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SUBSPACES

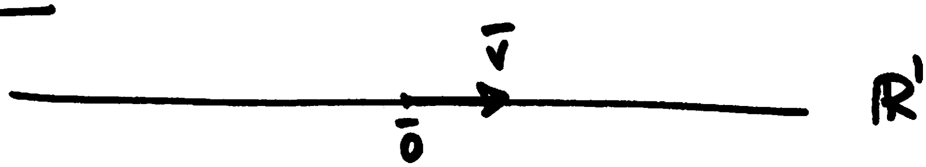
Formal definition: A subspace V of $\mathbb{R}^n$ is a subset satisfying 3 conditions:

(1) $\bar{0} \in V$

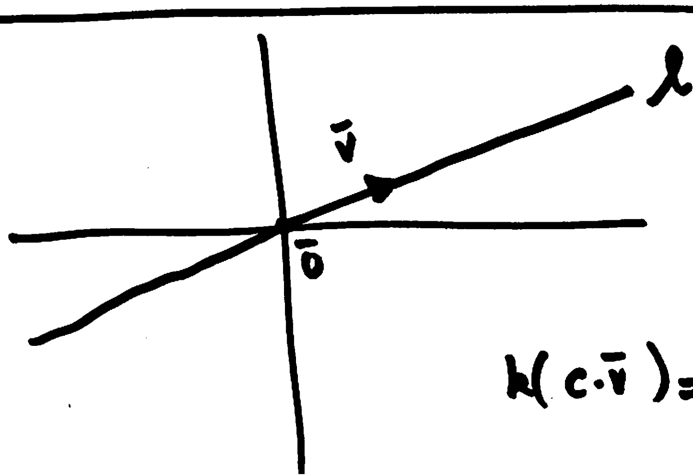
(2) If $\bar{v} \in V$, then any multiple $c\bar{v}$, $c \in \mathbb{R}$ is also in V

(3) If $\bar{v}_1, \bar{v}_2 \in V$, then $\bar{v}_1 + \bar{v}_2 \in V$

Examples:



$\left\{ \begin{array}{l} 0 = \{\bar{0}\} \text{ is a subspace of } \mathbb{R}^1. \\ \mathbb{R}^1 \text{ is a subspace of } \mathbb{R}^1 \end{array} \right.$ This is everything!



$\mathbb{R}^2$ $\quad 0 = \{\bar{0}\}$

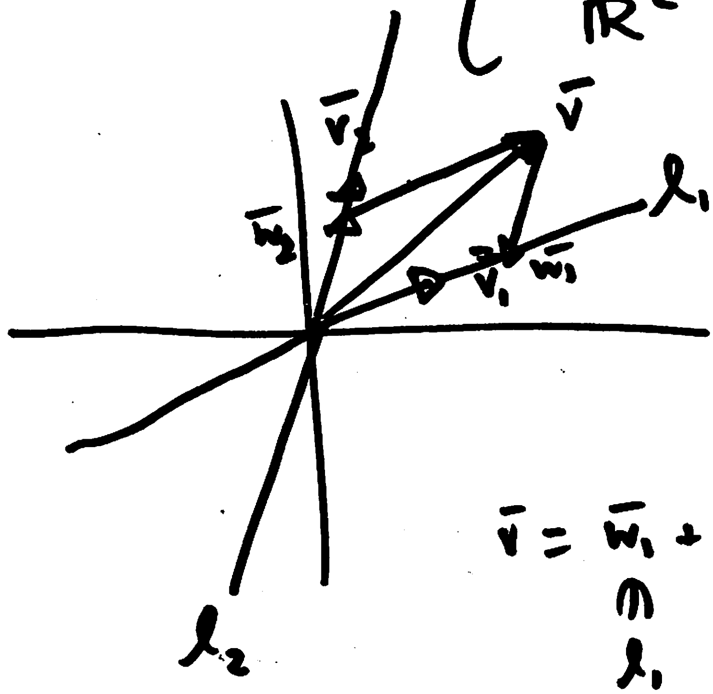
l is a subspace of $\mathbb{R}^2$

$\quad = \{c \cdot \bar{v} \mid c \in \mathbb{R}\}$

$k(c \cdot \bar{v}) = (kc)\bar{v}$, $\quad c_1\bar{v} + c_2\bar{v}$
$\quad = (c_1 + c_2)\bar{v}$

$\mathbb{R}^2$

$$\begin{cases} 0 = \{\bar{0}\} \\ \ell = \text{a line through the origin} \\ \mathbb{R}^2 \end{cases}$$



$\bar{v}_1 \in V$

$\ell_1 \subseteq V$

$\ell_2 \subseteq V$

$$\bar{v} = \underset{\ell_1}{\underset{\cap}{\bar{w}_1}} + \underset{\ell_2}{\underset{\cap}{\bar{w}_2}} \qquad V = \mathbb{R}^2$$

$$\underset{\mathbb{R}^m}{\overset{\ker(L)}{\cap}} \xrightarrow{L} \underset{\mathbb{R}^n}{\overset{\mathrm{Im}(L)}{\cap}} \qquad A \underset{n \times m}{\begin{bmatrix} & & \end{bmatrix}}$$

$\ker(L) = \{ \bar{v} \in \mathbb{R}^m \,/\, L(\bar{v}) = \bar{0} \} \subseteq \mathbb{R}^m$

$\mathrm{Im}(L) = \{ L\bar{v} \,/\, \bar{v} \in \mathbb{R}^m \} \subseteq \mathbb{R}^n$

<u>Key fact:</u> $\ker(L)$ is a subspace of $\mathbb{R}^m$

$\mathrm{Im}(L)$ " " " of $\mathbb{R}^n$

$$L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

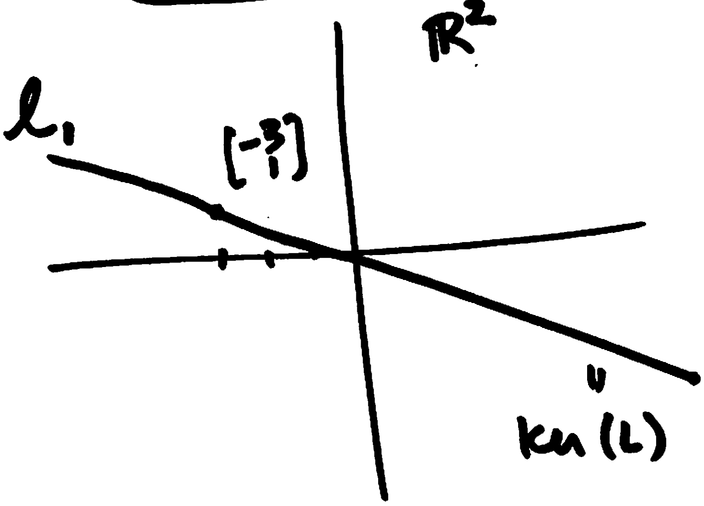
$$L(\bar{x}) = A\bar{x} \qquad A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$$

- Find the kernel & image of L
- Draw pictures of them.

$$\left[\begin{array}{cc|c} 1 & 3 & 0 \\ 2 & 6 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

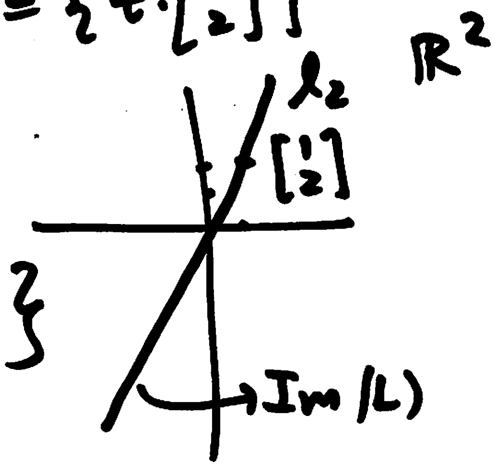
$$\boxed{x_1 + 3x_2 = 0} \qquad x_1 = -3x_2 \qquad x_2 = s$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3s \\ s \end{bmatrix} = s\begin{bmatrix} -3 \\ 1 \end{bmatrix}$$



$$\left\{ x_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2\begin{bmatrix} 3 \\ 6 \end{bmatrix} \right\}$$

$$= \left\{ t \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$



$$\left\{ A\bar{x} = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\bar{v}_1 + x_2\bar{v}_2 \right\}$$

$$\underset{\bar{v}_1}{\begin{bmatrix}1\\2\end{bmatrix}} \quad \underset{\bar{v}_2}{\begin{bmatrix}3\\6\end{bmatrix}}$$

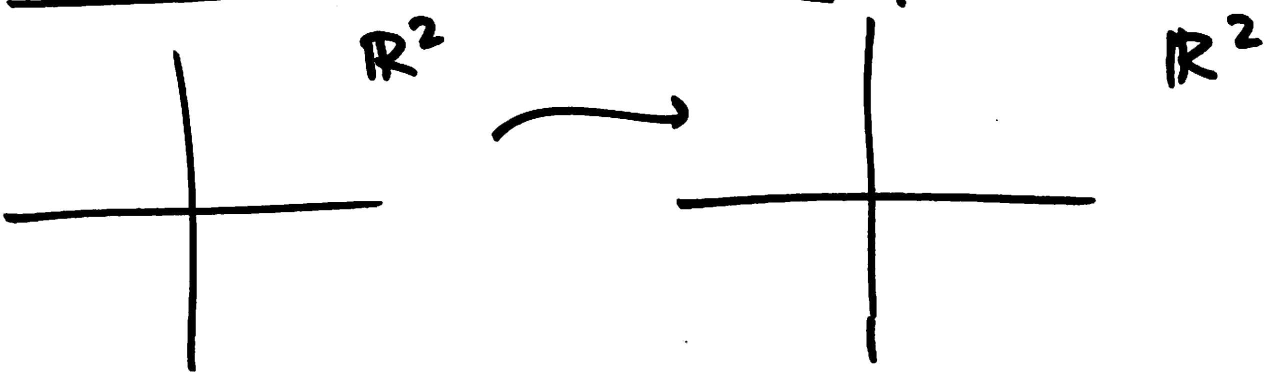
$$\ker = \{\bar{x} \,/\, A\bar{x} = 0\}.$$

$$x_1\bar{v}_1 + x_2\bar{v}_2 = 0$$

$$\begin{bmatrix}-3\\1\end{bmatrix} \Rightarrow -3\cdot\bar{v}_1 + 1\cdot\bar{v}_2 = \bar{0}$$

$\in \ker$ $\quad\Rightarrow \bar{v}_2 = 3\bar{v}_1$

Im = span of $\bar{v}_1$



ker vs Im $\qquad [\bar{v}_1 \ \bar{v}_2]$

ker = line , Im = line.

How to describe a subspace.

Defn: $\text{span}(\bar{v}_1, -, \bar{v}_m) = \{ c_1\bar{v}_1 + \dots + c_m\bar{v}_m / c_i \in \mathbb{R} \}$.

Linearly independent.

$\bar{v}_1, \to \bar{v}_r$ are linearly independent

if. $c_1\bar{v}_1 + \dots + c_r\bar{v}_r = \bar{0} \Rightarrow c_1 = 0, c_2 = 0, \to c_r = 0$.

i.e. only solns to $\begin{bmatrix} | & & | \\ \bar{v}_1 & \cdot & \bar{v}_r \\ | & & | \end{bmatrix} \begin{bmatrix} c_1 \\ \cdot \\ c_r \end{bmatrix} = \bar{0}$ is $\bar{0}$

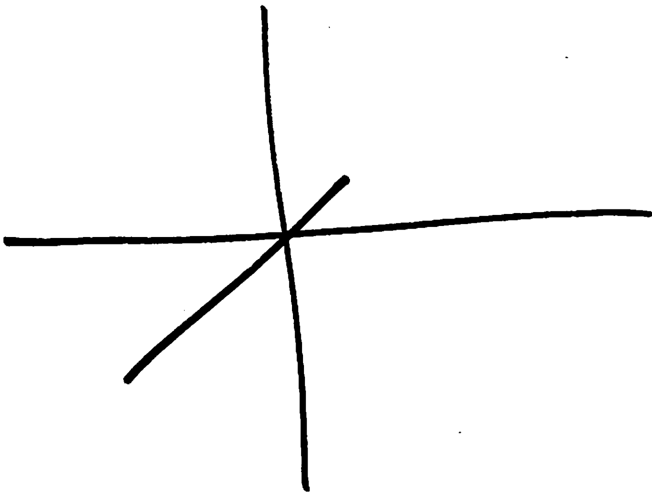
$\bar{v}_1, \bar{v}_2, \bar{v}_3$

- $\bar{v}_1$ is not a l.c. of $\bar{v}_2$ & $\bar{v}_3$
- $\bar{v}_2$ ... " $\bar{v}_1$ & $\bar{v}_3$
- $\bar{v}_3$ .... $\bar{v}_1$ & $\bar{v}_2$

If $c_1\bar{v}_1 + c_2\bar{v}_2 + c_3\bar{v}_3 = \bar{0} \Rightarrow c_1 = 0, c_2 = 0, c_3 = 0$.

$\bar{v}_1 = 3\bar{v}_2 + 5\bar{v}_3$ $\qquad$ $\bar{v}_1 - 3\bar{v}_2 + 5\bar{v}_3 = 0$

$\mathbb{R}^3 \quad V: \; x_1 + x_2 + x_3 = 0$



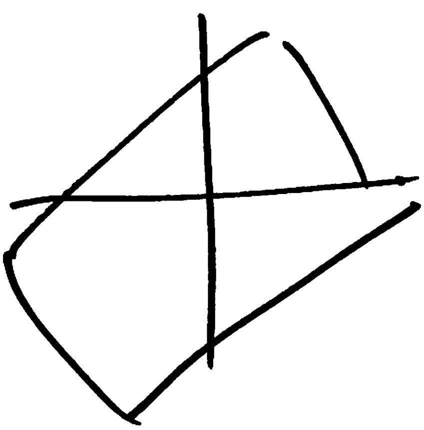
$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \end{array}\right]$

$x_2 = s$

$x_3 = t$

$x_1 = -s - t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -s-t \\ s \\ t \end{bmatrix} = s \underset{\bar{w}_1}{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}} + t \cdot \underset{\bar{w}_2}{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}$$



$$\boxed{\underset{\bar{w}_1}{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}, \underset{\bar{w}_2}{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}, \underset{\bar{w}_3}{\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}}}$$

Span V

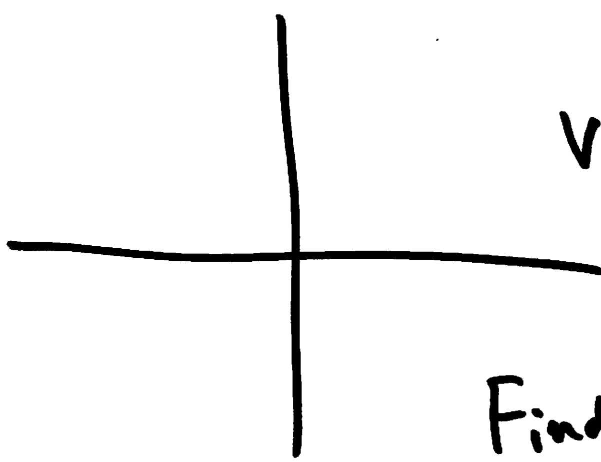
But not lin. ind.

A basis of a subspace

$V \subseteq \mathbb{R}^n$

subspace

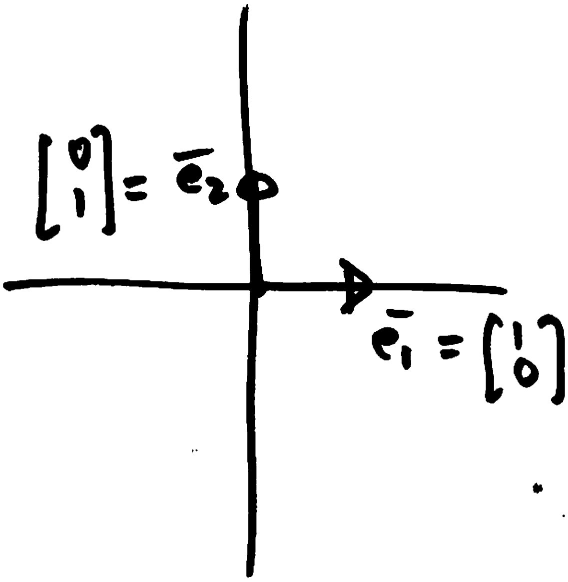
Defn: A set of vectors $(\bar{v}_1, \ldots, \bar{v}_r)$ in V is said to be a basis for V if they span V & they are linearly independent.

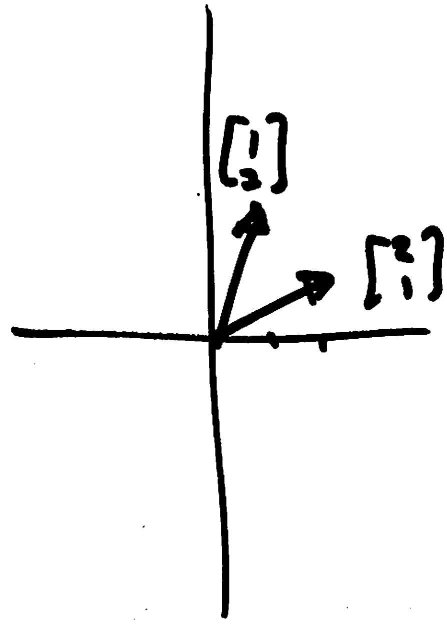


$V = \mathbb{R}^2$

Find a basis for $V = \mathbb{R}^2$

A basis is not unique!

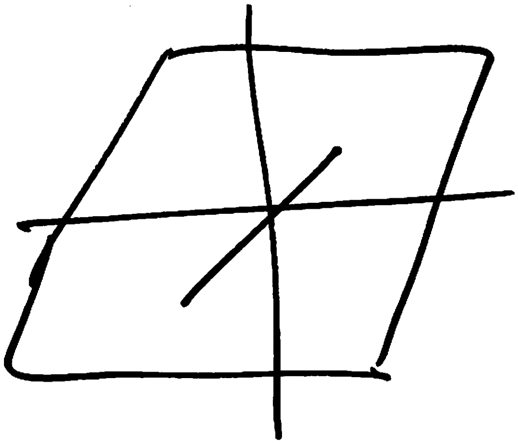




<u>Key Fact:</u> While a basis of a subspace is not unique, the number of elements in a basis is independent of the choice of basis.

$$V \subseteq \mathbb{R}^3$$

$$= \left\{ \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \middle/ x_1 + x_2 + x_3 = 0 \right\}$$



$$\bar{v}_1 = \begin{bmatrix}3\\-1\\-2\end{bmatrix} \quad \bar{v}_2 = \begin{bmatrix}0\\3\\-3\end{bmatrix}$$

$(\bar{v}_1, \bar{v}_2)$ is a basis for V

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \end{array}\right]$$

$$x_1 = -x_2 - x_3$$

dep. vari   free variables

$x_2 = s$

$x_3 = t$

$x_1 = -s-t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -s-t \\ s \\ t \end{bmatrix} = s\underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}_{\bar{w}_1} + t\underbrace{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}_{\bar{w}_2}$$

$$\bar{w}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \bar{w}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$(\bar{w}_1, \bar{w}_2)$ span V.

More generally, $\mathbb{R}^m \xrightarrow{L} \mathbb{R}^n$

$$L(\bar{x}) = A\bar{x}$$

$$\ker(L) = \ker(A) \subseteq \mathbb{R}^m$$

$$\mathrm{Im}(L) = \mathrm{Im}(A) \subseteq \mathbb{R}^n$$

Q: How do we find a basis for ker & Im?

Example: $L: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$L(\bar{x}) = A\bar{x}$,

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

$\bar{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\bar{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

Find a basis for the kernel & image of L (A).

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pivots

$$B = RREF(A) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Kernel: solve $A\bar{x} = \bar{0}$ is the same thing as solving $B\bar{x} = \bar{0}$.

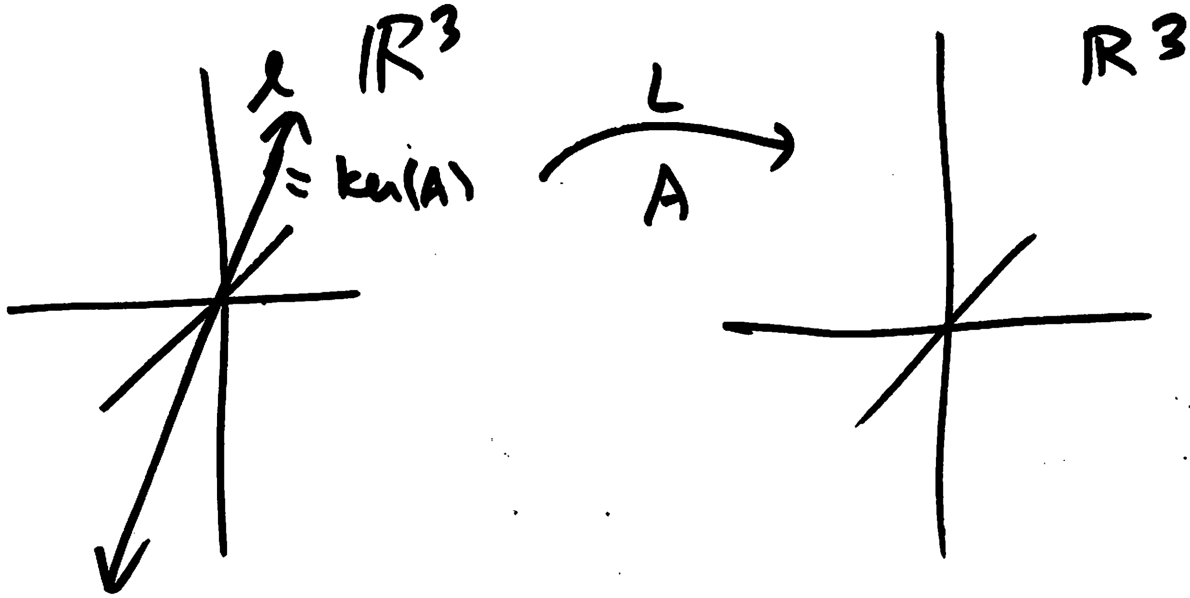
x_1, x_2 : dep. vars. x_3 free var.

$x_1 = x_3$ $x_3 = s$

$x_2 = -2x_3$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ -2s \\ s \end{bmatrix}$$

$$= s \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$



$$A = \begin{bmatrix} | & | & | \\ \bar{v}_1 & \bar{v}_2 & \bar{v}_3 \\ | & | & | \end{bmatrix}$$

$$A\bar{x} = x_1\bar{v}_1 + x_2\bar{v}_2 + x_3\bar{v}_3$$

$$1\cdot\bar{v}_1 + (-2)\bar{v}_2 + 1\cdot\bar{v}_3 = \bar{0}$$

$$\boxed{\bar{v}_3 = (-1)\bar{v}_1 + (2)\bar{v}_2}$$

$$\begin{aligned} \operatorname{Im}(A) &= \{ x_1\bar{v}_1 + x_2\bar{v}_2 + x_3\bar{v}_3 \,/\, x_1, x_2, x_3 \in \mathbb{R} \} \\ &= \{ x_1\bar{v}_1 + x_2\bar{v}_2 \,/\, x_1, x_2 \in \mathbb{R} \} \end{aligned}$$

$(\bar{v}_1, \bar{v}_2)$ is a basis for $\operatorname{Im}(A)$

$$A = \begin{bmatrix} \bar{v}_1 & \bar{v}_2 & \bar{v}_3 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \qquad \underset{\text{RREF}(A)}{B} = \begin{array}{c} \begin{array}{ccc} \bar{w}_1 & \bar{w}_2 & \bar{w}_3 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\bar{v}_3 = (-1)\bar{v}_1 + 2\bar{v}_2 \quad \Leftarrow \quad \bar{w}_3 = (-1)\bar{w}_1 + 2\bar{w}_2$$

Basis for the image:

Look at RREF(A) & identify the columns containing the pivots. These columns are lin. ind, & the other columns are lin combinations of them.

$\Rightarrow$ Same thing holds for A.

$\Rightarrow$ the corresponding cols in A form a basis for Im(A).

$$\underset{n \times m}{\begin{bmatrix} A \end{bmatrix}} \xrightarrow[\text{RR.}]{} \begin{bmatrix} B \end{bmatrix} \quad \mathbb{R}^m \to \mathbb{R}^n$$

$$B = \text{RREF}(A).$$

Defn: dim(V) = number of elements in a basis.

dim (ker A) = number of free vars.

dim (Im A) = number of dep vars.

$$\dim(\ker A) + \dim(\operatorname{Im} A) = \# \text{ of cols in } A = m$$

Rank-Nullity Thm.