrix} \cdot \begin{bmatrix} 3/4 & \sqrt{3}/4 \\ \sqrt{3}/4 & 1/4 \end{bmatrix} = \left[\frac{3}{4}\begin{bmatrix}1/2\\ \sqrt{3}/2\end{bmatrix} + \frac{\sqrt{3}}{4}\begin{bmatrix}\overset{-\sqrt{3}/2}{\cancel{1/2}}\\ \underset{1/2}{\cancel{\sqrt{3}/2}}\end{bmatrix} \;\middle|\; \frac{\sqrt{3}}{4}\begin{bmatrix}\frac{1}{2}\\ \frac{\sqrt{3}}{2}\end{bmatrix} + \frac{1}{4}\begin{bmatrix}\frac{\sqrt{3}}{2}\\ \frac{1}{2}\end{bmatrix}\right.$$