9/16/22 §2.4.
- Invertability
- Inverse matrices
- Checking if a transformation/matrix is invertable

Recall A function $f: X \to Y$ is called invertable if $\forall y \in Y$ $\exists$ unique $x \in X$ s.t. $f(x) = y$

Inverse function: $f^{-1}: Y \to X$

$$f(x) = y \iff f^{-1}(y) = x$$

$$\cancel{f}(\cancel{f^{-1}}(y)) = y \qquad \cancel{f^{-1}}(\cancel{f}(x)) = x$$

for all $y \in Y$ and $x \in X$

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eg. The linear function $T: \mathbb{R}^2 \to \mathbb{R}^2$ given by the matrix $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ is invertible: i.e. given any $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \mathbb{R}^2$, there exists a unique $\vec{x} \in \mathbb{R}^2$ with $T(\vec{x}) = \vec{y}$. Namely, $\vec{x} = \begin{bmatrix} y_1/2 \\ y_2 \end{bmatrix}$.

$$\boxed{T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} 2a \\ b \end{bmatrix}}$$

Check: $T(\vec{x}) = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} y_1/2 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$.

Also, there are no other vectors $\vec{x}'$ such that $T(\vec{x}') = \vec{y}$.

Inverse function: $T^{-1}: \mathbb{R}^2 \to \mathbb{R}^2$, $T^{-1}\left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) = \begin{bmatrix} y_1/2 \\ y_2 \end{bmatrix}$

Notice: T^{-1} is linear! matrix: $\begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}$

In general, if $T: \mathbb{R}^m \to \mathbb{R}^n$ is linear and invertible, then $T^{-1}: \mathbb{R}^n \to \mathbb{R}^m$ must also be linear. (§2.2 exc #29)

Q let $T: \mathbb{R}^m \to \mathbb{R}^n$ be an invertible linear function, let A be the corresponding matrix.
What can we say about A?

T invertible $\iff$ $\forall \vec{y}\ \exists$ unique $\vec{x}$ s.t. $\underbrace{T(\vec{x}) = \vec{y}}_{A\vec{x} = \vec{y}}$

$\iff$ for all $\vec{y}$, the linear system of eqs, $A\vec{x} = \vec{y}$, has a unique solution $\vec{x}$.

$\iff$ rref(A) has a pivot in every column

(otherwise, you'd get ∞ many solutions for some $\vec{y}$s: eg $A = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
then $A\vec{x} = \vec{y}$ always has ∞ solutions.)

AND rref(A) has a pivot in every row.

(otherwise, $A\vec{x} = \vec{y}$ will have no solutions for some choices of $\vec{y}$. eg $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. $A\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ has no solutions!)

In summary:

THM ~~TX invert~~ Given linear function $T: \mathbb{R}^n \to \mathbb{R}^n$,

with matrix A,

T invertible $\iff$ rref (A) has a pivot in each row and each column

$$\iff m = n \text{ and } \operatorname{rref}(A) = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I_n$$

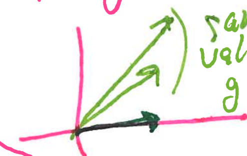
Def If $T: \mathbb{R}^n \to \mathbb{R}^n$ is invertible linear function w/ matrix A, then the matrix $T^{-1}: \mathbb{R}^n \to \mathbb{R}^n$ is denoted A^{-1}, "inverse matrix of A"

Check: $T(T^{-1}(\vec{x})) = T^{-1}(T(\vec{x})) = \vec{x}$

$$\Rightarrow A \cdot A^{-1} = A^{-1} \cdot A = I_n$$

Problem 1 Which of the following transformations are invertible?

- $f : \mathbb{R}^3 \to \mathbb{R}^3$, which rotates every vector clockwise by $\pi/3$ around the y-axis, viewed from the positive y direction Yes! f^{-1}: rotate CCW by $\pi/3$
- $T : \mathbb{R}^2 \to \mathbb{R}^2$, which reflects each vector across the y-axis → Yes! T^{-1}: reflect again
- $g : \mathbb{R}^2 \to \mathbb{R}^2$, which projects each vector onto the x-axis → multiple vectors project to the same thing:



same value for $g(\vec{v})$

- $h : \mathbb{R}^2 \to \mathbb{R}^3$, given by

$$\cancel{g}\,h\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

$h\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ has no sol'ns

- $u : \mathbb{R}^3 \to \mathbb{R}^2$, given by

$$u\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}$$

$u\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ has ∞ many solutions

not invertible: Domain and range have diff. dims.

Problem 2 Which of the following matrices are invertible?

- $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ $\quad \mathrm{rref}(A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ yes
- $B = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ $\quad \mathrm{rref}(B) = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$ no!
- $C = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 7 & 10 \end{bmatrix}$ not square. no!
- $D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad \mathrm{rref}(D) = D$ no!

Exit ticket

1. Give a (new!) example of a linear transformation which is not invertible
2. Ask me a question!