Final Project Details

Math 451 Sections 001 and 002

Winter 2022

For your Math 451 final project, you will write a paper about analysis. This is an opportunity for you to solidify your understanding of the material in our course, to explore an aspect of analysis that interests you personally, and to demonstrate what you learned this semester on your own terms. As mentioned in the syllabus, the final project is worth 25% of your final course grade.

A good final project should:

- demonstrate a thorough understanding of concepts discussed in Math 451 and/or involve independent study (i.e. learning material not taught in class),
- be well-written and understandable to any of your classmates,
- include at least one precise proof, and
- be a fun learning experience for you!

This document explains the logistics of the final project and my expectations for people's projects. At the end I provide a list of suggested topics you could write about, but I am happy to let you do something else for your project that you're excited about—see section 5.

1 Expectations

You need to submit three things for your projects:

- A project proposal, due March 29th
- A project draft, due April 14th
- The final draft, due April 26th at midnight

You only need one person from your team to submit each of these three things.

1.1 Project Proposal

Your project proposal should briefly explain what you plan to do for your project as well as who you will work with on your team. If you're choosing one of the projects below, you can just say which project you're choosing. If you want to come up with your own project idea though, you should write a paragraph explaining what specific theorems or ideas you wish to cover and what resources you can use to learn about them. **If you are coming up with your own final project topic, please make an appointment to talk with me about it before submitting your project proposal.** I will post a google form on canvas where your team will submit your proposal. Submitting your proposal on time is worth 1% of your course grade.

It is expected, but not required, that you work in a team of 2 or 3 people on your project. This is for a number of reasons—for one, I think that being able to collaborate well is an important skill for a mathematician to have and that working in a team will allow you to have a more insightful and fulfilling project. Also I would much rather grade 35 projects than 70 :). I understand some people have good reasons for working alone though, so I'm not forcing anyone to work in a team.

Your teammate(s) can be enrolled in either section of the course. You can also let me know in your project proposal if you'd like to have a teammate but haven't been able to find one. More on that below.

1.2 Project draft

The point of the project draft is to make sure that you're on track to finish your final project and that you understand the expectations. Your project draft should include an outline of the whole project and at least one completed page of writing that you expect to go into the project unedited. Your outline should describe in detail what work you still have to do in order to complete your project—for instance what theorems you plan to include where in your paper. Your team will submit your draft on gradescope. This component is worth 4% of your course grade.

1.3 Final submission

Final projects will be submitted on gradescope. They are due by midnight on April 26th (though I will accept submissions as late as April 27th at 9am, because that's when I'll actually start grading them. No need to ask for an extension). This will be worth the final 20% of your grade.

There is no length requirement for this project, but I expect most projects to be about 2500-5000 words (in other words, 5-10 typed single-spaced pages) in length. Everything you submit should be original work: it's inevitable that your project will include proofs you found somewhere else, but you should rewrite these proofs in your own words. Be sure to cite your sources as well—tell your reader where you found your proofs and who deserves credit! I don't really care what sort of citation style you use (MLA/APA/etc) as long as it's easy for me to find the document you're referring to.

As mentioned, you should write in an expository style. The "Guidelines for Good Mathematical Writing" document on canvas has some good, well, guidelines.

Your final draft will be graded out of 20 points. They will be graded based on the quality of writing (5 points), on the breadth of material covered (5 points) and on the depth/precision/soundness of the mathematics (10 points).

2 On typesetting mathematics

I expect final projects to be typed up. By and large, the mathematical community uses the LaTeX markup language to type mathematical documents on the computer, though there are ways to get the job done using Microsoft Word or Google Docs. I suggest using LaTeX. If you're new to LaTeX, I would suggest pairing up with someone who has used it before. I'm also happy to work with you if you have trouble using the software.

I also suggest using something like `overleaf.com` to collaborate on a single LaTeX document and to avoid the (somewhat confusing) process of installing a LaTeX compiler on your own computer. That website also has great resources for learning LaTeX: see the link "Learn LaTeX in 30 minutes" on canvas.

That said, please let me know if typing your paper is a serious hurdle for you and we can figure something out.

3 Finding teammates

Due to the hybrid nature of our course, a lot of us haven't been able to meet many potential teammates. For this reason, I'm opening a discussion board on canvas where you can post if you're looking for a teammate. I will also have an option in the project proposal form where you can indicate if you're looking for a teammate but haven't been able to find one. In that case, I'll try to match you with a classmate who had a similar project proposal. I can't guarantee I'll be able to find a match for everyone though.

4 Project ideas

The following is a list of potential final project ideas. I ordered them roughly in terms of how much new material you'd have to learn to do them.

4.1 Summarize the course

Go through chapters 1 through 4 of our textbook and answer the following: what were the main theorems and definitions from each section? How did the theorems build on each other? Draw a diagram showing how each result builds on earlier results.

Look back at the proofs we saw in these chapters. Were there any proof techniques or tricks that came up often? Point out these techniques and give them descriptive names. For instance, one technique that we've used a few times is that, in order to prove $a \leq b$, we instead showed $a \leq b + \varepsilon$ for all $\varepsilon > 0$. We could call this technique "giving yourself an ε of wiggle room" or something like that. Write proofs demonstrating each of these techniques (these can be proofs we've seen in class, just explained in your own words).

Completing this project will give you a great document to refer back to in the future!

4.2 Teach Math 451

Come up with three lesson plans for teaching three topics in analysis in one hour each. For instance, one of your topics can be the notion of a subsequence—in that case, you'd want to think about what are the most important things to know about subsequences and how you can teach them in one hour. Your project should include your lecture notes or any worksheets/activities you would have the class work on.

Your project should also include a brief discussion of each of your lectures: how did you choose what to cover for each topic? What skills would you want students to have after completing your lessons? How would you assess whether your students learned these skills and offer them feedback?

You may wish to look at some other analysis textbooks for alternate ways to explain the material.

4.3 Construction of the real numbers

In our class we defined the real numbers as a set, $\mathbb{R}$, with binary operations $+$ and $\cdot$ and a binary relation $\leq$ satisfying 15 axioms. Everything in our course is proved on the basis of these 15 axioms. However, it's not obvious that such a set even exists!

This project is about showing that the real numbers do in fact exist by describing any real number in terms of the rational numbers. One description, due to Dedekind, is to define a "Dedekind cut" to be a set $A \subseteq \mathbb{Q}$ such that:

- $A \neq \emptyset$ and $A \neq \mathbb{Q}$,
- if $r \in A$ and $x < r$, then $x \in A$, and
- A has no maximum element.

Show that the set of Dedekind cuts forms a complete ordered field by explaining how to add, multiply, and order Dedekind cuts in a way that satisfies the 15 axioms of a complete ordered field. Make sure to include some concrete examples illustrating these operations.

If you prefer though, you can define a real number as a Cauchy sequences of rational numbers; see [Kem]. This approach makes it much easier to explain how to add and multiply real numbers, but it requires the added machinery of "equivalence classes."

One good way to do this project is to follow the presentation of Section 8.6 of [Abb15] and do all the exercises.

4.4 Defining exp and log

We've all seen expressions like e^x and 2^π, but upon close inspection it's not clear what this really means—if exponentiation is repeated multiplication, how can you raise a number to an irrational power? How would you define this based on the 15 axioms of $\mathbb{R}$ (i.e. based on things we've seen in this class)? This project is about putting these functions on firm foundations.

There are many ways to do this. For instance, you can define $\exp(x)$ using a power series, show this function is invertible, and define $\log(x)$ to be the inverse. Another approach is to start by defining $\log x =$

$\int_1^x \frac{1}{t}dt$, showing that this function is invertible, and defining $\exp(x)$ to be the inverse function.[1] Spivak's book [Spi06] is a great reference for the latter approach.

No matter what approach you choose, you should carefully define the functions $\exp(x)$, $\log(x)$, and b^x (where x is an arbitrary real number and b is an arbitrary positive real number). You should also prove, using your definitions, the familiar facts:

- $\log(xy) = \log x + \log y$
- $\exp(x + y) = \exp x \cdot \exp y$
- $b^{xy} = (b^x)^y$ and $b^{x+y} = b^x \cdot b^y$
- For any positive number $p \in \mathbb{R}$, the function $f(x) = x^p$ is a continuous increasing function on its domain
- $\exp'(x) = \exp(x)$

4.5 Prove that π and e are transcendental

We discussed early in the semester how $\overline{\mathbb{Q}}$, the set of algebraic numbers, is an "algebraic" way to add "missing" numbers into the rationals. It turns out that you miss a lot of real numbers this way though! In this project, you can demonstrate this fact by proving that e is transcendental, meaning $e \notin \overline{\mathbb{Q}}$. Of course you should look up the proof somewhere, but make sure to write it in your own words and to show all the details in your project. Follow your proof with a discussion: how would you break it down into a few main steps that you can remember? What were the key ideas and techniques involved? What were the hardest parts of the proof?

If you have time, you can also address π. The proof that π is transcendental is a lot more involved that e, so for this I recommend stating the (rather deep) Lindemann-Weierstrass theorem and using this theorem to show that π is transcendental. But if you prefer, you can also write a proof that only uses material we learned in the class, such as [Niv39]

4.6 Metric spaces

A metric space is a set, S, along with some notion of distance—see section 13 for a precise definition. Metric spaces provide a natural setting to which we can generalize many of the theorems of analysis, and are also important to modern topology and geometry. For this project, you should introduce the idea of a metric space and describe a few examples. Describe the concept of an open set, closed set, and compact set in a metric space, and again show various examples. Prove the Heine-Borel theorem: any subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded. Of course, before you do this, you should explain what each of those words means.

If you have time, you could next define what it means for a function between two metric spaces to be continuous. Prove that the continuous image of a connected set is connected, and use this to show a quick proof of the intermediate value theorem. Explain also why any continuous function from a compact set to $\mathbb{R}$ has a maximum.

Feel free to state some technical lemmas without proving them if it helps you get to the more interesting material. Just make sure you're very clear about which facts you're taking for granted.

Our textbook explains all of this material in some of the sections that we skipped. Another good reference is [Rud76].

5 Coming up with your own project idea

There are many other directions in analysis you could explore! For one, there are tons of more "advanced" topics in analysis you could explore, such as p-adic numbers, point-set topology, measure theory, or Hilbert spaces. You could explore some more niche topics like Stirling's formula or the Cantor set. You can delve

[1] Here, by $\log(x)$ I mean the natural logarithm—this is a common notational convention in pure mathematics.

into the history of analysis and explain how the ideas of calculus have evolved over time. You could explore some applications of analysis to other disciplines: for instance, if your major is not mathematics, you can write about applications of analysis to your field.

I am also open to final project ideas that are more artistic/creative than a traditional term paper. For instance, maybe writing isn't really your thing and you'd really prefer to do an interpretive dance explaining some theorems from the course[2]. Or maybe you like art and you want to make illustrations explaining some of the proofs we've seen in class. Or maybe you like games and you want to create an analysis-themed board game. I'm happy as long as your project demonstrates that you've engaged seriously with the material of our course and includes some sort of written component explaining your thinking.

If you wish to explore your own idea for the final project, please make an appointment to talk to me about your idea before submitting your project proposal.

References

[Abb15] Stephen Abbott. *Understanding analysis*. Second. Undergraduate Texts in Mathematics. Springer, New York, 2015, pp. xii+312. ISBN: 978-1-4939-2711-1; 978-1-4939-2712-8. DOI: 10.1007/978-1-4939-2712-8. URL: https://doi-org.proxy.lib.umich.edu/10.1007/978-1-4939-2712-8.

[Kem] Todd Kemp. https://mathweb.ucsd.edu/~tkemp/140A/Construction.of.R.pdf. Accessed March, 2022.

[Niv39] Ivan Niven. "The transcendence of π". In: *Amer. Math. Monthly* 46 (1939), pp. 469–471. ISSN: 0002-9890. DOI: 10.2307/2302515. URL: https://doi-org.proxy.lib.umich.edu/10.2307/2302515.

[Rud76] W. Rudin. *Principles of Mathematical Analysis*. International series in pure and applied mathematics. McGraw-Hill, 1976. ISBN: 9780070856134. URL: https://books.google.com/books?id=kwqzPAAACAAJ.

[Spi06] M. Spivak. *Calculus*. Calculus. Cambridge University Press, 2006. ISBN: 9780521867443. URL: https://books.google.com/books?id=7JKVu%5C_9InRUC.

[2] This would not be unprecedented. See, for instance: https://vimeo.com/47049144