Finish §3.3, Start 3.4

Exam: Oct 10

Recap: 3.2: A Subspace of $\mathbb{R}^n$ is the span of some vectors.

- $\{\vec{v}_1, \dots, \vec{v}_r\}$ is "linearly independent" if no vector is contained in the span of the others.

  $\rightsquigarrow$ $\text{Span}\{\vec{v}_1, \dots, \vec{v}_{r-1}\} \neq \text{Span}\{\vec{v}_1, \dots, \vec{v}_r\}$

See "Summary 3.2.9"

- $\{\vec{v}_1, \dots, \vec{v}_r\}$ is lin indep $\iff \ker\left[\vec{v}_1, \dots, \vec{v}_r\right] = \{\vec{0}\}$

  $\iff$ if $c_1\vec{v}_1 + \dots + c_r\vec{v}_r = 0$, then $c_1, \dots, c_r = 0$

Def (Basis) Let $V \subseteq \mathbb{R}^n$ be a subspace. Then a set of vectors $\vec{w}_1, \dots, \vec{w}_r \in V$ is called a basis of V if: $\vec{w}_1, \dots, \vec{w}_r$ are lin. indep. and $\text{span}(\vec{w}_1, \dots, \vec{w}_r) = V$

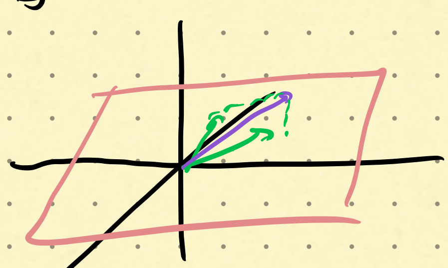
3.3 Dimension

Let $V \subseteq \mathbb{R}^n$ be a subspace.

(Thm 3.3.4) (part of it) Deep fact: Max size of a lin. indep. set of vectors in V = min size of any set of vectors which spans V.

This # is called the dimension of V.

eg. Consider a plane in $\mathbb{R}^3$:



- you can't span the plane w/ fewer than 2 vectors.
- any set of more than 2 vectors in this

plane will be lin. dependent!

So this plane is a 2-dimensional subspace of $\mathbb{R}^3$.

If $V \subseteq \mathbb{R}^n$ is a subspace, then any basis for V must have exactly $\dim(V)$ elements.

(e.g. any basis of that plane must consist of 2 vectors)

Exercise: let $A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}$

find a basis for $\mathrm{im}(A)$ and a basis for $\ker(A)$.

Solve using row reduction.

$\ker(A) = ?$ Solve $A\vec{x} = \vec{0} \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \end{array}\right]$

RREF $\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightsquigarrow \begin{matrix} x_1 + 2x_3 = 0 \\ x_2 + 2x_3 = 0 \end{matrix}$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_3 \\ -2x_3 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$\Rightarrow \ker(A)$ is spanned by $\begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$; Also, ~~and~~

these two vectors are lin. indep.

$\Rightarrow \begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ is a basis for $\ker(A)$.

so these are a basis for $\mathrm{im}(A)$.

pivot cols.

$\downarrow \downarrow$ $\downarrow \downarrow$

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix} \qquad RREF(A) = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

im(A) is spanned by $\begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\end{bmatrix}, \begin{bmatrix}2\\2\\2\\2\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$

But, this not lin. indep! So we need to throw out some vectors to get a basis for im(A).

<u>Here</u>: $2\begin{bmatrix}1\\0\\1\\0\end{bmatrix} + 2\begin{bmatrix}0\\1\\0\\1\end{bmatrix} = \begin{bmatrix}2\\2\\2\\2\end{bmatrix}$, so $\begin{bmatrix}2\\2\\2\\2\end{bmatrix}$ is redundant.

$0 \cdot \begin{bmatrix}1\\0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$

We're left with $\left\{\begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\end{bmatrix}\right\}$, and these are lin. indep!

In general, look at the pivot columns of A.

3.4

In Exercises 1 through 18, determine whether the vector $\vec{x}$ is in the span V of the vectors $\vec{v}_1, \dots, \vec{v}_m$ (proceed "by inspection" if possible, and use the reduced row-echelon form if necessary). If $\vec{x}$ is in V, find the coordinates of $\vec{x}$ with respect to the basis $\mathfrak{B} = (\vec{v}_1, \dots, \vec{v}_m)$ of V, and write the coordinate vector $[\vec{x}]_{\mathfrak{B}}$.

1. $\vec{x} = \begin{bmatrix}2\\3\end{bmatrix}; \vec{v}_1 = \begin{bmatrix}1\\0\end{bmatrix}, \vec{v}_2 = \begin{bmatrix}0\\1\end{bmatrix}$

2. $\vec{x} = \begin{bmatrix}23\\29\end{bmatrix}; \vec{v}_1 = \begin{bmatrix}46\\58\end{bmatrix}, \vec{v}_2 = \begin{bmatrix}61\\67\end{bmatrix}$

3. $\vec{x} = \begin{bmatrix}31\\37\end{bmatrix}; \vec{v}_1 = \begin{bmatrix}23\\29\end{bmatrix}, \vec{v}_2 = \begin{bmatrix}31\\37\end{bmatrix}$

4. $\vec{x} = \begin{bmatrix}3\\-4\end{bmatrix}; \vec{v}_1 = \begin{bmatrix}0\\1\end{bmatrix}, \vec{v}_2 = \begin{bmatrix}1\\0\end{bmatrix}$

5. $\vec{x} = \begin{bmatrix}7\\16\end{bmatrix}; \vec{v}_1 = \begin{bmatrix}2\\5\end{bmatrix}, \vec{v}_2 = \begin{bmatrix}5\\12\end{bmatrix}$

Translation:

- Does the system $[\vec{v}_1 \ \vec{v}_2]\begin{bmatrix}c_1\\c_2\end{bmatrix} = \vec{x}$ (unknowns.) have a solution $\begin{bmatrix}c_1\\c_2\end{bmatrix}$?
- If so, what is the solution?

eg #4 $\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}3\\-4\end{bmatrix} \rightarrow c_2 = 3, \ c_1 = -4$

$\begin{bmatrix}c_2\\c_1\end{bmatrix}$

$\rightarrow [\vec{x}]_B = \begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}-4\\3\end{bmatrix}$

<u>3.3</u>

$$A = \begin{array}{ccccc} \left[\begin{matrix} 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{matrix}\right] \\ \begin{matrix} v_1 & \;\; v_2 & \; v_3 & \;\; v_4 & \; v_5 \end{matrix} \end{array}$$

Find bases for $\mathrm{im}(A)$, $\ker(A)$ "by inspection"

Basis for $\mathrm{im}(A) = ?$ The vectors $v_1, \ldots, v_5$ span $\mathrm{im}(A)$,

v_2 and v_4 look redundant: $\boxed{\begin{aligned} v_2 &= -2v_1 \\ v_4 &= -v_1 + 5v_3 \end{aligned}}$

Another argument: $\mathrm{span}(\vec{v}_1, \vec{v}_3, \vec{v}_5) = \mathbb{R}^3$, so $\vec{v}_1, \vec{v}_3, \vec{v}_5$ span im A.

Also, $\underline{\vec{v}_1, \vec{v}_3, \vec{v}_5}$ are lin. indep, so this is a basis for im A.

$\ker(A) = ?$ $\quad 2\vec{v}_1 + \vec{v}_2 = 0$. $\rightsquigarrow \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in \ker(A)$,

$\vec{v}_1 + \vec{v}_4 - 5\vec{v}_3 = 0$ $\quad \begin{bmatrix} 1 \\ 0 \\ -5 \\ 1 \\ 0 \end{bmatrix} \cancel{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ -5 \end{bmatrix}} \in \ker(A)$,

and they're lin. indep.

Are we done? Rank-nullity thm:

$$\dim(\ker(A)) = \underbrace{\#\text{cols of } A}_{5} - \underbrace{\dim(\mathrm{im}(A))}_{3}$$

$= 2$.

So we already found our 2 basis vectors.