9/2/22

Agenda: vocab, # of solutions, matrix algebra.

Sys of lin eqs:

$$a_{11}x_1 + \dots + a_{1n}x_n = b_1$$
$$\vdots$$
$$a_{m1}x_1 + \dots + a_{mn}x_n = b_n$$

$$\longrightarrow \begin{array}{c} \begin{array}{ccc} x_1 & \cdots & x_n \end{array} \\ \left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_n \end{array}\right] \end{array}$$

$$\Big\downarrow \text{RREF}$$

pivots

$$\begin{bmatrix} 1 & \cdots & & & & \\ & 1 & \cdots & & & \\ & & 1 & \cdots & & \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = A$$

<u>Vocab</u>: the <u>rank</u> of a matrix is the # of pivots it has after being put in RREF

<u>e.g.</u> $\operatorname{rank}(A) = 3$

Columns of A which have pivots, after being put in RREF, are called <u>pivot columns</u>.

The corresponding variables are called <u>pivot variables</u> (or "leading variables"). Other variables are called "free variables" eg.

system of eqs:

$$\begin{aligned} x_1 + x_2 + 5x_3 &= -3 \\ x_2 + 3x_3 &= -4 \\ x_1 + 2x_3 &= 1 \end{aligned} \Rightarrow A = \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & 5 & -3 \\ 0 & 1 & 3 & -4 \\ 1 & 0 & 2 & 1 \end{array}\right] \end{array} \xrightarrow{\text{RREF}} \begin{array}{c} \begin{array}{ccc} \downarrow & \downarrow & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$\text{rank}(A) = 2$

First two columns of A are pivot columns

x_1 and x_2 are pivot variables

x_3 is a free variable.

<u>Solutions to this system</u>:

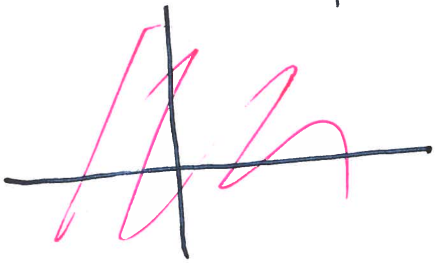
$$\begin{aligned} x_1 + 2x_3 &= 1 \\ x_2 + 3x_3 &= -4 \end{aligned} \rightsquigarrow \begin{aligned} x_1 &= 1 - 2x_3 \\ x_2 &= -4 - 3x_3 \end{aligned}$$

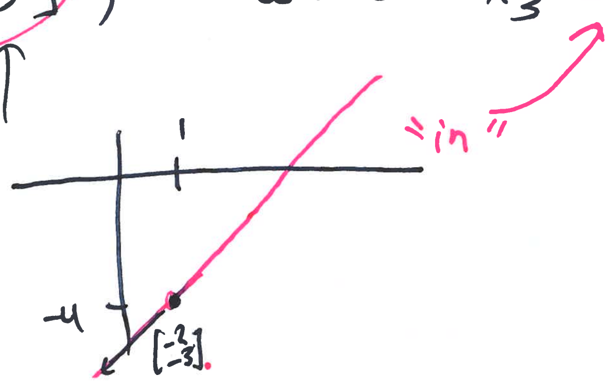
where x_3 is any real #.

<u>In vector notation</u>:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 - 2x_3 \\ -4 - 3x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \end{bmatrix} + \begin{bmatrix} -2x_3 \\ -3x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ -3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ -3 \end{bmatrix}, \quad \text{where } x_3 \in \mathbb{R}$$





\# of solutions

Q If you do RREF and you get ~~a~~ the matrix ~~of the form~~

$$\begin{bmatrix} 1 & 0 & 2 & \vdots & \cancel{3}\ 0 \\ 0 & 1 & 1 & \vdots & \cancel{4}\ 0 \\ 0 & 0 & 0 & \vdots & 1 \end{bmatrix}$$

pivot

$x_1 + 2x_3 = 0$
$x_2 + x_3 = 0$
rank $= 3$

\# pivot variables $= 2$

Q: how many solutions are there to the sys of eqs?

0 why? last row: $0 \cdot x_1 + 0 \cdot x_2 + 0\, x_3 = 1$

can't happen.

"inconsistent system of equations"

Thm (p.25)

A sys has no solutions $\iff$ the row $[0 \cdots 0 \,\vdots\, 1]$ shows up in the RREF $\iff$ rightmost column is a pivot column.

Otherwise, it has a unique sol'n if and only if there are no free variables. $\iff$ each column of the augmented matrix to the left of $\vdots$ has a pivot. $\iff$ rank of augmented matrix = \# variables in the system.

In this case, RREF looks like:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & a_1 \\ 0 & 1 & 0 & 0 & a_2 \\ 0 & 0 & 1 & 0 & a_3 \\ 0 & 0 & 0 & 1 & a_4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

If the system is consistent ("$0 \neq 1$") then you have ∞ many solutions $\Leftrightarrow$ rank of augmented matrix $<$ # variables

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Matrix algebra:

Matrix addition:

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 2 \end{bmatrix}}_{2\times 3} + \underbrace{\begin{bmatrix} 6 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}}_{2\times 3} = \begin{bmatrix} 1+6 & 2+1 & 3+3 \\ 0+1 & 4+1 & 2+1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 3 & 6 \\ 1 & 5 & 3 \end{bmatrix}$$

Scalar multiplication:

$$6 \cdot \begin{bmatrix} 2 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 6\cdot 2 & 6\cdot 2 \\ 6\cdot 1 & 6\cdot 4 \end{bmatrix} = \begin{bmatrix} 12 & 12 \\ 6 & 24 \end{bmatrix}$$

Multiplication of 2 matrices (§2.3)

- Matrix times a vector: (§1.3):

$$\underbrace{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}}_{n \text{ columns}} \cdot \left.\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}\right\} n \text{ rows} = v_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + v_2 \begin{bmatrix} a_{12} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + v_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

e.g.

$$\begin{bmatrix} 1 & -2 & 4 \\ 0 & 3 & 5 \end{bmatrix}\begin{bmatrix} 6 \\ 2 \\ 4 \end{bmatrix} = 6\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} -2 \\ 3 \end{bmatrix} + 4\begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ 0 \end{bmatrix} + \begin{bmatrix} -4 \\ 6 \end{bmatrix} + \begin{bmatrix} 16 \\ 20 \end{bmatrix}$$

$$= \begin{bmatrix} 6-4+16 \\ 0+6+20 \end{bmatrix} = \begin{bmatrix} 18 \\ 26 \end{bmatrix}$$

- Matrix times matrix:

Let $A = \underbrace{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}}_{n \text{ columns}}$ $\qquad B = \left.\begin{bmatrix} b_{11} & \cdots & b_{1\ell} \\ \vdots & & \\ b_{n1} & \cdots & b_{n\ell} \end{bmatrix}\right\} n \text{ rows}$

$m \times \underline{n}$ matrix $\qquad \underline{n} \times \ell$ matrix

Then $A \cdot B = \underbrace{\begin{bmatrix} A \cdot \begin{bmatrix} b_{11} \\ \vdots \\ b_{n1} \end{bmatrix} & A \cdot \begin{bmatrix} b_{12} \\ \vdots \\ b_{n2} \end{bmatrix} & \cdots & A \cdot \begin{bmatrix} b_{1\ell} \\ \vdots \\ b_{n\ell} \end{bmatrix} \end{bmatrix}}_{m \times \ell \text{ matrix}}$