BOREL SUPERRIGIDITY FOR ACTIONS OF LOW RANK LATTICES

BY SCOTT SCHNEIDER

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ABSTRACT OF THE DISSERTATION

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Dissertation Director: Simon Thomas

A major recent theme in Descriptive Set Theory has been the study of countable Borel equivalence relations on standard Borel spaces, including their structure under the partial ordering of Borel reducibility. We shall contribute to this study by proving Borel incomparability results for the orbit equivalence relations arising from Bernoulli, profinite, and linear actions of certain subgroups of $PSL_2(\mathbb{R})$. We employ the techniques and general strategy pioneered by Adams and Kechris in [3], and develop purely Borel versions of cocycle superrigidity results arising in the dynamical theory of semisimple groups.

Specifically, using Zimmer's cocycle superrigidity theorems [58], we will prove Borel superrigidity results for suitably chosen actions of groups of the form $PSL_2(\mathcal{O})$, where $\mathcal{O}$ is the ring of integers inside a multi-quadratic number field. In particular, for suitable primes $p \neq q$, we prove that the orbit equivalence relations arising from the natural actions of $PSL_2(\mathbb{Z}[\sqrt{q}])$ on the p-adic projective lines are incomparable with respect to Borel reducibility as p, q vary. Furthermore, we also obtain Borel non-reducibility results for orbit equivalence relations arising from Bernoulli actions of the groups $PSL_2(\mathcal{O})$. In particular, we show that if E_p denotes the orbit equivalence relation arising from a nontrivial Bernoulli action of $PSL_2(\mathbb{Z}[\sqrt{p}])$, then E_p and E_q are incomparable with respect to Borel reducibility whenever $p \neq q$.

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Chapter 1

Introduction to countable Borel equivalence relations

This thesis is a contribution to the study of Borel equivalence relations on standard Borel spaces. The organizing framework for this theory comes from Descriptive Set Theory, which classically has consisted of the study of definable sets and functions in complete, separable metric spaces. Recently a new direction of research has emerged in the field, aimed at understanding the definable equivalence relations on these spaces together with their resulting quotients. As we will see, such quotients arise naturally as spaces of invariants for classification problems in many different areas of mathematics. When realized in this way as definable equivalence relations, such classification problems admit a natural notion of "relative complexity" that induces a partial pre-ordering on the collection of all such definable equivalence relations. While applications to the complexity theory of classification problems constitute an important, and indeed motivating, branch of the subject, much recent attention has been focused on understanding the structure of the partial ordering of the complexity classes itself. This thesis contributes to the study of this structure, and to the development of techniques for its continued study.

In this introductory chapter we will develop preliminary notions and provide a context for our results by highlighting some elements of the basic theory of countable Borel equivalence relations. We begin with some elementary definitions and facts from Descriptive Set Theory.

1.1 Background from Descriptive Set Theory

Formally, a *Polish space* is a topological space that admits a complete, separable metric, and a *standard Borel space* is a measurable space $(X, \mathcal{B})$ that admits a Polish topology whose σ-algebra of Borel sets is $\mathcal{B}$. If $(X, \mathcal{B})$ and $(Y, \mathcal{C})$ are standard Borel spaces, then a function $f : X \to Y$ is called *Borel* if $f^{-1}(C) \in \mathcal{B}$ for every $C \in \mathcal{C}$. A *Borel isomorphism* is a bijective function $f : X \to Y$ such that both f and f^{-1} are Borel. By a classical theorem due to Kuratowski, any two uncountable standard Borel spaces are Borel isomorphic.

Examples of standard Borel spaces include the unit interval $[0, 1]$, the analytic spaces $\mathbb{R}$, $\mathbb{R}^n$, and $\mathbb{C}$, the p-adic numbers $\mathbb{Q}_p$, and also the descriptive set-theoretic spaces such as Baire space $\mathbb{N}^{\mathbb{N}}$ and Cantor space $2^{\mathbb{N}} = \mathcal{P}(\mathbb{N})$. Any Borel set $A \subseteq X$ in a standard Borel space $(X, \mathcal{B})$ can be regarded as a standard Borel space in its own right with the induced σ-algebra

$$\mathcal{B} \restriction A = \{B \cap A \mid B \in \mathcal{B}\},$$

although in general it may be necessary to take a different underlying Polish topology. For instance, the open unit interval $(0, 1) \subseteq \mathbb{R}$ with its subspace σ-algebra is standard Borel, even though it is not complete in the subspace topology.

Informally, we think of Borel sets and functions as those that are "explicitly" constructed or defined, and we think of the unique (up to isomorphism) uncountable standard Borel space as the abstract setting in which "explicit" or "concretely definable" mathematics takes place. Working in the Borel setting will often resemble working without the full power of the Axiom of Choice.

If $(X, \mathcal{B})$ and $(Y, \mathcal{C})$ are standard Borel spaces, then $X \times Y$ is a standard Borel space in the product σ-algebra $\mathcal{B} \times \mathcal{C}$, which coincides with the Borel σ-algebra of the product topology $\tau \times \sigma$ for any Polish topologies τ and σ that generate $\mathcal{B}$ and $\mathcal{C}$, respectively. In particular, an equivalence relation $E \subseteq X \times X$ on the standard Borel space X is called *Borel* if E is Borel in the product space $X \times X$. Furthermore, if X and Y are standard Borel spaces then a function $f : X \to Y$ is Borel if and only if its graph is Borel as a subset of $X \times Y$ (for instance, see [49, 4.5.2]). For general background from Descriptive Set Theory see [30] or [49].

1.2 Classification problems and equivalence relations

Abstractly, we might say that a "classification problem" in mathematics consists of:

(1) a collection of mathematical objects to be classified, and

(2) some notion of equivalence up to which the classification is to be made.

For example, we might wish to classify some collection of groups up to group isomorphism, or graphs up to graph isomorphism, or topological spaces up to homeomorphism. The study of definable equivalence relations on standard Borel spaces is motivated by the fact that for many such abstract classification problems, the collection of objects to be classified can be given the structure of a standard Borel space, on which the given notion of equivalence is a definable equivalence relation.

As an example of how this might be done, let us consider the problem of classifying all countable groups up to group isomorphism. At first glance it might seem impossible to form the "space" of all countable groups, since this collection is a proper class. However, since we wish only to consider groups up to isomorphism, without loss of generality we may take each such group to have underlying set $\mathbb{N}$. In fact, up to isomorphism, the countable group $(G, \circ_G)$ is completely determined by the information contained in its multiplication function, $\circ_G$. Taking G to have underlying set $\mathbb{N}$, this multiplication function can be regarded formally as an element of the Polish space $\mathbb{N}^{\mathbb{N}\times\mathbb{N}}$. In this way, the set

$$X = \{\, \circ_G \in \mathbb{N}^{\mathbb{N}\times\mathbb{N}} \mid \text{the group axioms hold}\}$$

can be viewed as the collection of all countable groups. As a Borel subset of $\mathbb{N}^{\mathbb{N}\times\mathbb{N}}$, X is a standard Borel space in its own right. Moreover, two groups $\circ_G$ and $\circ_H$ in X will be isomorphic as groups if and only if there is some permutation $\pi : \mathbb{N} \to \mathbb{N}$ such that $\pi(m) \circ_H \pi(n) = \pi(m \circ_G n)$ for all $(m, n) \in \mathbb{N} \times \mathbb{N}$. Hence a given abstract group will appear many times in X, and the equivalence relation E on X defined by

$$E(\circ_G, \circ_H) \iff \circ_G \text{ and } \circ_H \text{ are isomorphic as groups}$$

is a definable equivalence relation on X. Understanding this equivalence relation amounts to understanding the problem of classifying the countable groups up to isomorphism.

In fact, this example is a special case of a general construction that goes back to Friedman-Stanley [15] and Hjorth-Kechris [22]. Suppose $\mathcal{L}$ is any countable relational language, and let σ be a sentence in the infinitary logic $\mathcal{L}_{\omega_1,\omega}$ in which countable conjunctions and disjunctions are allowed. (Since functions and constants may always be viewed as relations, there is no loss of generality in assuming $\mathcal{L}$ to be relational). Then as discussed in Hjorth-Kechris [22], the collection of $\mathcal{L}$-structures

$$\mathrm{Mod}\,(\sigma) = \{\mathcal{M} \mid \text{the universe of } \mathcal{M} \text{ is } \mathbb{N} \text{ and } \mathcal{M} \models \sigma\}$$

is a standard Borel space, and the isomorphism relation $\cong_\sigma$ on $\mathrm{Mod}\,(\sigma)$ is a definable equivalence relation. This construction shows that a great many naturally occurring classification problems can be viewed as definable equivalence relations on standard Borel spaces. Indeed the primary restriction for viewing a classification problem in this way is that the objects one wishes to classify must be determined by a countable amount of data, or else they will be too numerous to collect together into a separable space.

Recall that a set A in a standard Borel space X is *analytic* if there is a Borel set $B \subseteq X$ and a Borel function $f : X \to X$ such that $A = f(B)$. In the setting of the previous paragraph, the isomorphism relation $\cong_\sigma$ is an analytic equivalence relation on $\mathrm{Mod}\,(\sigma)$ that need not be Borel, although in many important special cases it is (for the analyticity of $\cong_\sigma$ see [22]). Every definable equivalence relation on a standard Borel space that we shall consider in this thesis will be Borel, and hence from now on, and for definiteness, we speak simply of *Borel* rather than "definable" equivalence relations.

1.3 Borel reducibility

Suppose that a classification problem has been represented as a Borel equivalence relation E on a standard Borel space X. Then we might say that the classification problem is "solved" by finding a complete set of invariants for E; that is, by assigning to each

$x \in X$ an invariant $i(x)$ such that $i(x) = i(y)$ if and only if $x \mathrel{E} y$. For this to be a reasonable solution to the classification problem, both the assignment i and the invariants $i(x)$ should be as concrete as possible. For example, the quotient map $x \mapsto [x]_E$ assigning E-classes as invariants usually cannot be considered a satisfactory solution, since the space X/E of E-classes is typically an extremely complicated object. Indeed, for all but the simplest equivalence relations E on a Polish space X, the collection X/E of E-classes cannot be viewed in any reasonable way as a definable set inside a Polish space; in particular, X/E can be viewed as a standard Borel space in its own right only if the equivalence relation E is "trivial" in the following sense:

Definition 1.3.1. *Let E be a Borel equivalence relation on the standard Borel space X. Then E is* concretely classifiable*, or* smooth *for short, if there is a standard Borel space Y and a Borel function $f : X \to Y$ such that for all $x, x' \in X$,*

$$x \mathrel{E} x' \iff f(x) = f(x').$$

Informally, then, we might consider a classification problem to be fully and concretely "solved" if we can assign, in a Borel fashion, complete invariants which can be viewed as elements in a suitable standard Borel space.

Of course, not every classification problem can be expected to admit such a solution. Vastly generalizing the notion of smoothness as in Friedman-Stanley [15], suppose we are given a pair of Borel equivalence relations E and F on the standard Borel spaces X and Y, respectively. Then a *Borel reduction* of E to F is a Borel function $f : X \to Y$ on the underlying spaces such that for all $x, x' \in X$,

$$x \mathrel{E} x' \iff f(x) \mathrel{F} f(x').$$

We call E *Borel reducible* to F, and write $E \leq_B F$, if there is a Borel reduction of E to F. If f is such a reduction, then any set of invariants for F may also be regarded, via composition with f, as a set of invariants for E. Hence one can interpret $E \leq_B F$ to mean that the classification problem associated to E is at most as complicated as that associated to F, and so $\leq_B$ can be thought of as a complexity comparison on classification problems. Additionally one might interpret $E \leq_B F$ to mean that the elements of X can be classified up to E by choosing elements of the quotient space Y/F as complete invariants; and hence in particular, the quotient space X/E, considered as a space of potential invariants, is no more complicated than Y/F.

Of course, another way of saying this is that Y/F is *at least* as complicated as X/E, which sounds discouraging if we are to regard the assignment of complete invariants chosen from Y/F as a "solution" to the classification problem corresponding to E. After all, what good is a reduction if the complexity of the invariants can only increase, or at best remain the same? For this reason, it is sometimes more useful to consider an assignment of *incomplete* invariants that respects E-equivalence but does not necessarily distinguish between E-inequivalent elements. We may define such an assignment as follows. If E and F are Borel equivalence relations on the standard Borel spaces X and Y, respectively, then a *Borel homomorphism* from E to F is a Borel function $f : X \to Y$ such that for all $x, x' \in X$,

$$x \, E \, x' \quad \Longrightarrow \quad f(x) \, F \, f(x').$$

We can think of a Borel homomorphism as a "partial" solution to a classification problem that assigns invariants which may not be complete, but at least stand a chance of being less complicated than the collection of objects to be classified. Clearly every reduction is in particular a homomorphism.

The relation $\leq_B$ is easily seen to define a partial pre-order on the collection of all Borel equivalence relations on standard Borel spaces. We say that Borel equivalence relations E and F are *Borel bireducible*, and write $E \sim_B F$, if $E \leq_B F$ and $F \leq_B E$. Then $\leq_B$ induces a partial order on the collection of $\sim_B$-classes of Borel equivalence relations, which we also denote by $\leq_B$. (Indeed, we shall sometimes ignore the distinction between an equivalence relation E and its $\sim_B$-class, especially when referring, by a slight abuse of terminology, to "the partial order $\leq_B$ on the collection of Borel equivalence relations." If we want to emphasize the distinction between E and $[E]_{\sim_B}$, we sometimes call $[E]_{\sim_B}$ the *Borel complexity* of E.) Finally, we write $E <_B F$ if $E \leq_B F$ but $F \nleq_B E$, and $E \perp_B F$ if E and F are $\leq_B$-incomparable. The structure of $\leq_B$ is of great interest, and will be the subject of the next two sections.

1.4 Borel equivalence relations and the $\leq_B$ partial order

For $n < \omega$ let n denote also the standard Borel space containing n elements, and let $\Delta(X)$ be the identity relation on the standard Borel space X. The partial order $\leq_B$ begins with the uninteresting initial segment

$$\Delta(1) <_B \cdots <_B \Delta(n) <_B \cdots <_B \Delta(\mathbb{N})$$

consisting of the Borel equivalence relations with countably many classes. These relations are characterized up to Borel bireducibility by the number of classes, and henceforth will be omitted from consideration. By a dichotomy theorem due to Silver that predates the subject, a Borel equivalence relation with uncountably many classes admits a perfect set of pairwise inequivalent elements, and hence has continuum many classes [47]. Any such relation E admits an injective Borel reduction of $\Delta(\mathbb{R})$. The smooth relations are precisely those that reduce to $\Delta(\mathbb{R})$, which then can be thought of as the least complex Borel equivalence relations with uncountably many classes.

A second dichotomy theorem due to Harrington, Kechris, and Louveau establishes the existence of an immediate $\leq_B$-successor to $\Delta(\mathbb{R})$, denoted E_0 and defined on Cantor space as follows: for every $\alpha, \beta \in 2^{\mathbb{N}}$, let

$$\alpha \, E_0 \, \beta \quad \iff \quad (\exists N)(\forall n \geq N) \, \alpha(n) = \beta(n),$$

so that E_0 is the equivalence relation of eventual equality on binary sequences. Generalizing an earlier dichotomy result of Glimm-Effros, Harrington, Kechris, and Louveau showed in [19] that if F is any nonsmooth Borel equivalence relation, then $E_0 \leq_B F$, so that

$$\Delta(1) <_B \cdots <_B \Delta(n) <_B \cdots <_B \Delta(\mathbb{N}) <_B \Delta(\mathbb{R}) <_B E_0$$

is an initial segment of the partial order $\leq_B$. Beyond E_0 this linearity breaks down, and the structure of $\leq_B$ becomes extremely complicated, as we shall see.

In particular the partial ordering $\leq_B$ admits no greatest element among the Borel equivalence relations: by a construction due to Friedman and Stanley, for every Borel equivalence relation E there is a Borel equivalence relation F such that $E <_B F$

[15, Section 1.2]. There do, however, exist analytic equivalence relations (necessarily non-Borel) that are universal for a significant subclass of Borel equivalence relations. Here we say that a definable equivalence relation F is *universal* for the collection $\mathcal{C}$ of definable equivalence relations if $F \in \mathcal{C}$ and $E \leq_B F$ for every $E \in \mathcal{C}$.

Recall that if σ is an $\mathcal{L}_{\omega_1,\omega}$-sentence for some countable relational language $\mathcal{L}$, then we write $\cong_\sigma$ for the isomorphism relation on the standard Borel space $\operatorname{Mod}(\sigma)$. An equivalence relation E on the standard Borel space X is said to be *classifiable by countable structures* if there is some σ such that $E \leq_B \cong_\sigma$. Following Friedman-Stanley [15], we call an equivalence relation *Borel complete* if it is universal for the collection of equivalence relations that are classifiable by countable structures. It is shown in [15] that there exist sentences σ such that the isomorphism relation $\cong_\sigma$ is Borel complete. For example, isomorphism of countable groups and isomorphism of countable graphs are Borel complete. As indicated above, no such equivalence relation can be Borel. Classification problems such as these whose corresponding equivalence relations are Borel complete should be regarded as totally intractable.

1.5 Countable Borel equivalence relations

We now restrict our attention to an important subclass of Borel equivalence relations that will be our central focus throughout the remainder of this thesis. A Borel equivalence relation E on the standard Borel space X is said to be *countable* if each of its equivalence classes is countable, and *essentially countable* if there is a countable Borel equivalence relation F on some standard Borel space Y such that $E \sim_B F$. Unlike the collection of all Borel equivalence relations, the subclass of countable relations does admit a universal element, which we now describe.

If G is any countable group, define the *left shift* action of G on its powerset 2^G by

$$(g \cdot x)(h) = x(g^{-1}h).$$

If $\mathbb{F}_2$ is the 2-generator free group, then the orbit equivalence relation E_∞ arising from the left shift action of $\mathbb{F}_2$ on its powerset is universal for countable Borel equivalence relations in the sense that $E \leq_B E_\infty$ for every countable Borel equivalence relation E

(for instance, see [10]). Hence the nonsmooth countable Borel equivalence relations all lie in the $\leq_B$-interval $[E_0, E_\infty]$.

Important natural examples of countable Borel equivalence relations include:

(1) $\Delta(\mathbb{R})$ and E_0;

(2) the isomorphism relation on finitely generated groups;

(3) the Turing equivalence relation on $\mathcal{P}(\mathbb{N})$ defined by $A \equiv_T B$ iff A and B lie in the same Turing degree;

(4) the orbit equivalence relation arising from a Borel action of a countable group on a standard Borel space.

In fact, there is an intimate relationship between countable Borel equivalence relations and Borel actions of countable groups. Let us introduce some notation and terminology which will be used throughout this thesis. If G is any Polish group, then by a *standard Borel G-space* we mean a standard Borel space X together with a Borel action of G on X. If X is a standard Borel G-space, then we denote by E_G^X the corresponding orbit equivalence relation arising from the action of G on X. In this case if G is countable, then clearly E_G^X is countable Borel, as in (4) above. More surprisingly, if G is locally compact then E_G^X is essentially countable by Kechris [29]. But perhaps the single most striking feature of the countable relations is that the converse of (4) holds: by a representation theorem due to Feldman and Moore [14], if E is an *arbitrary* countable Borel equivalence relation on the standard Borel space X, then there is a countable group Γ together with a Borel action of Γ on X such that $E = E_\Gamma^X$.

Unfortunately, the group Γ and its action on X are not canonically given in the proof of the Feldman-Moore Theorem. Indeed, while the Turing relation $\equiv_T$ is countable Borel, there is no known "natural" group action $\Gamma \curvearrowright \mathcal{P}(\mathbb{N})$ for which $\equiv_T$ arises as the orbit equivalence relation $\equiv_T = E_\Gamma^{\mathcal{P}(\mathbb{N})}$. Nevertheless, the representation theorem of Feldman and Moore provides a crucial starting point for studying the structure of the $\leq_B$ partial order on the collection of countable Borel equivalence relations.

1.6 Superrigidity: a first glance

Indeed, by the Feldman-Moore Theorem, in studying countable Borel equivalence relations we are essentially studying the orbit spaces of countable group actions. This suggests the following strategy for determining the structure of $\leq_B$ on the countable relations: perhaps we might try $\leq_B$-distinguishing $E = E_\Gamma^X$ and $E' = E_\Lambda^Y$ by distinguishing the groups Γ, Λ and actions $\Gamma \curvearrowright X$, $\Lambda \curvearrowright Y$ that give rise to them. For this strategy to be at all successful, the equivalence relation $E = E_\Gamma^X$ must "remember," or encode information about the group action $\Gamma \curvearrowright X$ from which it came. Hence the fundamental question becomes: to what extent does the equivalence relation $E = E_\Gamma^X$ (or more precisely its Borel complexity) determine the group Γ and its action on X?

In general the answer is not at all. For instance, by Dougherty-Jackson-Kechris [10], if the countable group Γ acts freely on the standard Borel Γ-space X and fails to preserve a Borel probability measure on X, then *any* countable group containing Γ admits a free Borel action on X that induces the same orbit equivalence relation as that of $\Gamma \curvearrowright X$. And by Miller [38], for any countable Borel equivalence relation E with all classes infinite, there are continuum many pairwise nonisomorphic groups Γ that realize E through some Borel action that is faithful on each orbit. Hence if there is to be any hope of recovering the group Γ and its action on X from the Borel complexity of E_Γ^X alone, we must at a minimum assume that Γ acts freely and preserves a probability measure.

However, if these hypotheses are satisfied, then in certain cases a significant amount of information about Γ and its action on X can be recovered from the Borel complexity of the equivalence relation E_Γ^X alone. This phenomenon is referred to as Borel superrigidity. For anything like this to hold, very strong hypotheses must be imposed upon the groups and the actions involved. As as example of what we have in mind, we state here a consequence appearing in Thomas [53] of a superrigidity theorem due to Popa [43]. Here E_G is the orbit equivalence relation arising from the restriction of the left shift action (defined above) of G on its powerset to the invariant subset of 2^G on which this action is free.

Theorem 1.6.1 ([53, 3.8]). *Suppose that S is a countable group with no nontrivial finite normal subgroups and let $G = SL_3(\mathbb{Z}) \times S$. If H is any countable group, then $E_G \leq_B E_H$ if and only if G embeds into H.*

Let us call a countable Borel equivalence relation F *essentially free* if F is Borel bireducible with a countable Borel equivalence relation $E = E_\Gamma^X$ that arises from a *free* action $\Gamma \curvearrowright X$. As a consequence of the above theorem, Thomas was able to show [53] that the collection of essentially free countable Borel equivalence relations does not admit a universal element, and hence in particular E_∞ is not essentially free.

Indeed, virtually everything we know about the structure of countable Borel equivalence relations under the partial ordering of Borel reducibility is based upon the phenomenon of superrigidity. In this thesis we will prove Borel superrigidity theorems for Bernoulli, profinite, and linear actions of certain subgroups of $PSL_2(\mathbb{R})$. These theorems are stated in Chapter 2. Our proofs are based on the same cocycle superrigidity theorems of Zimmer that were used in the first application of superrigidity to the field of Borel equivalence relations in the ground-breaking work of Adams and Kechris [3]. We will discuss superrigidity generally at greater length in Section 5.1, and examine Zimmer's theorem in particular in Section 5.4. But first we conclude this introductory chapter with a brief account of the structure of the countable Borel equivalence relations under $\leq_B$ as it is currently understood, followed by some comments on the organization of the remainder of this thesis.

1.7 The basic structure of countable Borel equivalence relations under Borel reducibility

We have already seen that there is a universal countable Borel equivalence relation E_∞, and that every nonsmooth countable Borel equivalence relation lies in the interval $[E_0, E_\infty]$. We now discuss what is known about the structure of this interval under $\leq_B$, beginning with some important definitions.

A countable Borel equivalence relation E on the standard Borel space X is called *hyperfinite* if E can be expressed as an increasing union of finite Borel equivalence relations

$$E = \bigcup_n F_n, \quad F_1 \subseteq F_2 \subseteq \cdots,$$

where an equivalence relation is called *finite* if each of its classes is finite. The countable Borel equivalence relation E is called *treeable* if there is a Borel acyclic graph on X whose connected components are precisely the E-classes. Again let $\mathbb{F}_2$ denote the 2-generator free group, and let $(2)^{\mathbb{F}_2}$ be the invariant subset of $2^{\mathbb{F}_2}$ on which the left shift action of $\mathbb{F}_2$ on its powerset is free. Denote by E_{T_∞} the orbit equivalence relation arising from the (restricted) shift action of $\mathbb{F}_2$ on $(2)^{\mathbb{F}_2}$. Then E_{T_∞} is a treeable countable Borel equivalence relation that is universal for treeable relations; ie, if E is any treeable countable Borel equivalence relation, then $E \leq_B E_{T_\infty}$ (for instance, see [26]).

It has long been known (see [26, Section 3.5]) that $E_0 <_B E_{T_\infty} <_B E_\infty$. This result is ultimately based on the work of Adams, which can be adapted to show that the product relation $E_{T_\infty}^2$ is non-treeable and non-universal. Here the product E^2 of an equivalence relation E on X with itself is the equivalence relation defined on X^2 by

$$(x, x')\, E^2\, (y, y') \quad \iff \quad x\, E\, y \text{ and } x'\, E\, y'.$$

For a long time this remained all that was known about the interval (E_0, E_∞). In particular, it was unknown whether there existed infinitely many $\sim_B$-distinct countable Borel equivalence relations, or whether two countable Borel equivalence relations could be incomparable under $\leq_B$. Then in 2000, Adams and Kechris made use of Zimmer Cocycle Superrigidity [58] to show that the interval (E_0, E_∞) is as complicated as it could possibly be.

Theorem 1.7.1 ([3, Theorem 1])**.** *Let X be any standard Borel space, and $\mathcal{B}$ its σ-algebra of Borel sets. Then the partial ordering $(X, \subseteq)$ embeds into the partial ordering $((E_0, E_\infty), \leq_B)$.*

Subsequently Adams provided in [2] the first known example of a pair of countable Borel equivalence relations $E \subseteq F$ such that $E \not\leq_B F$. In [53] Thomas showed that there are uncountably many countable Borel equivalence relations (considered up to $\sim_B$) that are essentially free, and uncountably many that are not essentially free. Many questions concerning equivalence relations situated near the top of the interval $[E_0, E_\infty]$ remain open. For instance, the question of whether $\equiv_T$ is universal remains open, as does the related question of whether a countable Borel equivalence relation that contains a universal relation is itself universal. A positive answer to either of these questions would imply the failure of the Martin Conjecture on degree invariant Borel maps (for recent work concerning Martin's Conjecture, Turing equivalence, and countable Borel equivalence relations see Thomas [54]).

Returning to the lower end of the interval $[E_0,\ E_\infty]$, we have the following characterization of hyperfiniteness: the countable Borel equivalence relation E on the standard Borel space X is hyperfinite *iff* $E \leq_B E_0$ *iff* there exists a Borel $\mathbb{Z}$-action on X such that $E = E^X_{\mathbb{Z}}$ (for instance, see [10]). In fact, the orbit equivalence relation induced on *any* standard Borel $\mathbb{Z}$-space is hyperfinite, and it is an important open question to determine the widest class of groups for which this holds. It is known to hold for (countable) abelian groups [17], as well as for finitely generated groups of polynomial growth [26]. Conceivably it could hold for all countable amenable groups, but at present this remains open.

Every hyperfinite countable Borel equivalence relation is treeable, but the universal treeable relation E_{T_∞} mentioned above is not hyperfinite. Any orbit equivalence relation arising from a free action of a countable free group is treeable, and any treeable countable Borel equivalence relation is Borel bireducible with such an orbit equivalence relation [26]. Very little is known about the interval $(E_0,\ E_{T_\infty}]$ of non-hyperfinite treeable countable Borel equivalence relations. It has been shown only very recently by Hjorth that this interval contains uncountably many $\leq_B$-distinct countable Borel equivalence relations [21]. Hjorth's proof, however, is nonconstructive, and there remains no known infinite, pairwise $\leq_B$-distinct family of "naturally occurring" treeable relations. Conjectural candidates for such a family will be described below in Chapter 2.

For additional material on the general theory of Borel equivalence relations see Kanovei [28], and for an account with an emphasis on the countable Borel equivalence relations see Jackson-Kechris-Louveau [26].

1.8 Outline of thesis

The remainder of this thesis will be organized as follows. In Chapter 2 we will motivate and state our main results, as well as fix some terminology and notational conventions. In Chapters 3 – 5 we will develop the background material that will be needed in the proofs of our theorems. Chapter 3 will be devoted to algebra, and in particular some notions from the theory of algebraic groups. In Chapter 4 we develop the necessary background material from dynamics and ergodic theory, and in Chapter 5 we discuss superrigidity, including a brief introduction to the notion of a cocycle. Finally, in Chapters 6 and 7 we will prove our main results and suggest directions for further research.

Chapter 2

Precise statements of main results

2.1 Profinite and linear actions of $SL_2(\mathcal{O})$

For p a rational prime, let $\mathbb{Q}_p$ denote the field of p-adic numbers, and $\mathbb{Z}_p$ the ring of p-adic integers. Further let $PG(n-1, \mathbb{Q}_p)$ denote the $(n-1)$-dimensional projective space over $\mathbb{Q}_p$ for each $n \geq 2$, so that $PG(n-1, \mathbb{Q}_p)$ is the space of 1-dimensional vector subspaces of the n-dimensional vector space $\mathbb{Q}_p^n$. If $\mathcal{O}$ is any subring of $\mathbb{Z}_p$, then $SL_n(\mathcal{O})$ admits a natural Borel action on $PG(n-1, \mathbb{Q}_p)$ arising from the linear action of $SL_n(\mathcal{O})$ on $\mathbb{Q}_p^n$. Furthermore, $SL_n(\mathcal{O})$ also acts by translations as a subgroup on the profinite group $SL_n(\mathbb{Z}_p)$. In [51], Thomas proved the following Borel incomparability results for orbit equivalence relations arising from $SL_n(\mathbb{Z})$-actions, for $n \geq 3$.

Theorem 2.1.1 (Thomas [51, 5.1])**.** *Suppose $n \geq 3$, and let J_1 and J_2 be nonempty sets of primes. For $i = 1, 2$, denote by E_{J_i} the orbit equivalence relation arising from the translation action (as a subgroup via the diagonal embedding) of $SL_n(\mathbb{Z})$ on*

$$K(J_i) = \prod_{p \in J_i} SL_n(\mathbb{Z}_p).$$

If $J_1 \neq J_2$, then E_{J_1} and E_{J_2} are incomparable with respect to Borel reducibility.

Theorem 2.1.2 (Thomas [51, 6.7])**.** *Suppose $n \geq 3$, and for each prime p let E_p be the orbit equivalence relation arising from the natural action of $SL_n(\mathbb{Z})$ on $PG(n-1, \mathbb{Q}_p)$. If $p \neq q$, then E_p and E_q are incomparable with respect to Borel reducibility.*

The proofs of these theorems depended essentially upon Zimmer's measure theoretic superrigidity results [58] for lattices in higher rank simple Lie groups. Unfortunately, the analogue of Zimmer's theorem fails for the low rank Lie group $SL_2(\mathbb{R})$, and hence there is no hope of applying the ideas of Thomas [51] to orbit equivalence relations arising from analogous actions of $SL_2(\mathbb{Z})$. This is especially unfortunate considering that analogues of the above theorems for $n = 2$ would yield a concrete example of an infinite family of treeable countable Borel equivalence relations that are pairwise incomparable with respect to Borel reducibility, thus solving the important open problem in the field of countable Borel equivalence relations mentioned at the end of Section 1.7.

Indeed, Thomas conjectured ([51, 5.7], [51, 6.10]) that Theorems 2.1.1 and 2.1.2 should still hold for $n = 2$. With this context in mind, Thomas' subsequent results in [52] may be viewed as attempts to extend Theorems 2.1.1 and 2.1.2 as far as possible in the direction of Conjectures [51, 5.7] and [51, 6.10], while still appealing to Zimmer superrigidity. Specifically, in [52] Thomas proved the following analogues of Theorems 2.1.1 and 2.1.2 for actions of lattices of the form

$$\Theta_S = SL_2(\mathbb{Z}[S]),$$

where $S = \{p_1, \ldots, p_s\}$ is a finite, nonempty set of primes, and

$$\mathbb{Z}[S] := \mathbb{Z}[1/p_1, \ldots, 1/p_s]$$

is the ring generated over $\mathbb{Z}$ and inside $\mathbb{Q}$ by the reciprocals of the primes in S.

Theorem 2.1.3 (Thomas [52, 1.2])**.** *Suppose that S_1 and S_2 are finite nonempty sets of primes and that J_1, J_2 are (possibly infinite) nonempty sets of primes such that $S_1 \cap J_1 = S_2 \cap J_2 = \emptyset$. For $i = 1, 2$, let $E_{S_i}^{J_i}$ be the orbit equivalence relation arising from the translation action (as a subgroup via the diagonal embedding) of Θ_{S_i} on*

$$K(J_i) = \prod_{p \in J_i} SL_2(\mathbb{Z}_p).$$

If $(J_1, S_1) \neq (J_2, S_2)$, then $E_{S_1}^{J_1}$ and $E_{S_2}^{J_2}$ are incomparable with respect to Borel reducibility.

Theorem 2.1.4 (Thomas [52, 1.1])**.** *Suppose that p, q are primes and that S, T are finite nonempty sets of primes such that $p \notin S$ and $q \notin T$. Let E_S^p be the orbit equivalence relation arising from the action of Θ_S on $PG(1, \mathbb{Q}_p)$. If $(p, S) \neq (q, T)$, then E_S^p and E_T^q are incomparable with respect to Borel reducibility.*

Here again the proofs of Theorems 2.1.3 and 2.1.4 in [52] were based upon Zimmer's Cocycle Superrigidity Theorem [58, 10.1.6]. To see why Zimmer's theorem might be relevant to actions of these "low-dimensional" matrix groups, notice that Θ_S may be realized, via identification with its image under the diagonal embedding, as an irreducible lattice in the *higher*-rank product of real and p-adic Lie groups

$$\mathcal{G}_{S^+} = SL_2(\mathbb{R}) \times SL_2(\mathbb{Q}_{p_1}) \times \cdots \times SL_2(\mathbb{Q}_{p_s}).$$

Furthermore, we remark that the proofs of Theorems 2.1.1 and 2.1.2 in [51] depended upon the fact that for $n \geq 3$, $SL_n(\mathbb{Z})$ is Kazhdan. Unfortunately, $SL_2(\mathbb{Z})$ is *not* Kazhdan, and in fact neither are any of the groups $\Theta_S = SL_2(\mathbb{Z}[S])$; but, fortunately, the groups Θ_S do possess the weaker Property(τ), and in fact this turns out to be enough to push through, in [52], the line of argument used by Thomas in [51]. We shall have more to say about Property (τ) and its connection to one of the hypotheses of Zimmer's theorem below in Section 5.5.

Our first pair of results may be viewed as a continuation of the study initiated in [52], ie, as a further attempt to extend Theorems 2.1.1 and 2.1.2 in the direction of Conjectures [51, 5.7] and [51, 6.10] while still relying upon Zimmer superrigidity. We shall consider actions of lattices of the form

$$\Gamma_S = SL_2(\mathcal{O}_S),$$

where again $S = \{p_1, \ldots, p_s\}$ is a finite nonempty set of rational primes, and where $\mathcal{O}_S$ is the ring of integers inside the algebraic number field

$$\mathbb{Q}(\sqrt{p_1}, \ldots, \sqrt{p_s}).$$

We shall obtain the following Γ_S-analogues of Theorems 2.1.3 and 2.1.4.

Theorem 2.1.5. *For $i = 1, 2$, let S_i and J_i be nonempty sets of primes, with $|S_1| = |S_2|$ finite. Suppose that $\sqrt{p} \in \mathbb{Z}_q$ for all $p \in S_i$ and $q \in J_i$, so that Γ_{S_i} is a subgroup of $K(J_i) = \prod_{p \in J_i} SL_2(\mathbb{Z}_p)$ via the diagonal embedding. Let $E_{S_i}^{J_i}$ be the orbit equivalence relation arising from the left translation action of Γ_{S_i} on $K(J_i)$. Then $E_{S_1}^{J_1}$ and $E_{S_2}^{J_2}$ are incomparable with respect to Borel reducibility whenever $(S_1, J_1) \neq (S_2, J_2)$.*

Theorem 2.1.6. *For $i = 1, 2$, let S_i be a finite, nonempty set of primes such that $|S_1| = |S_2|$. Further let p_i be a prime such that $\sqrt{q} \in \mathbb{Z}_{p_i}$ for all $q \in S_i$, and consider the action of $\Gamma_{S_i} \leq SL_2(\mathbb{Z}_{p_i})$ on $PG(1, \mathbb{Q}_{p_i})$ as a group of fractional linear transformations. Then writing $E^{p_i}_{S_i}$ for the orbit equivalence relation arising from this action, we have that $E^{p_1}_{S_1}$ and $E^{p_2}_{S_2}$ are incomparable with respect to Borel reducibility whenever $(p_1, S_1) \neq (p_2, S_2)$.*

These results will be shown to follow from a more general Borel superrigidity theorem, which we state in the next section as Theorem 2.2.1. Much of our effort in this thesis shall go into proving 2.2.1.

We make some comments now regarding the statements of Theorems 2.1.5 and 2.1.6, and in particular regarding the hypothesis that $|S_1| = |S_2|$. As we will see in Chapter 6, it will follow from the proofs of these theorems that the following results are also true.

Theorem 2.1.7. *Assume all the hypotheses of Theorem 2.1.5, except for the hypothesis that $|S_1| = |S_2|$. If $E^{J_1}_{S_1} \leq_B E^{J_2}_{S_2}$, then $|S_1| \leq |S_2|$.*

Theorem 2.1.8. *Assume all the hypotheses of Theorem 2.1.6, except for the hypothesis that $|S_1| = |S_2|$. If $E^{p_1}_{S_1} \leq_B E^{p_2}_{S_2}$, then $|S_1| \leq |S_2|$.*

As easy corollaries of these results we obtain:

Corollary 2.1.9. *Assume all the hypotheses of Theorem 2.1.5, except for the hypothesis that $|S_1| = |S_2|$. Then $E^{J_1}_{S_1} \sim_B E^{J_2}_{S_2}$ if and only if $(S_1, J_1) = (S_2, J_2)$.*

Corollary 2.1.10. *Assume all the hypotheses of Theorem 2.1.6, except for the hypothesis that $|S_1| = |S_2|$. Then $E^{p_1}_{S_1} \sim_B E^{p_2}_{S_2}$ if and only if $(S_1, J_1) = (S_2, J_2)$.*

Remark 2.1.11. It is very likely that Theorems 2.1.5 and 2.1.6 are still true without having to suppose $|S_1| = |S_2|$. However, in order to prove this, it will be necessary to overcome a technical difficulty in their proofs that at present seems intractable.

2.2 A Borel superrigidity theorem

In this section, we state the Borel superrigidity theorem that we shall prove in Chapters 6 and 7, and obtain from it Theorems 2.1.5 and 2.1.6 as easy corollaries. In particular, we fix here the following notation that will be used throughout this thesis.

Given a finite, nonempty set of rational primes $S = \{p_1, \ldots, p_s\}$, write $\mathcal{O}_S$ for the ring of integers inside the multi-quadratic algebraic number field

$$\mathbb{Q}(\sqrt{p_1}, \ldots, \sqrt{p_2}),$$

and define

$$\Gamma_S = SL_2(\mathcal{O}_S) \quad \text{and} \quad \Lambda_S = PSL_2(\mathcal{O}_S).$$

For each $A \subseteq S$, let

$$\sigma_A^S : \mathbb{Q}(\sqrt{p_1}, \ldots, \sqrt{p_s}) \hookrightarrow \mathbb{R}$$

be the field embedding that maps $\sqrt{p_i} \mapsto -\sqrt{p_i}$ if $p_i \in A$, and $\sqrt{p_i} \mapsto \sqrt{p_i}$ if $p_i \in S \setminus A$, so that in particular $\sigma_\emptyset^S$ is the inclusion embedding. Then, identifying as usual $\mathcal{P}(S) = 2^S$, define

$$\sigma^S : \Gamma_S \to \prod_{2^S} SL_2(\mathbb{R}) \quad \text{by} \quad \sigma^S(\gamma) = \langle \gamma^{\sigma_A^S} \rangle_{A \subseteq S},$$

where $\gamma^{\sigma_A^S}$ has the obvious meaning and 2^S is ordered, merely for definiteness, lexicographically on the natural ordering for S itself.

Then if we identify Γ_S with its image under σ^S, we have by Margulis [36, IX(1.7v)] that

$$\Gamma_S \leq G_S = \prod_{2^S} SL_2(\mathbb{R})$$

is an irreducible lattice in the higher rank connected semisimple Lie group G_S. By an abuse of notation, we shall also denote by σ^S the corresponding embedding

$$\sigma^S : \Lambda_S \leq H_S = \prod_{2^S} PSL_2(\mathbb{R}),$$

which realizes Λ_S as an irreducible lattice in H_S. Throughout we shall identify $\sigma^S(\Gamma_S)$, $\sigma^S(\Lambda_S)$ with Γ_S, Λ_S, and rely upon the context to distinguish them.

Now, given a nonempty (and possibly infinite) set of primes J, let

$$K(J) = \prod_{p \in J} SL_2(\mathbb{Z}_p).$$

Then provided $\sqrt{p} \in \mathbb{Z}_q$ for every $p \in S$ and $q \in J$, we may regard Γ_S as a subgroup of $K(J)$ via the diagonal embedding. (This occurs, for instance, whenever $S \cap J = \emptyset$, $2 \notin J$, and p is a quadratic residue modulo q for each $p \in S$, $q \in J$).

We then have the following Borel superrigidity theorem for the translation action of Λ_S on full measure subsets of that part of $K(J)/Z(\Gamma_S)$ on which Λ_S acts freely. (For the definition of virtual isomorphism used here, see Definition 4.3.3).

Theorem 2.2.1. *Suppose that S_1, S_2 are finite nonempty sets of primes and that J_1, J_2 are (possibly infinite) nonempty sets of primes. Suppose that $|S_1| = |S_2|$. For $i = 1, 2$, suppose that:*

- *$\sqrt{p} \in \mathbb{Z}_q$ for each $p \in S_i$ and $q \in J_i$;*
- *$L_i \leq K(J_i)$ is closed, contains $Z(\Gamma_{S_i})$, and satisfies $\mu_{J_i}(F_{S_i}(J_i, L_i)) = 1$, where μ_{J_i} is Haar probability measure on $K(J_i)/L_i$, and $F_{S_i}(J_i, L_i)$ is the subset of $K(J_i)/L_i$ on which Λ_{S_i} acts freely;*
- *X_i is a μ_{J_i}-conull, Λ_{S_i}-invariant Borel subset of $F_{S_i}(J_i, L_i)$; and*
- *E_i is the Λ_{S_i}-orbit equivalence relation on X_i.*

Suppose that $f : X_1 \to X_2$ is a Borel reduction from E_1 to E_2. Then

(1) $S_1 = S_2$, and

(2) $(K(J_1)/L_1, \Lambda_{S_1}, \mu_{J_1})$ and $(K(J_2)/L_2, \Lambda_{S_2}, \mu_{J_2})$ are virtually isomorphic.

We shall present the proof of Theorem 2.2.1 in Chapters 6 and 7. At this point we immediately prove Theorems 2.1.5 and 2.1.6 from 2.2.1. These arguments are essentially identical to those of Thomas in [52].

Proof of Theorem 2.1.5. Assuming the hypotheses of the theorem, suppose $E_{S_1}^{J_1} \leq_B E_{S_2}^{J_2}$. For $i = 1, 2$, let $Z_i = Z(\Gamma_{S_i})$, and let E_i denote the orbit equivalence relation arising from the free action of Λ_{S_i} on $K(J_i)/Z_i$. Clearly $E_1 \leq_B E_2$. Thus by Theorem 2.2.1, we have that $S_1 = S_2$ and that $(K(J_i)/Z_i, \Lambda_{S_i}, \mu_{J_i})$ are virtually isomorphic. Arguing as in the proof of Thomas [51, 5.1], we see that $K(J_1)/Z_1$ and $K(J_2)/Z_2$ contain open subgroups which are isomorphic as topological groups. By Gefter-Golodets [18, A.6], it follows that $J_1 = J_2$. $\square$

Proof of Theorem 2.1.6. Assume the hypotheses of the theorem, and suppose that $f : PG(1, \mathbb{Q}_{p_1}) \to PG(1, \mathbb{Q}_{p_2})$ is a Borel reduction from $E^{p_1}_{S_1}$ to $E^{p_2}_{S_2}$. Recall that for $i = 1, 2$, $E^{p_i}_{S_i}$ is the orbit equivalence relation arising from the action of Γ_{S_i} on $PG(1, \mathbb{Q}_{p_i})$. Of course, since $Z(\Gamma_{S_i})$ acts trivially on $PG(1, \mathbb{Q}_{p_i})$, $E^{p_i}_{S_i}$ is also the orbit equivalence relation arising from the action of Λ_{S_i} on $PG(1, \mathbb{Q}_{p_i})$. We now show how to apply Theorem 2.2.1 to this action.

For each $i = 1, 2$, we regard $PG(1, \mathbb{Q}_{p_i})$ as the space of 1-dimensional subspaces of the 2-dimensional vector space $\mathbb{Q}^2_{p_i}$. By Thomas [51, 6.1], $SL_2(\mathbb{Z}_{p_i})$ acts transitively on $PG(1, \mathbb{Q}_{p_i})$. Thus fixing $x_{p_i} \in PG(1, \mathbb{Q}_{p_i})$ with (closed) stabilizer $L_{p_i} \leq SL_2(\mathbb{Z}_{p_i})$, we may identify $SL_2(\mathbb{Z}_{p_i})/L_{p_i}$ as an $SL_2(\mathbb{Z}_{p_i})$-set with the homogeneous space $PG(1, \mathbb{Q}_{p_i})$. Regarding Γ_{S_i} as a subgroup of $SL_2(\mathbb{Z}_{p_i})$, we clearly have $Z(\Gamma_{S_i}) \leq L_{p_i}$. Now let

$$PG^*(1, \mathbb{Q}_{p_i}) = \{x \in PG(1, \mathbb{Q}_{p_i}) \mid \gamma \cdot x \neq x \text{ for all } \gamma \in \Gamma_{S_i} \setminus Z(\Gamma_{S_i})\}$$

be the subset of $PG(1, \mathbb{Q}_{p_i})$ on which Λ_{S_i} acts freely. Then $PG^*(1, \mathbb{Q}_{p_i})$ is Λ_{S_i}-invariant. Moreover, for every $x \in PG(1, \mathbb{Q}_{p_i}) \setminus PG^*(1, \mathbb{Q}_{p_i})$, there is $\gamma \in \Gamma_{S_i} \setminus Z(\Gamma_{S_i})$ such that x is an eigenspace of γ. This shows that $PG(1, \mathbb{Q}_{p_i}) \setminus PG^*(1, \mathbb{Q}_{p_i})$ is countable, and hence

$$\mu_i(PG^*(1, \mathbb{Q}_{p_i})) = 1,$$

where μ_i is the Haar probability measure on $PG(1, \mathbb{Q}_{p_i}) = SL_2(\mathbb{Z}_{p_i})/L_{p_i}$. As f is countable-to-one, the set

$$\{x \in PG^*(1, \mathbb{Q}_{p_1}) \mid f(x) \in PG(1, \mathbb{Q}_{p_2}) \setminus PG^*(1, \mathbb{Q}_{p_2})\}$$

is countable, and so there is a μ_1-measure one subset X_0 of $PG^*(1, \mathbb{Q}_{p_1})$ such that $f(X_0) \subseteq PG^*(1, \mathbb{Q}_{p_2})$. As f is a Borel reduction and $PG^*(1, \mathbb{Q}_{p_2})$ is Λ_{S_2}-invariant, X_0 is necessarily Λ_{S_1}-invariant.

We have now verified all the hypotheses of Theorem 2.2.1, and so applying it to the Borel reduction f we obtain that $S_1 = S_2$ and that for $i = 1, 2$, the standard Borel systems $(PG(1, \mathbb{Q}_{p_i}), \Lambda_{S_i}, \mu_{p_i})$ are virtually isomorphic. By the proof of Thomas [51, 6.3], this implies that $p_1 = p_2$. $\square$

We remark that the proof of Theorem 2.2.1 will also show the following:

Theorem 2.2.2. *Assume all the hypotheses of Theorem 2.2.1, except for the hypothesis that* $|S_1| = |S_2|$. *If* $E_1 \leq_B E_2$, *then* $|S_1| \leq |S_2|$.

From this we immediately obtain the following easy corollary.

Theorem 2.2.3. *Assume all the hypotheses of Theorem 2.2.1, except for the hypothesis that* $|S_1| = |S_2|$. *If* $E_1 \sim_B E_2$, *then*

(1) $S_1 = S_2$*; and*

(2) $(K(J_1)/L_1,\, \Lambda_{S_1},\, \mu_{J_1})$ *and* $(K(J_2)/L_2,\, \Lambda_{S_2},\, \mu_{J_2})$ *are virtually isomorphic.*

Proof. Suppose $E_1 \sim_B E_2$. From Theorem 2.2.2 it follows that $|S_1| = |S_2|$. Hence we may apply Theorem 2.2.1, and the conclusion follows. $\square$

Theorems 2.1.7 and 2.1.8 easily follow from 2.2.2, together with an examination of the proofs of 2.1.5 and 2.1.6.

2.3 Bernoulli actions of $PSL_2(\mathcal{O})$

The argument we use to prove Theorem 2.2.1 can also be applied, with some modifications, to Bernoulli actions of the groups Λ_S.

Let (Y, ν) be a nontrivial (and possibly countable or finite) standard Borel probability space, where by "nontrivial" we simply mean that the Borel probability measure ν does not concentrate on a single point. If Γ is any countable (discrete) group, we denote by $(Y, \nu)^\Gamma = (Y^\Gamma, \nu^\Gamma)$ the product of identical copies of (Y, ν) indexed by Γ, so that Y^Γ is the space of functions $x : \Gamma \to Y$ which we think of as sequences (x_γ) in Y, and ν^Γ is the product measure on the product σ-algebra of Y^Γ. We may then define a left action of Γ on $(Y,\nu)^\Gamma$ by

$$(\gamma_0 \cdot x)(\gamma) = x(\gamma_0^{-1}\gamma) \quad \text{for } \gamma_0 \in \Gamma,\ x \in Y^\Gamma.$$

We call any such action a *Bernoulli action* of Γ. We are now ready to state the second principal result of this thesis. Its proof, given below in Chapters 6 and 7, will overlap significantly with the proof of Theorem 2.2.1 stated above.

Theorem 2.3.1. *For $i = 1, 2$, let (Y_i, ν_i) be a non-trivial standard Borel probability space, and let S_i be a finite, nonempty set of primes. Let $X_i \subseteq Y_i^{\Lambda_{S_i}}$ be the subset of $Y_i^{\Lambda_{S_i}}$ on which Λ_{S_i} acts freely as a group of Bernoulli shifts. Let E_i be the orbit equivalence relation arising from this free Bernoulli action of Λ_{S_i} on X_i. Then $E_1 \leq_B E_2$ implies $S_1 \subseteq S_2$. In particular, if each of $S_1 \setminus S_2$ and $S_2 \setminus S_1$ is nonempty, then E_1 and E_2 are incomparable with respect to Borel reducibility.*

We remark that since the state space (Y_i, ν_i) is allowed to vary in the statement of Theorem 2.3.1, it is not necessarily the case that $S_1 \subseteq S_2$ will imply $E_1 \leq_B E_2$. However, if we fix a single state space (Y, ν) and let X_i be the free part of the Bernoulli action of Λ_{S_i} on $Y^{\Lambda_{S_i}}$, then we may prove that $S_1 \subseteq S_2$ implies $E_{\Lambda_{S_1}}^{X_1} \leq_B E_{\Lambda_{S_2}}^{X_2}$ just as in Thomas [53, 3.3]. Hence we obtain the following.

Corollary 2.3.2. *Let (Y, ν) be a nontrivial standard Borel probability space, and E_S the orbit equivalence relation arising from the restriction of the Bernoulli action $\Lambda_S \curvearrowright (Y,\nu)^{\Lambda_S}$ to the subset of Y^{Λ_S} on which Λ_S acts freely. If T is any other nonempty set of primes, then $E_S \leq_B E_T$ if and only if $S \subseteq T$.*

Proof. If $S \subseteq T$, then Λ_S naturally embeds into Λ_T, and hence we may proceed exactly as in [53, 3.3]. Specifically, fix some point $y_0 \in Y$, and for each $\alpha \in (Y)^{\Lambda_S}$ define a corresponding element $\alpha^* \in Y^{\Lambda_T}$ by

$$\alpha^*(\lambda) = \begin{cases} \alpha(\gamma) & \text{if } \lambda \in \Lambda_S \\ y_0 & \text{otherwise} \end{cases}.$$

It is easily shown that $\alpha^* \in (Y)^{\Lambda_T}$ and that the assignment $\alpha \mapsto \alpha^*$ is a Borel reduction from E_S to E_T. $\square$

2.4 A more abstract Borel superrigidity theorem

In this section, we address the issue of determining the correct level of generality at which to state the above theorems. We have stated Theorems 2.1.5, 2.1.6, and 2.3.1 for actions of matrix groups over the rings of integers inside multi-quadratic number fields, but it is not until the final stages of our proofs that specific computational properties involving the rings $\mathcal{O}_S$ come into play. In fact, most of our arguments go through for the rings of integers in arbitrary totally real number fields.

Consequently, we find it appropriate to recast Theorem 2.2.1 and its proof in as abstract a setting as possible, not merely because it will help in uniformizing the proofs of Theorems 2.1.5, 2.1.6, and 2.3.1, but also because it seems the natural setting given the specific properties of the group actions that are involved. Hence the entirety of Chapter 6 will be devoted to proving the abstract and somewhat technical theorem stated below as Theorem 2.4.1. In order to state this theorem, we introduce notation which is similar to that used in Section 2.2, and which will be explained in greater detail in Section 3.5 below.

For k a totally real number field, let $\mathfrak{R}_k = \{\vartheta_1, \ldots, \vartheta_n\}$ be the set of embeddings $\vartheta_i : k \hookrightarrow \mathbb{R}$ of k into $\mathbb{R}$, with an arbitrary but fixed ordering. Let $\mathcal{O}_k$ be the ring of integers in k, and for each $\gamma \in SL_2(\mathcal{O}_k)$ and $\vartheta \in \mathfrak{R}_k$, write γ^ϑ for the matrix obtained by replacing each entry of γ with its image under ϑ. Define the embedding

$$\sigma^k : \Gamma_k = SL_2(\mathcal{O}_k) \to G_k = \prod_{\vartheta_i \in \mathfrak{R}_k} SL_2(\mathbb{R})$$

by

$$\sigma^k(\gamma) = \langle \gamma^{\vartheta_1}, \ldots, \gamma^{\vartheta_n} \rangle.$$

Then by Margulis [36, IX(1.7v)], $\sigma^k(\Gamma_k)$ is an irreducible lattice in G_k. Denote also by σ^k the corresponding embedding that realizes $\Lambda_k = PSL_2(\mathcal{O}_k)$ as an irreducible lattice in

$$H_k = \prod_{\vartheta_i \in \mathfrak{R}_k} PSL_2(\mathcal{O}_k).$$

In Chapter 6 we will prove the following theorem.

Theorem 2.4.1. *Suppose that $\mathbb{K}$ and $\mathbb{F}$ are totally real number fields properly extending $\mathbb{Q}$, let $\mathcal{O}_{\mathbb{K}}$ and $\mathcal{O}_{\mathbb{F}}$ be their rings of integers, and let $\Lambda_{\mathbb{K}} = PSL_2(\mathcal{O}_{\mathbb{K}})$ and $\Lambda_{\mathbb{F}} = PSL_2(\mathcal{O}_{\mathbb{F}})$. Let X be a standard Borel $\Lambda_{\mathbb{K}}$-space with invariant ergodic probability measure μ_1, and let Y be a free standard Borel $\Lambda_{\mathbb{F}}$-space with invariant ergodic probability measure μ_2. Let E_1 and E_2 denote the orbit equivalence relations arising from the actions of $\Lambda_{\mathbb{K}}$ and $\Lambda_{\mathbb{F}}$ on X and Y, respectively. Further suppose that the following conditions are satisfied:*

(1) E_1 is E_0-ergodic.

(2) The induced $H_{\mathbb{K}}$-space $\hat{X} = X \times H_{\mathbb{K}}/\Lambda_{\mathbb{K}}$ is irreducible.

Suppose that $E_1 \leq_B E_2$. Then there exist

- *a $\Lambda_{\mathbb{K}}$-invariant Borel set $X_0 \subseteq X$ such that $\mu_1(X_0) = 1$,*
- *a Borel function $\tilde{f} : X \to \hat{Y}$, where $\hat{Y}$ is the $H_{\mathbb{F}}$-space induced from the action of $\Lambda_{\mathbb{F}}$ on Y, and*
- *an injective rational $\mathbb{R}$-homomorphism $\varphi : H_{\mathbb{K}} \to H_{\mathbb{F}}$*

such that

- *$\tilde{f}$ is a Borel reduction from E_1 to $E^{\hat{Y}}_{H_{\mathbb{F}}}$,*
- *$\tilde{f}(\lambda x) = \varphi(\lambda)\tilde{f}(x)$ for all $x \in X_0$ and for all $\lambda \in \Lambda_{\mathbb{K}}$, and*
- *for all $\lambda \in \Lambda_{\mathbb{K}}$, each component of $\varphi(\sigma^{\mathbb{K}}(\lambda))$ is either λ^{ς}, or λ^{ς} with main diagonal scaled by -1, for some Galois automorphism $\varsigma \in$ Gal(n.c.($\mathbb{K}$)/$\mathbb{Q}$).*

We will prove Theorems 2.2.1 and 2.3.1 from 2.4.1 in Chapter 7.

2.5 Notation

For convenience, we record here some of the notation that has been introduced thus far, and that will be used through the remainder of this thesis.

$$S = \{p_1, \ldots, p_s\} \text{ is a finite, nonempty set of primes, and } S^+ = S \cup \{\infty\}$$

$$J \text{ is a nonempty, possibly infinite set of primes, and } K(J) = \prod_{p \in J} SL_2(\mathbb{Z}_p)$$

Each of the following diagonal embeddings realizes the countable discrete group on the left as an irreducible lattice in the connected real Lie group on the right.

$$\begin{array}{lllllllll}
\sigma^S & : & \Gamma_S & = & SL_2(\mathcal{O}_S) & \leq & G_S & = & \prod_{2^S} SL_2(\mathbb{R}) \\
\sigma^S & : & \Lambda_S & = & PSL_2(\mathcal{O}_S) & \leq & H_S & = & \prod_{2^S} PSL_2(\mathbb{R}) \\
\sigma^k & : & \Gamma_k & = & SL_2(\mathcal{O}_k) & \leq & G_k & = & \prod_{\vartheta \in \mathfrak{R}_k} SL_2(\mathbb{R}) \\
\sigma^k & : & \Lambda_k & = & PSL_2(\mathcal{O}_k) & \leq & H_k & = & \prod_{\vartheta \in \mathfrak{R}_k} PSL_2(\mathbb{R})
\end{array}$$

Chapter 3

Some preliminaries from the theory of algebraic groups

In this chapter we recall some basic definitions and facts that we shall need concerning algebraic groups, lattices inside semisimple Lie groups, and totally real number fields. In the final section we will prove a measure-classification result that will be needed in the proofs of Theorems 2.2.1 and 2.3.1. However, this is the only result from this chapter that will be needed in our proofs in Chapters 6 and 7, and the majority of this chapter should be considered background material that may be skipped on a first reading by the reader who is already familiar with it.

3.1 Polish groups and Haar measure

A topological group G is called a *Polish group* if its topology is Polish, ie, admits a complete separable metric. All of the topological groups we will consider will be locally compact Polish groups. Every such group admits a *left Haar measure*; ie, a regular Borel measure μ that is invariant under left translations. A group G with Haar measure μ is compact if and only if $\mu(G) < \infty$; in this case we typically normalize μ so that $\mu(G) = 1$. Recall that if μ is a left Haar measure on G, then $B \mapsto \mu(B^{-1})$ is a right Haar measure on G, and that if μ and ν are any two left Haar measures on G, then there is $c \in \mathbb{R}^+$ such that $\nu = c\mu$. If Γ is a countable group, then Γ is Polish in the discrete topology; we assume all countable groups to be given the discrete topology unless otherwise stated.

Fix now a locally compact Polish group G with left Haar measure μ. For each $x \in G$, the measure μ_x defined by $\mu_x(B) = \mu(Bx)$ is again a left Haar measure on G, and hence there is $\Delta(x) \in \mathbb{R}^+$ such that $\mu_x = \Delta(x)\mu$. The function $x \mapsto \Delta(x)$ is called the *modular function* of G, and is easily seen to be independent of the choice of

μ. It is well known that $\Delta : G \to \mathbb{R}^+$ is a continuous homomorphism from G into the multiplicative group of positive reals. Evidently a left Haar measure on G (equivalently, *every* left Haar measure on G) is also right-invariant if and only if Δ is trivial; in this case G is called *unimodular*, and we speak simply of (bi-invariant) *Haar measure* on G. Discrete groups, abelian groups, and compact groups are unimodular. One easily checks that the commutator subgroup G' of G is contained in the kernel of the modular function Δ of G, and so groups G for which G/G' is finite are also unimodular. If G is a Lie group, then by [32, 8.27] we have $\Delta(g) = |\det \mathrm{Ad}\, g|$ for any $g \in G$. Neither subgroups nor quotients of unimodular groups need be unimodular.

Now suppose $H \leq G$ is a closed subgroup of the locally compact Polish group G, with μ a fixed left Haar measure on G. Denote by G/H the space of left cosets of H in G, and let $\pi_L : G \to G/H$ be the canonical surjection. By the Effros-Mackey cross-section theorem [49, 5.4.2], G/H carries a natural standard Borel structure that may be defined as follows:

$$A \subseteq G/H \text{ is Borel} \quad \Longleftrightarrow \quad \pi_L^{-1}(A) \text{ is Borel in } G.$$

Unless otherwise stated, whenever we refer to an action of a subgroup of G on G/H, we mean the natural action by left-translations. We call a nonzero Borel measure on G/H *invariant*, or a *Haar measure* on G/H, if it is invariant under left G-translations. There exists a Haar measure on G/H if and only if $\Delta_G(h) = \Delta_H(h)$ for all $h \in H$, and if such a measure exists, then it is unique up to a scalar multiple [42, Theorem 3.17]. In particular, such a measure exists if both G and H are unimodular. It is also clear that G/H admits a left Haar measure if H arises as the (necessarily closed) stabilizer of a Borel left action of G on some homogeneous standard Borel G-space (X, μ), with μ a G-invariant Borel probability measure on X.

Naturally, everything stated thus far in this section holds, *mutatis mutandis*, for right Haar measures and for right invariant measures on the space $H \setminus G$ of right cosets of H in G. In practice, however, we shall always deal with left actions, left Haar measures, and left coset spaces.

If G/H admits a *finite* invariant measure, then H is said to have *finite covolume* in G, and in this case there exists a unique invariant probability measure on G/H [42, Theorem 3.17]. A discrete subgroup $\Gamma \leq G$ of finite covolume in G is called a *lattice* in G. Below we shall see that if $\Gamma \leq G$ is a lattice, then we shall be able to obtain a more explicit description of the unique G-invariant probability measure on G/Γ. Any locally compact group that contains a lattice is necessarily unimodular [1, 2.4.2].

3.2 Semisimple Lie groups and lattice subgroups

Throughout we let $\mathbb{L}$ be a fixed algebraically closed field of characteristic zero containing $\mathbb{R}$ and all the p-adic fields $\mathbb{Q}_p$. Every algebraic group we shall consider will be a Zariski closed subgroup G of some $GL_n(\mathbb{L})$, so we shall simply agree now that by *algebraic group* we mean any group of this kind. If the field of definition of an algebraic group $G \leq GL_n(\mathbb{L})$ is some subfield $k \subseteq \mathbb{L}$, then we call G an *algebraic k-group*. For any subring $R \subseteq \mathbb{L}$ we set

$$GL_n(R) = \{(a_{ij}) \in GL_n(\mathbb{L}) \mid a_{ij} \in R \text{ and } \det{(a_{ij})}^{-1} \in R\},$$

and then define, for any algebraic group $G \leq GL_n(\mathbb{L})$,

$$G(R) = G \cap GL_n(R).$$

In particular, if G is an algebraic k-group for $k = \mathbb{R}$, $k = \mathbb{C}$, or $k = \mathbb{Q}_p$ for some prime p, then $G(k) \leq GL_n(k)$ is a locally compact Polish group in the *Hausdorff topology*, ie, the topology obtained by restricting the natural topology on k^{n^2} to $G(k)$. More generally, if k_α is $\mathbb{R}$ or some $\mathbb{Q}_p$ for all α in some finite set A, and if each G_α is an algebraic k_α-group, then

$$\prod_{\alpha \in A} G_\alpha(k_\alpha)$$

is a locally compact Polish group in the product of the Hausdorff topologies on $G_\alpha(k_\alpha)$, which we also call the Hausdorff topology. Any topological notions concerning groups of this kind will always refer to the Hausdorff topology unless explicitly stated otherwise.

An algebraic group G is said to be *simple* if it has no nontrivial proper normal algebraic subgroups, and *k-simple* if it has no nontrivial proper normal algebraic k-subgroups. G is called *semisimple* if its (solvable) radical is trivial, where the *solvable* *radical* of G is the maximal connected solvable normal algebraic subgroup of G. The groups G_S and H_S are semisimple. Every semisimple Lie group is unimodular [32, 8.31], and every connected semisimple noncompact Lie group contains a lattice [6].

Recall that a topological space is *irreducible* if it cannot be written as a union of a pair of nonempty closed subsets, and *connected* if it cannot be written as a *disjoint* union of a pair of nonempty closed subsets; in an algebraic group endowed with the Zariski topology, these two notions coincide. As usual, we call an algebraic group *connected* if it is connected (equivalently, irreducible) in the Zariski topology, and denote by G^0 the connected component of the identity inside an algebraic group G. It is well known that G^0 is a normal algebraic subgroup of finite index in G that is contained in all other finite index algebraic subgroups of G, and that the connected (equivalently, irreducible) components of G are the cosets of G^0 in G. In particular, every algebraic group decomposes as a finite disjoint union of its connected components.

If G is an algebraic $\mathbb{C}$-group, then $G(\mathbb{C})$ is connected in the Zariski topology if and only if it is connected in the Hausdorff topology; however, if $k \subseteq \mathbb{C}$ is not algebraically closed, then $G(k)$ may be Zariski connected without being Hausdorff connected. As mentioned above, if $k = \mathbb{R}$ or $\mathbb{C}$ or some $\mathbb{Q}_p$, then any topological notions concerning $G(k)$ will refer to the Hausdorff topology, including the notion of connectedness. Following Zimmer's notation (see [58, page 35]), if G is an algebraic $\mathbb{R}$-group, then $G^0_{\mathbb{R}}$ will denote the connected component of the identity in $G(\mathbb{R})$ *in the Hausdorff topology.* In this case, $G^0_{\mathbb{R}}$ need not equal $G(\mathbb{R})$ even when G is (Zariski) connected, but it will always be the case that $G^0_{\mathbb{R}}$ has finite index in $G(\mathbb{R})$. In particular, since $PSL_2(\mathbb{R})$ is Zariski connected (for instance see [50, 21.3.3 and 21.2.4(ii)]) and $PSL_2(\mathbb{R})$ has no subgroups of finite index, it follows that $PSL_2(\mathbb{R})$ is connected in the Hausdorff topology. This will be important in Chapter 6 when we apply Zimmer's Cocycle Superrigidity Theorem.

Next we will say a few words about the dimension of an algebraic group, and about rational homomorphisms between algebraic groups. For $k \subseteq \mathbb{L}$ an algebraically closed field and X an algebraic (ie, Zariski closed) set in k^n, we denote by $k[X]$ the k-algebra of k-valued polynomial functions defined on X with coefficients in k. If G and H

are algebraic groups, then we call a group homomorphism $\varphi : G \to H$ *rational*, or a *homomorphism of algebraic groups*, if φ is also a morphism of algebraic varieties; by which we mean that φ is Zariski continuous and has the property that for every $f \in k[H]$, $f \circ \varphi \in k[G]$. If we are dealing with matrix groups, this simply means that the entries of $\varphi(x)$ are polynomial functions, defined over k and independent of x, of the entries of $x \in G$. If these polynomial functions are defined over a subfield $K \subseteq k$, then φ is called *K-rational*. If the algebraic k-group G is connected, then $k[G]$ is an integral domain, in which case we may form its fraction field $k(G)$ of k-valued rational functions defined on G. We define the *dimension* of the algebraic k-group G, written dim G, to be the transcendence degree of $k(G^0)$ over k. (For basic facts, definitions, and background concerning the material of the last three paragraphs, see [48] or [50]).

We shall need the following facts about rational homomorphisms and dimension:

Proposition 3.2.1. *Suppose G and H are algebraic groups.*

(1) If $\varphi : G \to H$ is a homomorphism of algebraic groups, then $\varphi(G)$ is closed in H and dim $G =$ dim ker $\varphi +$ dim $\varphi(G)$ [48, 2.2.5 and 5.3.3, respectively].

(2) If G is a connected algebraic group and H is a closed proper subgroup of G, then dim $H <$ dim G [50, 14.1.6(ii)]. □

If $G \leq GL_n(\mathbb{L})$ is a semisimple algebraic k-group, $k \subseteq \mathbb{L}$, then the *k-rank* of G is defined to be the maximal dimension of an abelian k-subgroup of G which is k-split, ie, which can be diagonalized over k. In particular, if G is a connected semisimple real Lie group with trivial center, then there is a connected semisimple algebraic $\mathbb{Q}$-group $H \leq GL_n(\mathbb{C})$ such that G and $H^0_{\mathbb{R}}$ are isomorphic as Lie groups [58, 3.1.6], in which case the $\mathbb{R}$-rank of G is equal to the maximal dimension of an abelian $\mathbb{R}$-subgroup of H which can be diagonalized over $\mathbb{R}$. Hence the $\mathbb{R}$-rank of $PSL_n(\mathbb{R})$, for instance, is $n-1$, and if G is a finite product of algebraic k-groups G_α, then the k-rank of G is the sum of the k-ranks of the groups G_α. In particular, the groups G_S and H_S have rank at least 2.

Finally, we conclude this section with a discussion of lattices in semisimple Lie groups. Let G be a connected semisimple Lie group with finite center, and $\Gamma \leq G$

a discrete subgroup. Then both G and Γ are unimodular, so G/Γ admits a nonzero G-invariant Borel measure μ which is uniquely determined up to a scalar multiple, and which we call the *Haar measure* on G/Γ. In this case we may give an explicit description of μ as follows. Let ν be a Haar measure on G, and fix a Borel transversal $T \subseteq G$ for G/Γ, identifying $t \in T$ with the coset $t\Gamma$. Then the left-translation action of G on G/Γ induces, via this identification, a Borel action of G on T. One easily checks that the measure $\mu = \nu \upharpoonright T$ induced on G/Γ by ν is G-invariant and does not depend on the choice of T (though it does, of course, depend on ν). As above, Γ is said to be a *lattice* in G if $\mu(G/\Gamma) < \infty$, in which case we typically normalize μ so that $\mu(G/\Gamma) = 1$. A lattice $\Gamma \leq G$ is said to be *uniform*, or *cocompact*, if G/Γ is compact. Any discrete, cocompact subgroup of a unimodular group is a lattice. A lattice $\Gamma \leq G$ for which G/Γ is not compact is called *nonuniform.* For example, $\mathbb{Z}$ is a uniform lattice in $\mathbb{R}$ and $SL_n(\mathbb{Z})$ is a nonuniform lattice in $SL_n(\mathbb{R})$.

Again let G be a connected semisimple Lie group with finite center, and let Γ be a lattice in G. Then Γ is called *irreducible* if for every non-central normal subgroup N of G, Γ is dense when projected onto G/N. Intuitively, this notion is meant to rule out lattices of the form $\Gamma_1 \times \Gamma_2 \leq G_1 \times G_2$, where Γ_1 and Γ_2 are lattices in G_1 and G_2, respectively. For instance, $\mathbb{Z}^n$ is clearly a uniform lattice in $\mathbb{R}^n$ for $n \in \mathbb{N}$, but $\mathbb{Z}^n$ is not irreducible in $\mathbb{R}^n$ unless $n = 1$. Roughly speaking, rigidity results concerning lattices in simple Lie groups typically also hold inside semisimple Lie groups provided we assume the lattices involved to be *irreducible.*

In a closely related notion, an ergodic G-space X with invariant probability measure is called *irreducible* if for every non-central normal subgroup N of G, the restricted N-action on X is still ergodic. If G is simple, this notion reduces to mere ergodicity. Evidently a lattice $\Gamma \leq G$ is irreducible if and only if the action of G on G/Γ is irreducible. It is a consequence of the Howe-Moore Vanishing Theorem that if Γ is an irreducible lattice in G a finite product of connected non-compact simple Lie groups with finite center, then the action of G on G/Γ is strongly mixing (for example, see Adams [2, 6.3] together with Zimmer [58, 2.2.19 and 2.2.20]).

3.3 Amenability

In the next two sections we introduce a pair of group-theoretic properties that have played crucial roles in superrigidity theory and in the theory of orbit equivalence, and which in some sense may be thought of as antipodal properties. We begin with the notion of *amenability*.

Let G be a locally compact Polish group with fixed Haar measure μ. Denote by $L^\infty(G)$ the real vector space of essentially bounded real-valued measurable functions on G, identified if they agree μ-a.e. A *left-invariant mean* on G is a linear functional $m : L^\infty(G) \to \mathbb{R}$ satisfying:

(1) $m(f) \geq 0$ if $f \geq 0$;

(2) $m(\chi_G) = 1$, where χ_G is the constant function 1 on G;

(3) $m(g \cdot f) = m(f)$ for every $g \in G$ and $f \in L^\infty(G)$, where $(g \cdot f)(x) = f(g^{-1}x)$.

A right-invariant mean may be defined similarly. We call G *amenable* if G admits a left (equivalently, right) invariant mean. If G is discrete, then $L^\infty(G)$ consists of all bounded functions $f : G \to \mathbb{R}$, and hence one easily checks that a discrete group G is amenable if and only if G admits a left (equivalently, right) translation invariant, finitely-additive probability measure defined on $\mathcal{P}(G)$. Intuitively, a discrete group is amenable if there is a way to say what proportion of the entire group a given subset $A \subseteq G$ comprises.

It is a classical theorem of functional analysis that every abelian group is amenable. In fact all solvable groups are amenable, and all compact groups as well. Subgroups, quotients, and finite products of amenable groups are amenable, as are amenable extensions of amenable groups. Any discrete group containing a nonabelian free group is *not* amenable. Recall that every proper algebraic subgroup of $PSL_2(\mathbb{R})$ is solvable-by-finite (for instance see [25, Exercise 21.4.4]), and hence amenable; we shall use this below in Section 5.5.

The relevance of amenability to the theory of countable Borel equivalence relations and the theory of orbit equivalence stems from the fact that orbit spaces of Borel actions of amenable groups are relatively “well-behaved” with respect to these theories, whereas orbit spaces of nonamenable groups tend not to be. For instance, any measure-preserving, ergodic action of a countable amenable group on a standard Borel space is orbit equivalent to a $\mathbb{Z}$-action [8], [40]. On the other hand, it has recently been shown that any countable nonamenable group admits continuum many pairwise orbit inequivalent free, measure-preserving, ergodic standard actions [13].

In the context of countable Borel equivalence relations, the orbit equivalence relation arising from a probability measure-preserving, a.e.-free action of a countable nonamenable group is not hyperfinite [26, 1.7]. On the other hand, it is open whether every orbit equivalence relation arising from a Borel action of a countable amenable group is hyperfinite; however, for every standard Borel probability space (X, ν) and Borel action of a countable amenable group Γ on X, there is an invariant ν-conull subset $Y \subseteq X$ on which $E_\Gamma^X \upharpoonright Y$ is hyperfinite [26, 2.5 and 2.6]. In our proof of Theorem 2.2.1, we shall need a generalization of this fact to actions of arbitrary (ie, not necessarily countable) amenable locally compact Polish groups. We record this generalization below as Proposition 5.5.4, after we have discussed the notion of E_0-ergodicity in Chapter 4.

3.4 Property (T) and property (τ)

If G is a locally compact Polish group, then a *unitary representation* of G on a Hilbert space $\mathcal{H}$ is a continuous group homomorphism $\rho : G \to U(\mathcal{H})$ into the unitary group of $\mathcal{H}$, which of course may also be thought of as an action $G \curvearrowright \mathcal{H}$ by unitary transformations. As all our groups are separable we always assume $\mathcal{H}$ to be separable as well. If $\rho : G \to U(\mathcal{H})$ is a unitary representation and the closed subspace $V \leq \mathcal{H}$ is invariant under the G-action, then the restricted action is called a *subrepresentation.* A representation $\rho : G \to U(\mathcal{H})$ is called *irreducible* if its only subrepresentations are trivial, and *reducible* otherwise. By the *trivial one-dimensional representation* of G we mean the function $\rho_0 : G \to \mathbb{C}$ that maps every $g \in G$ to 1. We call two unitary representations $\rho : G \to U(\mathcal{H}_1)$ and $\pi : G \to U(\mathcal{H}_2)$ of G *equivalent*, or *isomorphic*, if there is a unitary map $\varphi : \mathcal{H}_1 \to \mathcal{H}_2$ such that $\pi\varphi = \varphi\rho$, ie, such that for all $g \in G$ and for all $x \in \mathcal{H}_1$,

$$\varphi(\rho(g)(x)) = \pi(g)(\varphi(x)).$$

Now fix a locally compact Polish group G and a unitary representation $\rho : G \to U(\mathcal{H})$. Of course, $\mathcal{H}$ may or may not contain nontrivial vectors invariant under $\rho(G)$. We shall define a notion of ρ admitting *almost invariant vectors.*

Specifically, for each $\epsilon > 0$ and for each compact subset $K \subseteq G$, we call a unit vector $u \in \mathcal{H}$ (ϵ, K)*-invariant* if

$$\|\rho(g)u - u\| < \epsilon \quad \text{for all } g \in K.$$

We then say that ρ *admits almost invariant vectors* if for all (ϵ, K), there exists an (ϵ, K)-invariant unit vector u in $\mathcal{H}$.

Finally, we say that G is *Kazhdan*, or has *Property (T)*, if the following equivalent properties hold (for their equivalence see [36, III.2.1]):

(1) Any unitary representation of G which admits almost invariant vectors actually admits a nontrivial invariant vector.

(2) There exists $\epsilon > 0$ and compact $K \subseteq G$ such that for every nontrivial irreducible unitary representation $\rho : G \to U(\mathcal{H})$ of G and for every unit vector $u \in \mathcal{H}$, $\|\rho(g)u - u\| \geq \epsilon$ for some $g \in K$.

If G is a locally compact group and if Γ is a lattice in G, then Γ has Property (T) if and only if G does ([35, 1.3]). For $n \geq 3$, the groups $SL_n(\mathbb{R})$ together with their lattice subgroups have Property (T). More generally, if G is a connected semisimple Lie group with finite center each of whose factors has $\mathbb{R}$-rank at least 2, then G has Property (T) [58, 7.1.4]. Some rank one real Lie groups have Property (T) while some do not; in particular, $SL_2(\mathbb{R})$ and its lattice subgroups do *not* [34, 3.1.9]. Nonabelian free groups are neither amenable nor Kazhdan [34, 3.1.7]. The groups $\mathbb{R}^n$ and $\mathbb{Z}^n$ do not have Property (T) either; in fact, an amenable group has Property (T) if and only if it is compact [34, 3.1.6]. In Section 5.5 below we shall see how the incompatibility between Kazhdan groups and amenable groups can be applied to arguments involving Zimmer's Cocycle Superrigidity Theorem.

One of Kazhdan's original reasons for introducing Property (T) was to show that (irreducible) lattices in higher rank (semi-)simple Lie groups are finitely generated; and indeed, it is well-known that if Γ is a countable discrete group with Property (T), then Γ is finitely generated and the compact set $K \subseteq \Gamma$ in (2) above can be taken to be any finite set of generators (for instance, see [35, 1.24] and [34, 3.2.5]). We define Property (τ) for a finitely generated, countable discrete group Γ by slightly weakening the definition of Property (T) as follows: we say that such a group Γ has *Property (τ)* if

(3) There exists $\epsilon > 0$ and a finite generating set $F \subseteq \Gamma$ such that for every nontrivial irreducible unitary representation $\rho : G \to U(\mathcal{H})$ of G with $[\Gamma : \ker \rho] < \infty$ and for every unit vector $u \in \mathcal{H}$, $\|\rho(\gamma)u - u\| \geq \epsilon$ for some $\gamma \in F$.

It is well-known that this definition does not depend on the choice of F ([34, 3.2.5]).

Of course every finitely generated countable Kazhdan group has Property (τ), but there exist many interesting examples of groups with Property (τ) that are not Kazhdan. By Zimmer [59, Corollary 19], every Kazhdan subgroup of $SL_2(\mathbb{C})$ is compact, and so the groups Γ_S and Θ_S are not Kazhdan. However, Γ_S is finitely generated [36, VIII.3.3] and has Property (τ), as discussed in Lubotzky [35, Section 4.1]. The groups Θ_S also have Property (τ) (see [52, Proof of 6.1]). In each case this is related to the fact that these groups have the congruence subgroup property (for instance, see [46]).

In Chapter 4 below we will discuss dynamical properties of actions of Property (T) and Property (τ) groups, and will later make use of these properties in setting up an application of Zimmer's Cocycle Superrigidity Theorem.

3.5 Totally real number fields

In this section, all of our fields are assumed to be subfields of $\mathbb{C}$. We write $\overline{\mathbb{Q}}$ for the algebraic closure of the rationals in $\mathbb{C}$. If $k \supseteq K$ are fields, then by a Galois automorphism of k over K we mean an element of $\mathrm{Gal}(k/K)$, ie, a field automorphism of k whose restriction to K is the identity.

Let $k \supseteq K$ be subfields of $\mathbb{C}$. As usual, by the *degree* $[k : K]$ of the extension $k \supseteq K$

we mean the vector space dimension $\dim_K k$ of k as a K-vector space. An element $u \in k \supseteq K$ is called *algebraic* over K if u is a root of some nonzero polynomial in $K[x]$; in this case u satisfies a unique monic polynomial of minimal degree in $K[x]$, called the *minimal* polynomial of u, whose degree is equal to the degree $[K(u) : K]$ of the extension $K(u) \supseteq K$. An extension $k \supseteq K$ is called *algebraic* if each element of k is algebraic over K, and *transcendental* otherwise. By a *number field* we mean an extension field $k \supseteq \mathbb{Q}$ of finite degree over $\mathbb{Q}$. Clearly every number field is an algebraic extension of $\mathbb{Q}$.

If every embedding of a number field $k \supseteq \mathbb{Q}$ into the complex numbers is real-valued, then k is called *totally real.* It is well known that every number field $k \supseteq \mathbb{Q}$ is generated over $\mathbb{Q}$ by a single algebraic number $u \in \overline{\mathbb{Q}}$, called a *primitive element* of the extension. If f is the minimal polynomial of a primitive element u of the number field $k \supseteq \mathbb{Q}$, then $k = \mathbb{Q}(u) \supseteq \mathbb{Q}$ is totally real if and only if all of the roots of f are real.

If $k = \mathbb{Q}(u) \supseteq \mathbb{Q}$ is a number field with minimal polynomial f in $\mathbb{Q}[x]$ of degree n, then the *conjugates* of u are the n roots of f in $\overline{\mathbb{Q}}$. In this case, the conjugates of u need not lie in k. A number field $k = \mathbb{Q}(u)$ is called *normal* if each conjugate of u *does* lie in k. Equivalently, a number field $k \supseteq \mathbb{Q}$ is normal if and only if every irreducible polynomial in $\mathbb{Q}[x]$ that has a root in k actually splits in $k[x]$, ie, has all of its roots in k. Furthermore, a number field $k \supseteq \mathbb{Q}$ is normal if and only if it is Galois over $\mathbb{Q}$, ie, if and only if for every $u \in k \setminus \mathbb{Q}$, there is a field automorphism of k fixing $\mathbb{Q}$ setwise and moving u. Every number field $k \supseteq \mathbb{Q}$ is contained in a unique minimal normal extension field n.c.(k) of finite degree over k with n.c.$(k) \subseteq \overline{\mathbb{Q}}$. We call n.c.$(k)$ the *normal closure* of k. If $k = \mathbb{Q}(u) \supseteq \mathbb{Q}$ is a number field and $\{u = u_1, u_2, \ldots, u_n\}$ is the set of conjugates of u, then every embedding $k \hookrightarrow \mathbb{C}$ is given by $u \mapsto u_i$. In particular, there are exactly n such embeddings.

Suppose $k \supseteq \mathbb{Q}$ is a totally real number field of degree n over $\mathbb{Q}$. In what follows we shall always associate to k a fixed primitive element $u \in k$ so that $k = \mathbb{Q}(u)$, together with an ordering $\{u = u_1, \ldots, u_n\}$ on the n conjugates of u in n.c.(k). Every embedding $k \hookrightarrow \mathbb{R}$ fixes $\mathbb{Q}$ and is completely determined by its action on u, which must be mapped to a conjugate. Thus for $1 \leq i \leq n$ we set $\vartheta_i : u \mapsto u_i$, and denote by $\mathfrak{R}_k = \{\vartheta_1, \ldots, \vartheta_n\}$

the (ordered) set of embeddings $\vartheta_i : k \hookrightarrow \mathbb{R}$ of k.

An element u in a number field $k \supseteq \mathbb{Q}$ is called an *integer* of k if u is a root of a *monic* polynomial in $\mathbb{Z}[x]$. We write $\mathcal{O}_k$ for the ring of integers in the number field k. If $\vartheta : k \hookrightarrow \mathbb{R}$ is one of the n complex embeddings of the totally real number field k of degree $n > 1$ over $\mathbb{Q}$, and if $\gamma \in SL_2(\mathcal{O}_k)$, then we write γ^ϑ for the matrix obtained by replacing each entry of γ with its image under ϑ. We then define the embedding

$$\sigma^k : \Gamma_k = SL_2(\mathcal{O}_k) \to G_k = \prod_{\vartheta_i \in \mathfrak{R}_k} SL_2(\mathbb{R})$$

by

$$\sigma^k(\gamma) = \langle \gamma^{\vartheta_1}, \ldots, \gamma^{\vartheta_n} \rangle.$$

By Margulis [36, IX(1.7v)], $\sigma^k(\Gamma_k)$ is an irreducible, non-cocompact lattice in G_k. By an abuse of notation we shall also denote by σ^k the corresponding embedding that realizes

$$\Lambda_k = PSL_2(\mathcal{O}_k)$$

as an irreducible lattice in

$$H_k = \prod_{\vartheta_i \in \mathfrak{R}_k} PSL_2(\mathcal{O}_k).$$

Throughout we shall identify $\sigma^k(\Gamma_k)$, $\sigma^k(\Lambda_k)$ with Γ_k, Λ_k, and rely upon context to distinguish them. Of course, the notation introduced in this paragraph is intended to be consistent with that already introduced in Chapter 2.

Indeed, we shall be especially concerned with multi-quadratic number fields and their integer rings. Suppose $S = \{p_1, \ldots, p_s\}$ is a finite, nonempty set of primes, and let $k_S = \mathbb{Q}(\sqrt{p_1}, \ldots, \sqrt{p_s})$. In this case, in order to simplify notation and to conform to the notation introduced in Chapter 2, we shall write $\mathcal{O}_S$, σ^S, Γ_S, G_S, Λ_S, and H_S in place of $\mathcal{O}_{k_S}$, σ^{k_S}, Γ_{k_S}, etc. Here the $2^{|S|}$ embeddings of k_S into $\mathbb{R}$ are given by sending each $\sqrt{p_i}$ either to itself or to $-\sqrt{p_i}$. Hence these embedding are naturally indexed by the subsets of S, and so we reiterate the following definitions from Chapter 2.

For each $A \subseteq S$, let

$$\sigma_A^S : k_S = \mathbb{Q}(\sqrt{p_1}, \ldots, \sqrt{p_s}) \hookrightarrow \mathbb{R}$$

be the field embedding that maps $\sqrt{p_i} \mapsto -\sqrt{p_i}$ if $p_i \in A$, and $\sqrt{p_i} \mapsto \sqrt{p_i}$ if $p_i \in S \setminus A$, so that in particular $\sigma^S_\emptyset$ is the inclusion embedding. Then, identifying as usual $\mathcal{P}(S) = 2^S$, define

$$\sigma^S : \Gamma_S \to \prod_{2^S} SL_2(\mathbb{R}) \quad \text{by} \quad \sigma^S(\gamma) = \langle \gamma^{\sigma^S_A} \rangle_{A \subseteq S},$$

where $\gamma^{\sigma^S_A}$ has the obvious meaning and 2^S is ordered, merely for definiteness, lexicographically on the natural ordering for S itself.

It is easily checked that $u = \sqrt{p_1} + \cdots + \sqrt{p_s}$ is a primitive element for k_S over $\mathbb{Q}$, and that the $2^{|S|}$ conjugates of u are the algebraic numbers $\sigma^S_A(u)$, $A \subseteq S$. Hence the minimal polynomial of k_S over $\mathbb{Q}$ is

$$\prod_{A \subseteq S} (x - \sigma^S_A(u)).$$

Notice that k_S is normal, ie, is equal to its normal closure.

Note also that the ring $\mathbb{Z}_S := \mathbb{Z}[\sqrt{p_1}, \dots, \sqrt{p_s}]$ generated over $\mathbb{Z}$ by adjoining the square roots of the primes in S is a subset of $\mathcal{O}_S$, in general a proper subset. The ring $\mathbb{Z}_S \subseteq \mathcal{O}_S$ is sometimes called an *order*. We have stated all our results for the groups $\Lambda_S = PSL_2(\mathcal{O}_S)$, but we remark that since $PSL_2(\mathbb{Z}_S)$ is a subgroup of finite index in $PSL_2(\mathcal{O}_S)$ — and hence still an irreducible lattice in H_S — and since we never use elements of $PSL_2(\mathcal{O}_S) \setminus PSL_2(\mathbb{Z}_S)$ in the proofs of our theorems, all of our results hold equally well for the groups $PSL_2(\mathbb{Z}_S)$ in place of $PSL_2(\mathcal{O}_S)$.

3.6 A measure classification theorem

In our proof of Theorems 2.2.1 and 2.3.1, we shall need to appeal to a measure classification result in order to obtain knowledge about the image of a particular Borel reduction. This technique goes back to Adams, who made use in [2] of Ratner's measure classification theorem [44]. Our approach is based on Thomas [52], who used a measure classification result essentially due to David Witte Morris [56, 5.8]. We recall first the definition of an *algebraic* probability measure.

Definition 3.6.1. *Let H be a locally compact Polish group and let L be a closed subgroup of H. Then a probability measure μ on H/L is said to be* algebraic *iff there exists* *a closed subgroup C of H such that*

1. *μ is C-invariant; and*

2. *μ is supported on a C-orbit; i.e., there exists $x \in H/L$ such that $\mu(Cx) = 1$.*

The remainder of this section will be devoted to proving the following Lemma.

Lemma 3.6.2. *Suppose $\mathbb{K}$ and $\mathbb{F}$ are totally real number fields properly extending $\mathbb{Q}$, let $\mathcal{O}_\mathbb{K}$ and $\mathcal{O}_\mathbb{F}$ be their integer rings, and let $\Lambda_\mathbb{K} = PSL_2(\mathcal{O}_\mathbb{K})$ and $\Lambda_\mathbb{F} = PSL_2(\mathcal{O}_\mathbb{F})$. Further let*

$$H_\mathbb{K} = \prod_{\vartheta \in \mathfrak{R}_\mathbb{K}} PSL_2(\mathbb{R}) \quad \text{and} \quad H_\mathbb{F} = \prod_{\theta \in \mathfrak{R}_\mathbb{F}} PSL_2(\mathbb{R}),$$

so that $\Lambda_\mathbb{K}$ and $\Lambda_\mathbb{F}$ are irreducible lattices in $H_\mathbb{K}$ and $H_\mathbb{F}$, respectively. Suppose that $\varphi : H_\mathbb{K} \to H_\mathbb{F}$ is an injective homomorphism of algebraic groups. Then every $\varphi(\Lambda_\mathbb{K})$-invariant, $\varphi(\Lambda_\mathbb{K})$-ergodic probability measure on $H_\mathbb{F}/\Lambda_\mathbb{F}$ is algebraic.

In order to prove this Lemma we shall need two results, the first of which is Theorem 3.1 in Margulis-Tomanov [37]:

Proposition 3.6.3 ([37, 3.1])**.** *Let G be a connected real Lie group, $\Gamma \leq G$ a closed subgroup, $H \leq G$ a subgroup generated by its unipotent algebraic subgroups, and μ an H-invariant, ergodic probability measure on G/Γ. Then μ is algebraic.* $\square$

This result will be used to verify the third hypothesis of Proposition 3.6.4 below. To set up the application of 3.6.4, assume the hypotheses of Lemma 3.6.2, write $H = \varphi(H_\mathbb{K}) \leq H_\mathbb{F}$, and let

$$\delta : H \to H \times H_\mathbb{F}$$

be the diagonal embedding. Supposing for the moment that $\delta(H)$ is generated by its unipotent algebraic subgroups inside $H \times H_\mathbb{F}$, it follows from 3.6.3 that every $\delta(H)$-invariant, $\delta(H)$-ergodic probability measure on

$$(H \times H_\mathbb{F})/(\varphi(\Lambda_\mathbb{K}) \times \Lambda_\mathbb{F}) = (H/\varphi(\Lambda_\mathbb{K})) \times (H_\mathbb{F}/\Lambda_\mathbb{F})$$

is algebraic; and this is precisely the third hypothesis needed in the following result of Witte Morris (see [52, A.1]):

Proposition 3.6.4 (Witte Morris)**.** *Let G be a locally compact Polish group and let H, L be closed subgroups of G. Suppose that:*

1. *Γ is a lattice in H;*

2. *$\Delta : H \to H \times G$ is the diagonal embedding; and*

3. *every ergodic $\Delta(H)$-invariant probability measure on $(H/\Gamma) \times (G/L)$ is algebraic.*

Then every ergodic Γ-invariant probability measure on G/L is algebraic. □

It follows immediately from this proposition that every $\varphi(\Lambda_{\mathbb{K}})$-invariant, $\varphi(\Lambda_{\mathbb{K}})$-ergodic probability measure on $H_{\mathbb{F}}/\Lambda_{\mathbb{F}}$ is algebraic, thus completing the proof of Lemma 3.6.2 as soon as we show that $\delta(H)$ is generated by its unipotent algebraic subgroups inside $H \times H_{\mathbb{F}}$.

Recall that an element

$$x \in G_{\mathbb{K}} = \prod_{\vartheta \in \mathfrak{R}_{\mathbb{K}}} SL_2(\mathbb{R})$$

is *unipotent* if $x - 1_{G_{\mathbb{K}}}$ is nilpotent, ie, if there is $n \in \mathbb{N}$ such that $(x-1)^n = 0$. If $\pi : G_{\mathbb{K}} \to H_{\mathbb{K}}$ is the canonical surjection, then by definition $\pi(x)$ is *unipotent* in $H_{\mathbb{K}}$ if either x or $-x$ is unipotent in $G_{\mathbb{K}}$. (For the general definition of unipotence in an algebraic group, see, for instance, [48, Section 2.4] or [25, Section 15.3]). Finally, an algebraic subgroup $A \leq H_{\mathbb{K}}$ is unipotent if each of its elements is unipotent.

Now, notice that $H_{\mathbb{K}}$ and $\delta(H) = (\delta \circ \varphi)(H_{\mathbb{K}})$ are isomorphic as algebraic groups via the injective rational homomorphism $(\delta \circ \varphi)$. Since unipotence is preserved by algebraic group homomorphisms (for instance, see [48, 2.8.8]), it will therefore suffice to show that $G_{\mathbb{K}}$ is generated by its unipotent algebraic subgroups. Recall that a *transvection* in $SL_2(\mathbb{R})$ is a matrix of the form $1 + e_{ij}$, where e_{ij} is the matrix with ij-entry 1 and all other entries equal to zero. Clearly every transvection is unipotent, and hence if each x_i is a transvection in $SL_2(\mathbb{R})$, then $\langle x_i \rangle \in G_{\mathbb{K}}$ is unipotent. But it is well known that every element of $SL_2(\mathbb{R})$ can be written as a product of transvections, and hence an arbitrary element $x \in G_{\mathbb{K}}$ can be written as a product of unipotent elements of $G_{\mathbb{K}}$. It follows that $G_{\mathbb{K}}$ is generated by its unipotent algebraic subgroups. This completes the proof of Proposition 3.6.4, and hence also that of Lemma 3.6.2.

Chapter 4

Some preliminaries from measurable dynamics

In this chapter we recall some basic facts about the dynamics of measure-preserving group actions. In particular we concentrate on the notion of ergodicity, which plays a crucial role in the theory of countable Borel equivalence relations. We also introduce several strengthenings of the ergodicity property, including strong and mild mixing, F-ergodicity for $E_0 \leq_B F$, and strong ergodicity. We then define precisely the notions of isomorphism, virtual isomorphism, factor, and quotient, which we will need in the proofs of Theorems 2.4.1 and 2.2.1. Finally, we conclude with a brief introduction to the theory of the Kolmogorov-Sinai entropy of a dynamical system, which we will use to characterize factors of Bernoulli systems in the proof of Theorem 2.3.1.

Throughout this chapter, as in the last, G will always be a locally compact Polish group and X a standard Borel G-space; though we remark that in practice our acting groups are usually countable discrete groups. Furthermore, all measures in this chapter and throughout this thesis are *Borel* measures, ie, measures whose domain includes the σ-algebra of Borel sets of the underlying Polish space. A Borel measure μ on a standard Borel space X is called *nonatomic* if for every Borel set $A \subseteq X$ of positive μ-measure, there exists a Borel subset $B \subseteq A$ such that $0 < \mu(B) < \mu(A)$. A measure is called *continuous* if it vanishes on singletons. In our case these two notions will always coincide (for instance, see [4]). All of our measures on *infinite* spaces will be assumed to be nonatomic.

4.1 Ergodicity

Recall that by a *standard Borel G-space* X we mean a standard Borel space X together with a Borel action of G on X, which we often write $G \curvearrowright X$. We say that a Borel set

$A \subseteq X$ is *G-invariant*, or invariant under the action of G, if $g \cdot A = A$ for all $g \in G$. If μ is a Borel measure on X, then we say that G *preserves* μ, or that μ is *G-invariant*, if for all Borel sets $B \subseteq X$ and for all $g \in G$, $\mu(g \cdot B) = \mu(B)$. Finally, if $G \curvearrowright (X, \mu)$ preserves μ, then we say that the action of G on (X, μ) is *ergodic* if every G-invariant Borel subset of X is μ-null or μ-conull.

We shall make frequent use of the following easy reformulation of ergodicity.

Proposition 4.1.1. *Let X be a standard Borel G-space with G-invariant Borel measure μ. Then the following are equivalent:*

(1) the action of G on (X, μ) is ergodic;

(2) for any standard Borel space Y and G-invariant Borel function $f : X \to Y$, f is μ-a.e. constant.

(3) every G-invariant Borel function $f : X \to [0, 1]$ is μ-a.e. constant. □

We shall also make use of a representation-theoretic formulation of ergodicity that is closely related to Proposition 4.1.1. Suppose that G is a locally compact Polish group and (X, μ) a standard Borel G-space with invariant probability measure μ. Define the associated left-regular unitary representation of G on the Hilbert space $L^2(X, \mu)$ by

$$g \cdot f(x) = f(g^{-1} \cdot x).$$

Then let $L^2_0(X, \mu)$ be the closed, G-invariant subspace of $L^2(X, \mu)$ given by

$$L^2_0(X, \mu) = \left\{ f \in L^2(X, \mu) \mid \int_X f \, d\mu = 0 \right\},$$

so that $L^2_0(X, \mu)$ is the orthogonal complement of the space of constant functions. Then the action of G on (X, μ) is ergodic if and only if the associated unitary action of G on $L^2_0(X, \mu)$ has no nonzero invariant vectors [58, 2.2.17]. Evidently if G is Kazhdan and acts ergodically on (X, μ), then furthermore $G \curvearrowright L^2_0(X, \mu)$ will not admit almost invariant vectors.

A measure-preserving action $G \curvearrowright (X, \mu)$ is called *essentially transitive* if it admits a conull orbit; clearly every such action is ergodic. An ergodic action that is not essentially transitive is called *properly ergodic*. A measure-preserving action $G \curvearrowright (X, \mu)$ is called *uniquely ergodic* if μ is the unique G-invariant probability measure on $(X, \mathcal{B})$. Every uniquely ergodic action is ergodic, for if $A \subseteq X$ is invariant with $0 < \mu(A) < 1$, then we may define a new invariant probability measure ν on X by

$$\nu(B) = \mu_A(B) = \mu(B \cap A)/\mu(A).$$

Ergodicity may be viewed as an irreducibility condition, for if X splits into positive measure invariant Borel subsets $X = A \sqcup B$, then we may decompose the action of G on X into component actions $G \curvearrowright (A, \mu \restriction A)$ and $G \curvearrowright (B, \mu \restriction B)$, thus simplifying our analysis of $G \curvearrowright (X, \mu)$. The relevance of ergodicity in the context of Borel equivalence relations lies in the fact that it provides a technique for showing that a given equivalence relation is not smooth. Specifically, we have the following:

Proposition 4.1.2. *Suppose that Γ is a countable group and that (X, μ) is a standard Borel Γ-space with μ a nonatomic Γ-invariant measure on X. If the action of Γ on (X, μ) is ergodic, then E_Γ^X is not smooth.*

Proof. Let $f : X \to [0, 1]$ be any Γ-invariant Borel function. By Proposition 4.1.1(3), f is μ-a.e. constant, and hence if f were a Borel reduction, then X would contain a μ-conull Γ-orbit. But each Γ-orbit is countable, and therefore μ-null, since μ is nonatomic. $\square$

In Section 4.2 below we will define stronger notions of ergodicity that can be used to rule out the possibility of a reduction existing between a given pair E, F of countable Borel equivalence relations.

Next we consider the decomposition of a space into ergodic components. Let Γ be a countable discrete group, and let (X, μ) be a standard Borel probability Γ-space. Suppose the action of Γ on X preserves μ and is ergodic, and let $\Lambda \leq \Gamma$ be a finite-index subgroup, say

$$[\Gamma : \Lambda] = N, \quad \text{with} \quad \Gamma = \bigsqcup_{i<N} \gamma_i \Lambda.$$

We will obtain a decomposition of X into finitely many pairwise disjoint positive measure subsets, called *ergodic components*, on which the restricted Λ-action is ergodic.

Suppose that the restricted action of Λ on X is not ergodic. Say $X_0 \subseteq X$ is Λ-invariant, with $0 < \mu(X_0) < 1$. Then of course $X_1 = X \setminus X_0$ is also Λ-invariant, and $0 < \mu(X_1) = 1 - \mu(X_0) < 1$. If the restricted actions of Λ on X_0 and X_1 are not both ergodic, then we continue decomposing into positive-measure Λ-invariant subsets. It is easily checked that at any stage $X = X_0 \sqcup X_1 \sqcup \cdots \sqcup X_n$ of this decomposition, the set

$$Y = \bigcup_{i<N} \gamma_i X_k$$

is Γ-invariant for each $0 \leq k \leq n$, and so $\mu(Y) = 1$. This implies that $\mu(X_k) \geq 1/N$, and so in particular we see that X cannot decompose into more than $N = [\Gamma : \Lambda]$ Λ-invariant subsets. Hence our decomposition process must terminate at some finite stage, at which point we will have

$$X = X_0 \sqcup \cdots \sqcup X_n,$$

with $n \leq N$, and $\mu(X_k) \geq 1/N$ for each $k \leq n$. We call the Λ-invariant sets X_i *ergodic components for the action of* Λ *on* X. It is easily checked that ergodic components are uniquely determined up to null sets, and that if the action of Γ on X is uniquely ergodic, then the actions of Λ on the ergodic components are also uniquely ergodic.

If $\Lambda \lhd \Gamma$ is a normal subgroup, then we can say much more. In this case Γ acts as a transitive permutation group on the collection $\{X_i\}$ of ergodic components. (To see this, note that $\lambda \cdot (\gamma \cdot X_i) = \lambda\gamma \cdot X_i = \gamma\lambda' \cdot X_i = \gamma \cdot X_i$, and hence $\gamma \cdot X_i$ is Λ-invariant, ie, $\gamma \cdot X_i = X_j$ for some j.) This implies that each ergodic component has the same measure, and that the number of components must divide the index of Λ in Γ.

Finally, we conclude this section with a discussion of equivariance and the image of a measure under a Borel function. Suppose that X and Y are standard Borel spaces, μ a Borel measure on X, and let $f : X \to Y$ be a Borel function. Then we can define a Borel measure $\nu = f_*\mu$ on Y, called the *image* of μ under f, by

$$\nu(B) = \mu(f^{-1}(B)) \text{ for all Borel } B \subseteq Y.$$

Clearly $f_*\mu$ is a probability measure if and only if μ is.

Now further suppose that X and Y are standard Borel G-spaces for G a locally compact Polish group, and assume that the G-action on X preserves μ. Then the Borel function $f : X \to Y$ is said to be G-*equivariant* if for all $g \in G$ and for μ-a.e. $x \in X$, $g \cdot f(x) = f(g \cdot x)$. In this case $f_*\mu$ is also G-invariant, and is ergodic if μ is.

We may slightly generalize these notions as follows. By a *permutation group* we mean simply a set X together with an action of a group G on X. Given a pair (X, G), (Y, H) of permutation groups, a *homomorphism of permutation groups* is a map $f : X \to Y$ together with a group homomorphism $\varphi : G \to H$ such that for all $x \in X$ and for all $g \in G$, $f(gx) = \varphi(g)f(x)$. We call (f, φ) an *isomorphism of permutation groups* if both f and φ are bijective. Of course, for our purposes G and H will be locally compact Polish groups, X and Y standard Borel spaces, and all actions and maps Borel. A homomorphism of permutation groups $(f, \varphi) : (X, G) \to (Y, H)$ is then a strong form of a Borel homomorphism from E_G^X to E_H^Y; it not only sends equivalent elements to equivalent elements, but preserves, via φ, witnesses to this equivalence. The following facts are easily checked:

Proposition 4.1.3. *Suppose that $(f, \varphi) : (X, G) \to (Y, H)$ is a homomorphism of permutation groups.*

(1) f is a Borel homomorphism from E_G^X to E_H^Y.

(2) If f is injective and φ is surjective, then f is a Borel reduction from E_G^X to E_H^Y.

(3) If f is injective and G acts faithfully on X, then φ is also injective.

(4) If H acts freely and φ is injective, then f is injective on G-orbits; in particular, if f is also a Borel reduction, then f is injective.

(5) If (f, φ) is an isomorphism, then f is a bijective Borel reduction and G acts freely if and only if H does.

Now, further suppose that X carries a G-invariant probability measure μ. In this case we typically relax the definition of homomorphism to require only that the identity $f(g \cdot x) = \varphi(g) \cdot f(x)$ hold μ-a.e. We then have the following easy result, which will be used in the proofs of Theorems 2.2.1 and 2.3.1.

Proposition 4.1.4. *Suppose G and H are locally compact Polish groups, and let X be a standard Borel G-space with invariant probability measure μ. Suppose*

$$(f, \varphi) : (X, G) \to (Y, H)$$

is a permutation group homomorphism into the standard Borel H-space Y. Let $\nu = f_* \mu$. Then ν is a $\varphi(G)$-invariant probability measure on Y; and, moreover, ν is $\varphi(G)$-ergodic if μ is G-ergodic. □

Of course, if (f, φ) is a permutation group homomorphism such that the identity $f(g \cdot x) = \varphi(g) \cdot f(x)$ only holds μ-a.e., then we much be very careful when drawing conclusions such as those in Proposition 4.1.3 about (f, φ). To recover Proposition 4.1.3 in this case, we must consider the restriction of the equivalence relation E_G^X to the (invariant) conull subset on which the permutation group homomorphism identity holds.

4.2 Strong ergodicity

If $T : X \to X$ is an invertible measure-preserving transformation of a probability space (X, μ), then it follows from the Birkoff-Khinchin Ergodic Theorem [55, 1.14] that the $\mathbb{Z}$-action induced by T on (X, μ) is ergodic if and only if for all Borel sets $A, B \subseteq X$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n-1} \mu(T^k A \cap B) = \mu(A)\mu(B).$$

Intuitively this means that T is ergodic if and only if every Borel set $A \subseteq X$ becomes, asymptotically on average, independent of any other Borel set $B \subseteq X$ as it moves through its orbit under T. If we strengthen this condition to say that the iterates $T^k A$ become independent of B in the limit outright, rather than on average, then we arrive at the notion of *strong mixing*.

Definition 4.2.1. *If G is a countably infinite discrete group and if X is a standard Borel G-space with G-invariant probability measure μ, then the action of G on (X, μ) is said to be* strongly mixing *if for any two Borel subsets $A, B \subseteq X$, if $\langle g_n \mid n \geq 0 \rangle$ is a sequence of distinct elements of G, then*

$$\lim_{n \to \infty} \mu(g_n A \cap B) = \mu(A)\mu(B).$$

It is clear that strong mixing implies ergodicity, for if $A \subseteq X$ is an invariant Borel set and $\langle g_n \mid n \geq 0 \rangle$ a sequence of distinct elements of G, then

$$\mu(A) = \lim_{n \to \infty} \mu(g_n A \cap A) = \mu(A)^2,$$

whence $\mu(A) = 0$ or 1. Strictly between the two notions is another mixing property we shall need, namely that of *mild mixing*.

Definition 4.2.2. *If G is a countably infinite discrete group and X is a standard Borel G-space with a G-invariant probability measure μ, then the action of G on (X, μ) is said to be* mildly mixing *if for every sequence $\langle g_n \mid n \geq 0 \rangle$ of distinct elements of G and for every Borel set $A \subseteq X$ with $0 < \mu(A) < 1$,*

$$\liminf_n \mu(A \bigtriangleup g_n A) > 0.$$

Again it is clear that mild mixing implies ergodicity. To see that strong mixing implies mild mixing, suppose $G \curvearrowright X$ is strongly mixing, let $\langle g_n \mid n \geq 0 \rangle$ be a sequence of distinct elements of G, and let $A \subseteq X$ be a Borel set that is neither null nor conull. Then

$$\liminf_n \mu(A \bigtriangleup g_n A) = 2\mu(A)(1 - \mu(A)) > 0.$$

In the proof of Theorem 2.3.1, we shall use the fact [20, 1.1] that if $G \curvearrowright X$ is mildly mixing, and if $G \curvearrowright Y$ is any other properly ergodic G-action on a standard Borel probability space (Y, ν), then the product action of G on $X \times Y$ is also ergodic. For more on mild mixing see [20].

Next we wish to introduce some further notions of strong ergodicity, generalizing this time not mixing properties but rather Proposition 4.1.1 and its consequence, 4.1.2. This latter proposition states that if $G \curvearrowright (X, \mu)$ is ergodic, then E_G^X is not smooth. We shall introduce, for (the $\sim_B$ class of) each countable Borel equivalence relation F, a notion of F-ergodicity which will have the property that if E is an F-ergodic countable Borel equivalence relation, then $E \not\leq_B F$.

Thus let E be a countable Borel equivalence relation on the standard Borel space X, and let μ be a Borel measure on X. Recall that by the Feldman-Moore Theorem there is a countable group Γ and a Borel action of Γ on X such that $E = E_\Gamma^X$. We say that μ is *E-invariant* if μ is Γ-invariant for some (equivalently every) countable group Γ and Borel action of Γ on X such that $E = E_\Gamma^X$.

Definition 4.2.3. *If E, F are Borel equivalence relations on the standard Borel spaces X, Y, respectively, and if μ is an E-invariant Borel measure on X, then (E, μ) (or simply E, if μ is understood) is said to be* F-ergodic *if for any Borel homomorphism $f : X \to Y$ from E to F, there exists a μ-conull subset of X that f maps into a single F-class.*

It is easily checked that if $F \sim_B F'$, then E is F-ergodic iff E is F'-ergodic, and that if E is F-ergodic and $F' \leq_B F$, then E is F'-ergodic. In light of Proposition 4.1.1, ergodicity is equivalent to $\Delta(Y)$-ergodicity, where $\Delta(Y)$ is the identity relation on any uncountable standard Borel space Y. We shall be especially concerned with E_0-ergodicity, where again E_0 denotes the *Vitali equivalence relation* defined on $2^{\mathbb{N}}$ by $\alpha\, E_0\, \beta$ iff $\alpha(n) = \beta(n)$ for all but finitely many n. Evidently if X is a standard Borel G-space with invariant Borel measure μ and E_G^X is E_0-ergodic, then the action of G on X is ergodic.

When G is countable, E_0-ergodicity is related to a representation-theoretic condition that will help to shed further light on the Kazhdan property. For Γ a countable group, suppose X is a standard Borel Γ-space with invariant probability measure μ, and again consider the left regular unitary representation of Γ on $L^2(X, \mu)$, restricted to the orthogonal complement $L_0^2(X, \mu)$ of the space of constant functions. Then as noted in Thomas [52, 5.3], if the action of Γ on $L_0^2(X, \mu)$ does not admit almost invariant vectors, then the action of Γ on (X, μ) is E_0-ergodic. It follows immediately that every ergodic action of a countable Kazhdan group Γ on a standard Borel probabilty Γ-space (X, μ) is actually E_0-ergodic.

Finally, we shall require one more strengthening of ergodicity. To say that the action of G on (X, μ) is ergodic is to say that (X, μ) has no nontrivial G-invariant sets. If we strengthen this and ask that (X, μ) have no nontrivial "almost" G-invariant sets, then we arrive at a notion that is closely related to E_0-ergodicity. Specifically, we make the following definition.

Definition 4.2.4. *If G is a countable discrete group and if X is a standard Borel G-space with G-invariant probability measure μ, then the action of G on (X, μ) has* almost invariant sets *if there is a sequence $\{A_n\}$ of Borel subsets of X with measures bounded away from 0 and 1 such that for all $g \in G$,*

$$\lim_{n \to \infty} \mu(g \cdot A_n \triangle A_n) = 0.$$

We shall use the following result of Jones and Schmidt [27] to prove that Bernoulli actions by non-amenable groups with possibly infinite state space are E_0-ergodic. For a clear discussion and proof of the following result, see Hjorth-Kechris [23, A2.2].

Proposition 4.2.5 ([27])**.** *Suppose G is a countable discrete group and X is a standard Borel G-space with G-invariant probability measure μ. If $G \curvearrowright (X, \mu)$ is ergodic, then E_G^X is E_0-ergodic if and only if $G \curvearrowright (X, \mu)$ does not have almost invariant sets.* $\square$

4.3 Isomorphism, factor, and quotient

In this section, we state what it means for a pair of Borel, measure-preserving group actions on standard Borel spaces to be isomorphic, and then we define the notions of *factor* and *quotient* of such an action. We take some care to state these definitions precisely, as there is some slight variation in the meaning of these terms across the literature, depending on context. Of course, the following definitions are closely related to those of homomorphism and isomorphism of permutation groups stated in the previous section, but here we emphasize the topological and measurable structures of the groups and spaces involved, whereas we tend to use the term 'permutation group' more loosely to refer simply to a G-set without emphasizing the additional structure.

Definition 4.3.1. *For $i = 1, 2$, suppose that G_i is a locally compact Polish group, and X_i a standard Borel G_i-space with G_i-invariant Borel measure μ_i. Then (X_1, G_1, μ_1) and (X_2, G_2, μ_2) are* isomorphic *if there exist*

- *μ_i-conull, G_i-invariant Borel subsets $Y_i \subseteq X_i$,*
- *a Borel group isomorphism $\varphi : G_1 \to G_2$, and*

- *a Borel isomorphism* $f : Y_1 \to Y_2$

such that

- $f_*\mu_1 = \mu_2$, *and*
- *for all* $g \in G_1$ *and for all* $x \in Y_1$, $f(g \cdot x) = \varphi(g) \cdot f(x)$.

Dropping the requirement that f is injective leads to the notion of a *factor*, and dropping additionally the requirement that φ is injective yields that of a *quotient*.

Definition 4.3.2. *For* $i = 1, 2$, *suppose that* G_i *is a locally compact Polish group, and* X_i *a standard Borel* G_i*-space with* G_i*-invariant Borel measure* μ_i*. Then* (X_2, G_2, μ_2) *is a* quotient *of* (X_1, G_1, μ_1) *if there exist*

- *a surjective Borel group homomorphism* $\varphi : G_1 \to G_2$, *and*
- *a Borel function* $f : X_1 \to X_2$

such that

- $f_*\mu_1 = \mu_2$, *and*
- *for all* $g \in G$ *and for* μ_1*-a.e.* $x \in X_1$, $f(g \cdot x) = \varphi(g) \cdot f(x)$.

If in addition φ *is an isomorphism, then we call* (X_2, G_2, μ_2) *a* factor *of* (X_1, G_1, μ_1).

Finally, in the proof of theorem 2.2.1 we shall require the following notion of *virtual isomorphism.* Recall here that if μ is a probability measure on the standard Borel space X, and if Y is a Borel subset of X such that $\mu(Y) > 0$, then μ_Y is the probability measure defined on Y by

$$\mu_Y(B) = \mu(B)/\mu(Y) \quad \text{for all Borel } B \subseteq Y.$$

Definition 4.3.3. *For* $i = 1, 2$, *suppose that* G_i *is a countable discrete group, and* X_i *a standard Borel* G_i*-space with* G_i*-invariant, ergodic Borel measure* μ_i*. Then* (X_1, G_1, μ_1) *and* (X_2, G_2, μ_2) *are* virtually isomorphic *if there exist*

- *finite index subgroups* $H_1 \leq G$ *and* $H_2 \leq G_2$*; and*

- *ergodic components $Y_1 \subseteq X_1$ and $Y_2 \subseteq X_2$ for the actions of H_1 and H_2 on X_1 and X_2, respectively*

such that $(Y_1, H_1, (\mu_1)_{Y_1})$ and $(X_1, H_2, (\mu_2)_{Y_2})$ are isomorphic.

4.4 Bernoulli actions

In this section, we recall basic definitions and facts about Bernoulli actions. Let (Y, ν) denote a nontrivial (and possibly countable or finite) standard Borel probability space. By "nontrivial" we mean simply that the Borel probability measure ν does not concentrate on a single point.

If Γ is any countable (discrete) group, we denote by $(Y, \nu)^\Gamma = (Y^\Gamma, \nu^\Gamma)$ the product of identical copies of (Y, ν) indexed by Γ, so that Y^Γ is the space of functions $x : \Gamma \to Y$ which we think of as sequences (x_γ) in Y, and ν^Γ is the product measure on the product σ-algebra of Y^Γ. We may then define a left action of Γ on $(Y, \nu)^\Gamma$ by

$$(\gamma_0 \cdot x)(\gamma) = x(\gamma_0^{-1}\gamma) \quad \text{ for } \gamma_0 \in \Gamma,\ x \in Y^\Gamma.$$

We call any such action a *Bernoulli action* of Γ. If $\Gamma \cong \mathbb{Z}$, then we call $\Gamma \curvearrowright (Y, \nu)^\Gamma$ a *Bernoulli automorphism.* As a special case we have $(Y, \nu) = \{0, 1\}$ with the coin-tossing measure; here the Bernoulli action $\Gamma \curvearrowright 2^\Gamma$ is the classical left shift action of Γ on its powerset. We often call a Borel action of a countable discrete group Γ on a standard Borel probability space *Bernoulli* if it is isomorphic (in the sense of Definition 4.3.1) to a Bernoulli action of Γ.

More generally, if I is any countable discrete Γ-set, then writing $\alpha : \Gamma \times I \to I$ for the Γ-action on I, we may define an action of Γ on $(Y, \nu)^I$ by

$$(g_0 \cdot x)(i) = x(\alpha(g^{-1}, i)).$$

We call any such action $\Gamma \curvearrowright (Y, \nu)^I$ a *generalized Bernoulli action* of Γ.

We now state several well-known facts about Bernoulli actions that will be needed below in the proof of Theorem 2.3.1.

First, it is well known that if $\Gamma \curvearrowright (Y, \nu)^\Gamma$ is a Bernoulli action, then the subset of Y^Γ on which Γ acts freely is Γ-invariant and ν^Γ-conull (for instance, see [31, 2.4]). For any Bernoulli action $\Gamma \curvearrowright (Y,\nu)^\Gamma$, we denote this subset of Y^Γ by $(Y)^\Gamma$. Frequently we will write $X = (Y)^\Gamma$ and $\mu = \nu^\Gamma \restriction X$, and by an abuse of terminology speak of the "free Bernoulli action" $\Gamma \curvearrowright (X,\mu)$.

Second, it is well-known that Bernoulli actions are strongly mixing. In fact, if $\Gamma \curvearrowright (Y,\nu)^I$ is a generalized Bernoulli action (with Γ, Y, ν, and I as above), then $\Gamma \curvearrowright (Y,\nu)^I$ is strongly mixing whenever Γ acts freely on I (for instance, see [31, 2.3]). In particular, free Bernoulli actions $\Gamma \curvearrowright (Y)^\Gamma$ are strongly mixing.

Third, we shall need the following result of Ornstein's, characterizing factors of Bernoulli automorphisms. Here we mean 'factor' in the sense of Definition 4.3.2. (See Ornstein [39], Walters [55, 4.29(ii)], or additionally the Corollary and discussion following Theorem 4.6 in Chapter I.3 of [7] for details concerning this result).

Theorem 4.4.1 (Ornstein)**.** *Let (Y,ν) be a nontrivial standard Borel probability space, and $\mathbb{Z} \curvearrowright (Y,\nu)^\mathbb{Z}$ a Bernoulli automorphism. Then any factor of $\mathbb{Z} \curvearrowright (Y,\nu)^\mathbb{Z}$ is also Bernoulli, i.e., is isomorphic to a Bernoulli automorphism.* □

Next we shall need the following easy result, which gives a condition for a generalized Bernoulli action to be Bernoulli.

Proposition 4.4.2. *Suppose Γ is a countable discrete group acting on a countable discrete index set I, and suppose $\Gamma \curvearrowright (Y,\nu)^I$ is the generalized Bernoulli Γ-action on $(Y,\nu)^I$. If the action of Γ on I is free, then $\Gamma \curvearrowright (Y,\nu)^I$ is isomorphic to the Bernoulli action $\Gamma \curvearrowright ((Y,\nu)^{I/\Gamma})^\Gamma$. In particular, $\Gamma \curvearrowright (Y,\nu)^I$ is Bernoulli.*

Proof. Let $T \subseteq I$ be a transversal for the orbit equivalence relation of the Γ-action on I, and identify T with I/Γ. Define a function

$$f : Y^I \to (Y^{I/\Gamma})^\Gamma \qquad \text{by} \qquad f(x)(\gamma)(t) = x(\gamma t).$$

It is easily checked that f gives the desired isomorphism. □

Finally, we shall need the fact that orbit equivalence relations arising from Bernoulli actions of nonamenable groups are E_0-ergodic. This is already well-known in the case of Bernoulli shifts $\Gamma \curvearrowright 2^\Gamma$ (for instance, see [23, A4]). Our proof in the case of a more general state space makes use of the notion of *almost invariant sets* mentioned at the end of Section 4.2. As we shall need this result at an important point in the middle of the proof of theorem 2.3.1, we postpone its statement until the relevant point in the proof of that theorem in Chapter 7, at which point its significance will be more clear. It is stated and proved below as Lemma 7.2.1.

4.5 Entropy

In this section, we will use the notion of the *Kolmogorov-Sinai entropy* of a dynamical system to show, loosely speaking, that in many cases Bernoulli systems do not have linear algebraic factors. This notion of entropy was initially defined for discrete-time measure preserving dynamical systems in [33]. For convenience we recall the definition and some basic facts below.

Let $(X, \mathcal{B}, \mu)$ be a standard Borel probability space, with $T : X \to X$ a Borel, measure-preserving transformation of X, so that $\mu(T^{-1}(A)) = \mu(A)$ for all $A \in \mathcal{B}$. Given a finite partition $\xi = \{C_1, \ldots, C_n\}$ of X into Borel subsets, define the *entropy of the partition* ξ to be the number

$$H(\xi) = -\sum_{C_i \in \xi} \mu(C_i) \log \mu(C_i),$$

where $\mu(C_i) \log \mu(C_i) = 0$ for C_i null. Next define the *entropy of the dynamical system* $(X, \mathcal{B}, \mu, T)$ *with respect to the partition* ξ to be

$$h(T, \xi) = \lim_{n \to \infty} \frac{1}{n} H(\xi \vee T^{-1}\xi \vee \cdots \vee T^{-n+1}\xi),$$

where the join $\vee_i \xi_i = \xi_1 \vee \cdots \vee \xi_n$ of a finite sequence of finite partitions ξ_i is simply the collection of all intersections of sets in $\bigcup_i \xi_i$. (For the fact that this limit exists, see [55, 4.9.1]). Finally, define the *Kolmogorov-Sinai entropy* of $(X, \mathcal{B}, \mu, T)$ to be

$$h(T) = \sup_{\xi} h(T, \xi),$$

where the supremem is taken over all finite partitions ξ of X into Borel sets. This supremum may be infinite, as we shall see momentarily.

Notice that if T is invertible with measurable inverse, then T induces a $\mathbb{Z}$-action on X defined by $n \cdot x = T^n(x)$. Generalizing this, we may view a measurable, measure-preserving $\mathbb{Z}^d$-action on a measure space $(X, \mathcal{M}, \mu)$ as a multi-parameter discrete-time dynamical system, and attempt to extend the definition of entropy to this setting. (See [45] for a survey of the rich theory of the dynamics of algebraic $\mathbb{Z}^d$-actions). In fact, it turns out that one may generalize Kolmogorov-Sinai entropy to measurable, measure-preserving actions of a large class of amenable groups; for the most general results see Ornstein and Weiss [41]. For clear accounts of the entropy of one-parameter dynamical systems see [7, I.3] or [55, Chapter 4].

As examples we consider various Bernoulli automorphisms $\mathbb{Z} \curvearrowright (Y, \nu)^{\mathbb{Z}}$. If the state space $Y = \{y_1, \ldots, y_n\}$ is finite with $\nu(y_i) = p_i$ for each $1 \leq i \leq n$, then by [55, 4.26], the entropy of the Bernoulli automorphism $\mathbb{Z} \curvearrowright \{y_1, \ldots y_n\}^{\mathbb{Z}}$ is

$$-\sum_{i=1}^{n} p_i \cdot \log p_i.$$

On the other hand if $Y = [0, 1]$ is the unit interval with ν Lebesgue measure, then the entropy of $\mathbb{Z} \curvearrowright [0, 1]^{\mathbb{Z}}$ is infinite [55, Example 8 of Section 4.7]. By Walters [55, 4.14.1] and the fact that Bernoulli actions are ergodic, the entropy of the Bernoulli automorphism $\mathbb{Z} \curvearrowright (Y, \nu)^{\mathbb{Z}}$ is nonzero whenever (Y, ν) is nontrivial, a fact that we shall use below in the proof of Lemma 4.5.2.

It is easy to check that if the dynamical systems $(X, \mathcal{B}, \mu, T)$ and $(Y, \mathcal{N}, \nu, S)$ are isomorphic, then they have the same entropy (for instance, see [55, 4.11]). What is quite remarkable, however, is Ornstein's celebrated result that in the context of Bernoulli actions, entropy is a *complete* isomorphism invariant. Again, the most general results can be found in Ornstein and Weiss [41], but we shall need only the following (see, for instance, [55, 4.28]):

Proposition 4.5.1 (Ornstein). *For $i = 1, 2$, let (Y_i, ν_i) be nontrivial standard Borel probability spaces, and $\mathbb{Z} \curvearrowright (Y_i, \nu_i)^{\mathbb{Z}}$ Bernoulli automorphisms. Then $(Y_i^{\mathbb{Z}}, \mathbb{Z}, \nu_i^{\mathbb{Z}})$ are isomorphic if and only if they have the same entropy.* $\square$

We are now ready to establish a result that will play a key role in the proof of Theorem 2.3.1.

Lemma 4.5.2. *Suppose Γ is a countable discrete group. Let (Y, ν) be a nontrivial standard Borel probability space, and I a countable discrete index set on which Γ acts freely. Let $\Gamma \curvearrowright (Y, \nu)^I$ be the generalized Bernoulli Γ-action. Suppose $(Z, \varphi(\Gamma), \zeta)$ is a nontrivial factor of $\Gamma \curvearrowright (Y, \nu)^I$. If $\gamma \in \Gamma$ has infinite order, then $\varphi(\gamma)$ acts on (Z, ζ) with nonzero Kolmogorov-Sinai entropy.*

Proof. Suppose $\varphi : \Gamma \to \Lambda$ is a Borel group isomorphism, with (Z, ζ) a nontrivial standard Borel Λ-space, and let $F : Y^I \to Z$ be a Borel function such that

(1) $F_* \nu^I = \zeta$, and

(2) for all $\gamma \in \Gamma$, $F(\gamma x) = \varphi(\gamma) F(x)$ for ν^I-a.e. $x \in Y^I$.

Suppose $\gamma \in \Gamma$ has infinite order, and consider the restriction of the generalized Bernoulli Γ-action on $(Y, \nu)^I$ to the cyclic subgroup $\mathbb{Z} \cong \langle \gamma \rangle \leq \Gamma$. By Proposition 4.4.2, this restricted action

$$\langle \gamma \rangle \curvearrowright (Y, \nu)^I$$

is isomorphic to the Bernoulli $\langle \gamma \rangle$-action

$$\langle \gamma \rangle \curvearrowright \left((Y, \nu)^{I / \langle \gamma \rangle} \right)^{\langle \gamma \rangle}.$$

Pushing forward through (F, φ) and using Proposition 4.4.1, we have that

$$\langle \varphi(\gamma) \rangle \curvearrowright (Z, \zeta)$$

is Bernoulli. That is, $\langle \varphi(\gamma) \rangle \curvearrowright (Z, \zeta)$ is isomorphic to a Bernoulli automorphism, say $\mathbb{Z} \curvearrowright (Z_0, \zeta_0)^{\mathbb{Z}}$, where (Z_0, ζ_0) must be nontrivial since (Z, ζ) is nontrivial. But this implies that the Kolmogorov-Sinai entropy of $\mathbb{Z} \curvearrowright (Z_0, \zeta_0)^{\mathbb{Z}}$ is nonzero, and as entropy is an isomorphism invariant for Bernoulli actions, it follows that the Kolmogorov-Sinai entropy of

$$\langle \varphi(\gamma) \rangle \curvearrowright (Z, \zeta)$$

is nonzero. This completes the proof of the lemma. $\square$

Chapter 5

Superrigidity

We begin in this chapter with a brief general discussion of Borel and orbit equivalence superrigidity, and then in Section 4 we state the particular cocycle superrigidity theorem, due to Zimmer, upon which our proof of Theorem 2.4.1 is based. We will introduce in Section 2 the notion of a Borel cocycle, and in Section 3 that of the space induced from an action of a lattice. Each of these notions is crucial for understanding and applying Zimmer's theorem. Finally, in Section 5, we will briefly discuss our strategy for proving Theorem 2.4.1, and then we will apply the techniques of Thomas [52] to establish that the actions appearing in the statement of 2.4.1 fit into the context of Zimmer's theorem.

5.1 Orbit equivalence and Borel reducibility

Recall that by the representation theorem of Feldman and Moore, every countable Borel equivalence relation on a standard Borel space arises as the orbit equivalence relation of a Borel action of a suitably chosen countable group. Let E be a countable Borel equivalence relation on the standard Borel space X, so that $E = E_\Gamma^X$ for some countable group Γ and Borel action $\Gamma \curvearrowright X$. In Chapter 1, we noted that in certain very special cases, the Borel complexity of E partially determines the group Γ and its action on X, a phenomenon which we referred to as *Borel superrigidity.* Similar rigidity phenomena arise in a number of different contexts, and the term "superrigidity" has come to have various related meanings including but by no means limited to the following:

(1) (Mostow-Margulis Superrigidity [58, 5.1.1 and 5.1.2]): an irreducible lattice in a semisimple Lie group determines, from its structure as an abstract group alone, the ambient Lie group in which it is a lattice.

(2) (Zimmer Orbit-Equivalence Superrigidity [58, 5.2.1]): for suitable Lie groups G, the orbit equivalence relation arising from a free, measure-preserving ergodic G-action fully determines G and its action from among the collection of actions of such Lie groups.

(3) (Furman Orbit-Equivalence Superrigidity [16]): for a lattice Γ in a suitable Lie group G, the orbit equivalence relation arising from a measure-preserving ergodic Γ-action determines Γ and its action up to a finite error.

(4) (Thomas Borel Superrigidity [51]): for suitable Lie groups G, the Borel complexity of the orbit equivalence relation E_Γ^X arising from a suitable action of the lattice $\Gamma \leq G$ determines Γ and its action from among the collection of actions of the various lattices in G, up to a finite error.

For Γ and Λ countable groups, suppose (X, μ) is a standard Borel Γ-space with an ergodic Γ-invariant probability measure μ, and suppose (Y, ν) is a standard Borel Λ-space with an ergodic Λ-invariant probability measure ν. Recall that the pair of orbit equivalence relations E_Γ^X and E_Λ^Y are said to be *orbit equivalent* if there exist conull subsets $X_0 \subseteq X$, $Y_0 \subseteq Y$ and a measure-space isomorphism $f : X_0 \to Y_0$ such that for all $x, x' \in X_0$,

$$\Gamma \cdot x = \Gamma \cdot x' \iff \Lambda \cdot f(x) = \Lambda \cdot f(x').$$

If the group Γ and its measure-preserving, ergodic action on X are to some extent determined by the orbit equivalence relation E_Γ^X and the measure μ, then informally we refer to this phenomenon as "orbit equivalence superrigidity."

Borel superrigidity and orbit equivalence superrigidity are often closely related. In particular, it is sometimes possible to reformulate the proof of an orbit equivalence superrigidity result so as to obtain a purely Borel version. This was the strategy of Adams and Kechris in [3] and of Thomas in [51], whose Borel superrigidity theorems are related to the orbit equivalence results of Zimmer [58] and Furman [16], respectively. It should be pointed out, however, that there are important differences between the orbit equivalence setting and the Borel setting. In the context of orbit equivalence, one is free to ignore null sets but must always deal with group actions and mappings that preserve the given measure structure. In the Borel setting, a reduction (or homomorphism) need not preserve a measure, and one cannot ignore null sets since the condition

$$x \, E \, x' \quad \Longleftrightarrow \quad f(x) \, F \, f(x')$$

for a reduction must hold everywhere. In fact, the complexity of a Borel equivalence relation often concentrates on a null set. Furthermore, there is rarely any reason to expect an orbit equivalence to be a Borel reduction, or vice versa.

In either context, of course, the hope is that groups which are in some sense incompatible with each other should give rise to incompatible orbit equivalence relations. It should be emphasized that strong hypotheses must be imposed upon the groups and their actions if there is to be any hope of this happening. For example, if Γ is a countable amenable group and if (X, μ) is a standard Borel Γ-space with invariant ergodic measure μ, then by Dye's theorem the orbit space E_Γ^X "remembers" nothing about Γ beyond its amenability. (In [11], [12] Dye proved that any two ergodic $\mathbb{Z}$-spaces are orbit equivalent; Ornstein and Weiss later extended this result to arbitrary countable amenable groups [40], [8]).

Each of the four examples of superrigidity mentioned above comes from the theory of discrete subgroups of semisimple Lie groups. The Borel superrigidity theorem stated as Theorem 1.6.1 is based on results of Popa in the theory of Von Neumann algebras. In both of these cases, the superrigidity results can be traced to a suitable *cocycle superrigidity theorem.* Indeed, as discussed in Popa [43], cocycle superrigidity results underlie a great many of the recent studies in orbit equivalence superrigidity theory. Furthermore, cocycles will play an important role in our proof of Theorem 2.4.1 in Chapter 6. We therefore spend much of the following two sections discussing the important notion of a cocycle.

5.2 Borel cocycles

We motivate the definition of a Borel cocycle by considering the context in which, for our purposes, cocycles shall always arise.

Thus let G and H be locally compact Polish groups, let X be a standard Borel

G-space, and suppose that Y is a free standard Borel H-space. Further suppose that the Borel function $f : X \to Y$ is a homomorphism from E_G^X to E_H^Y. Then for each pair $(g, x) \in G \times X$, there is an element $h \in H$ such that $h \cdot f(x) = f(g \cdot x)$. Moreover, since H acts freely on Y, this element $h \in H$ will be unique. Hence we may define a Borel function

$$\alpha : G \times X \to H$$

so that for all $g \in G$ and for all $x \in X$,

$$\alpha(g, x) = \text{ the unique element } h \in H \text{ such that } h \cdot f(x) = f(g \cdot x).$$

Now let $g_1,\ g_2 \in G$ and $x \in X$. From the fact that $(g_2\, g_1) \cdot x = g_2 \cdot (g_1 \cdot x)$, together with the definition of α, we have that both

$$\alpha(g_2\, g_1,\ x) \cdot f(x) = f(g_2\, g_1 \cdot x),$$

and also

$$\alpha(g_2, g_1 \cdot x) \cdot (\alpha(g_1, x) \cdot f(x)) = f(g_2\, g_1 \cdot x).$$

As H acts freely, this implies that

$$\alpha(g_2\, g_1,\ x) = \alpha(g_2, g_1 \cdot x)\, \alpha(g_1, x).$$

We take this last equality as the defining property of a cocycle.

Definition 5.2.1. *Let G and H be locally compact Polish groups, and suppose (X, μ) is a standard Borel G-space with invariant probability measure μ. A* cocycle *from the G-space (X, μ) into H is a Borel map $\alpha : G \times X \to H$ such that for all $g, h \in G$ and for μ-a.e. $x \in X$,*

$$\alpha(hg, x) = \alpha(h, gx)\, \alpha(g, x).$$

If for each $g, h \in G$ the cocycle identity holds for all $x \in X$, then we call α a strict cocycle*.*

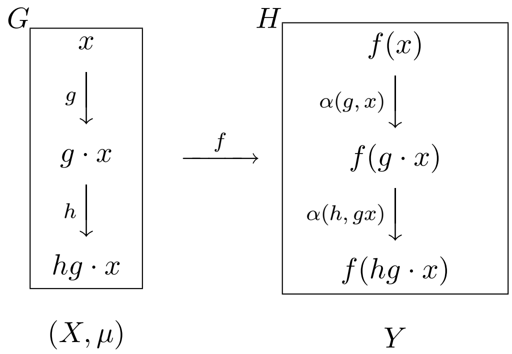
Remark 5.2.2. If G is countable, then the order of the quantifiers in the statement of the cocycle identity may be reversed; ie, one easily checks that the set

$$X_0 = \{x \in X \mid (\forall g, h \in G)\, \alpha(hg,\ x) = \alpha(h,\ gx)\, \alpha(g, x)\}$$

is G-invariant and μ-conull.

Thus whenever we have a standard Borel G-space X, a free standard Borel H-space Y, and a Borel homomorphism $f : X \to Y$ from E_G^X to E_H^Y, we may define the cocycle $\alpha : G \times X \to H$ as above, so that $\alpha(g, x) \cdot f(x) = f(g \cdot x)$ for all $g \in G$ and $x \in X$. In this case we call α the cocycle *corresponding to*, or *arising from*, f. Notice that such a cocycle is always strict. If for each $g \in G$, $\alpha(g, x) \cdot f(x) = f(g \cdot x)$ for μ-a.e. $x \in X$, then we say that α *weakly* corresponds to f. In general a cocycle need not correspond to a Borel homomorphism at all, but all of the cocycles in this thesis will arise in this manner.

The cocycle identity, together with the correspondence of a cocycle with a Borel homomorphism, may be pictured as follows:



Now suppose we have the cocyle $\alpha : G \times X \to Y$ from the standard Borel G-space X into the free standard Borel H-space Y corresponding to the Borel homomorphism $f : X \to Y$ from E_G^X to E_H^Y. If $b : X \to H$ is any Borel function, then we may define a new cocycle $\beta : G \times X \to H$ by

$$\beta(g, x) = b(gx)\, \alpha(g, x)\, b(x)^{-1},$$

and a new Borel function $\tilde{f} : X \to Y$ by

$$\tilde{f}(x) = b(x) \cdot f(x).$$

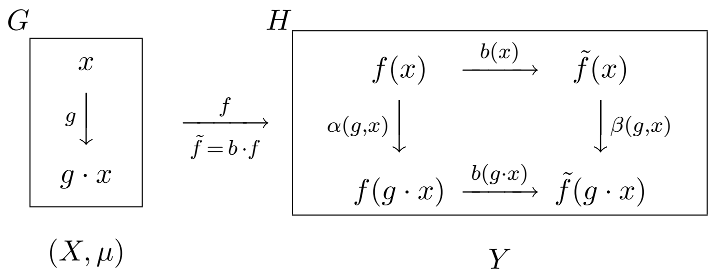
It is easily checked that β is indeed a cocycle and that $\tilde{f}$ is again a Borel homomorphism from E_G^X to E_H^Y, and furthermore that β is the cocycle corresponding to $\tilde{f}$. In this context we call $(\tilde{f}, \beta)$ a "strict adjustment" of (f, α), and note that $\tilde{f}$ is a Borel reduction from E_G^X to E_H^Y if and only if f is. This motivates the following definition, which is easily seen to define an equivalence relation on cocycles $\alpha : G \times X \to H$.

Definition 5.2.3. *Let G and H be locally compact Polish groups, suppose (X, μ) is a standard Borel G-space with invariant probability measure μ, and let $\alpha : G \times X \to H$ and $\beta : G \times X \to H$ be Borel cocycles. Then we say that α is* equivalent *to β if there is a Borel function $b : X \to H$ such that for all $g \in G$ and for μ-a.e. $x \in X$,*

$$\beta(g, x) = b(gx)\, \alpha(g, x)\, b(x)^{-1}.$$

We write $\alpha \sim \beta$ if α is equivalent to β.

If the cocycle α weakly corresponds to the Borel homomorphism $f : X \to Y$ from E_G^X to E_H^Y, and if $\alpha \sim \beta$ via the Borel function $b : X \to H$, then β weakly corresponds to the Borel function $\tilde{f} = b \cdot f$. In this context we call $(\tilde{f}, \beta)$ an *adjustment* of (f, α). The adjustment from (f, α) to $(\tilde{f}, \beta)$ via $b : X \to H$ may be pictured as follows:



Frequently in our arguments in Chapter 6 we will have to adjust a Borel reduction f together with its corresponding cocycle α so as to obtain a simpler pair $(\tilde{f}, \beta)$, where $\beta \sim \alpha$ and β weakly corresponds to $\tilde{f}$. Here, ideally, the goal is to adjust α to a cocycle β that is a function of its first variable only, in which case the cocycle identity reduces to the condition for a group homomorphism. Indeed, suppose there is a group homomorphism $\varphi : G \to H$ such that for all $g \in G$ and for all $x \in X$, $\beta(g, x) = \varphi(g)$. Then if β corresponds to the Borel homomorphism $\tilde{f}$, we will have that

$$\tilde{f}(g \cdot x) = \beta(g, x) \cdot \tilde{f}(x) = \varphi(g) \cdot \tilde{f}(x),$$

so that μ-a.e. $(\tilde{f}, \varphi)$ is a homomorphism of permutation groups. This will be very useful, as it is often far easier to rule out the possibility of a permutation group homomorphism $(G \curvearrowright X) \to (H \curvearrowright Y)$ than it is to rule out the possibility of a Borel reduction $E_G^X \leq_B E_H^Y$.

5.3 Induced spaces, actions, and cocycles

Our proof of Theorem 2.4.1 in Chapter 6 will involve the construction of the G-space *induced* from the action of a lattice $\Gamma \leq G$, where G is a semisimple Lie group. In this section we introduce the notions of induced space, induced action, and induced cocycle, and prove a cocycle reduction lemma that appears in Adams [2, 5.4] and will play a key role in Chapter 6.

Suppose G is a locally compact Polish group with Γ a lattice in G, so that G/Γ admits a G-invariant probability measure, ν. Fix a Borel transversal $T \subseteq G$ for G/Γ containing 1_G, and identify T with G/Γ via the natural identification $t \leftrightarrow t\Gamma$, so that we may view T as a G-space with invariant probability measure ν and natural G-action defined by

$$g.t \ = \ \text{the unique element of } T \text{ in the coset } gt\Gamma.$$

Also define the associated cocycle $\rho : G \times T \to \Gamma$ by

$$\rho(g,t) \ = \ \text{the unique } \gamma \in \Gamma \text{ such that } (g.t)\gamma = gt.$$

Now suppose that X is a standard Borel Γ-space with invariant Borel probability measure μ. We then define *the G-space induced from the action of* Γ *on* X, or more briefly the *induced G-space*, to be

$$(\hat{X}, \hat{\mu}) = (X \times T,\, \mu \times \nu),$$

with the associated *induced action* of G on $(\hat{X}, \hat{\mu})$ defined by

$$g * \langle x, t\rangle = \langle \rho(g,t) \cdot x,\, g.t\rangle.$$

It is easily checked (see [3]) that $\hat{\mu} = \mu \times \nu$ is a G-invariant probability measure on $\hat{X}$ that is ergodic if and only if μ is.

For $t \in T$, set $X_t = \{\langle x, t\rangle \mid x \in X\} \subseteq \hat{X}$. Intuitively, $\hat{X} = X \times T = X \times G/\Gamma$ consists of the various "twisted" copies X_t of X, indexed by the cosets $t\Gamma$. Note that X_t is a $t\Gamma t^{-1}$–invariant copy of X contained in $\hat{X}$, and that for $g = t\gamma t^{-1} \in t\Gamma t^{-1}$, we have

$$g * \langle x, t\rangle = t\gamma t^{-1} * \langle x, t\rangle = \langle \gamma \cdot x, t\rangle = \langle t^{-1}gt \cdot x, t\rangle.$$

Thus (X, Γ) and $(X_t, t\Gamma t^{-1})$ are isomorphic as permutation groups, and two points $x, y \in X$ lie in the same Γ-orbit if and only if the points $\langle x, t\rangle$, $\langle y, t\rangle \in \hat{X}$ lie in the same G-orbit, in which case any element $g \in G$ that sends $\langle x, t\rangle$ to $\langle y, t\rangle$ is in $t\Gamma t^{-1}$.

Now suppose H is another locally compact Polish group with Λ a lattice in H, and suppose (Y, η) is a standard Borel Λ-space with Λ-invariant probability measure η. Then we may form the induced H-space $(\hat{Y}, \hat{\eta})$, as above. Suppose further that $f : X \to Y$ is a Borel homomorphism from E_Γ^X to E_Λ^Y. Then we may define various Borel functions associated to f as follows:

$$
\begin{aligned}
{}^\wedge f : \hat{X} \to Y, &\qquad {}^\wedge f : \langle x, t\rangle \mapsto f(x)\\
f^\wedge : X \to \hat{Y}, &\qquad f^\wedge : x \mapsto \langle f(x), 1\rangle\\
{}^\wedge f^\wedge : \hat{X} \to \hat{Y}, &\qquad {}^\wedge f^\wedge : \langle x, t\rangle \to \langle f(x), 1\rangle
\end{aligned}
$$

It is easily checked that ${}^\wedge f$ is a Borel homomorphism from $E_G^{\hat{X}}$ to E_Λ^Y, that $f^\wedge$ is a Borel homomorphism from E_Γ^X to $E_H^{\hat{Y}}$, and that ${}^\wedge f^\wedge$ is a Borel homomorphism from $E_G^{\hat{X}}$ to $E_H^{\hat{Y}}$. Moreover, the homomorphisms ${}^\wedge f$, $f^\wedge$, and ${}^\wedge f^\wedge$ are reductions if and only if f is a reduction from E_Γ^X to E_Λ^Y.

Now, with G, Γ, (X, μ), and H as above, suppose that $\alpha : \Gamma \times X \to H$ is a *strict* cocycle into H. Then we call the map

$$\hat{\alpha} : G \times \hat{X} \to H$$

defined by

$$\hat{\alpha}(g, \langle x, t\rangle) = \alpha(\rho(g, t), x)$$

the *cocycle induced from* α. Notice that if Y is a free standard Borel H-space and if α corresponds to the Borel homomorphism $f : X \to Y$, then $\hat{\alpha}$ is simply the cocycle corresponding to the Borel homomorphism ${}^\wedge f : \hat{X} \to Y$ defined above. We shall recall these constructions and observations in Chapter 6, where they will appear in the proof of Theorem 2.4.1.

Finally, we conclude this section by proving a cocycle reduction result that we shall need in the the proof of Theorem 2.4.1. Our argument is based on Adams [2, 5.4], and corrects a slight error found there. As usual, and to improve notation, we will sometimes suppress reference to t when denoting elements of $\hat{X}$, and write $\hat{x}$ in place of $\langle x, t\rangle$ when no confusion can arise.

Lemma 5.3.1. *Let G and H be locally compact Polish groups, with Γ a lattice in G. Let (X, μ) be a standard Borel Γ-space with invariant probability measure μ, and let $(\hat{X}, \hat{\mu})$ be the G-spaced induced from the action of Γ on (X, μ). Suppose $\alpha : \Gamma \times X \to H$ is a cocycle into H, and let $\hat{\alpha} : G \times \hat{X} \to H$ be the cocycle induced from α. Suppose there is a group homomorphism $\varphi : G \to H$ such that $\hat{\alpha}$ is equivalent to a cocycle $\hat{\beta} : G \times \hat{X} \to H$ satisfying*

$$\hat{\beta}(g, \hat{x}) = \varphi(g) \text{ for all } g \in G \text{ and for } \hat{\mu}\text{-a.e. } \hat{x} \in \hat{X}.$$

Then α is equivalent to a cocycle $\beta : \Gamma \times X \to H$ satisfying

$$\beta(\gamma, x) = \varphi(\gamma) \text{ for all } \gamma \in \Gamma \text{ and for } \mu\text{-a.e. } x \in X.$$

Proof. Recall that $\hat{\mu} = \mu \times \nu$, where ν is the G-invariant probability measure on G/Γ that we regard as a measure on T via the identification of T with G/Γ. In order to improve notation in the following argument, we use the measure quantifier $\forall^*$ to mean $\hat{\mu}$-a.e., μ-a.e., or ν-a.e. depending on the context.

We begin with the precise statement of the equivalence $\hat{\alpha} \sim \hat{\beta}$. Thus let $\hat{b} : \hat{X} \to H$ be a Borel function such that

$$(\forall g \in G)(\forall^* \hat{x} \in \hat{X}) \;\; \varphi(g) = \hat{b}(g * \hat{x})\, \hat{\alpha}(g, \hat{x})\, \hat{b}(\hat{x})^{-1}.$$

By Fubini-Tonelli, this implies

$$(\forall g \in G)(\forall^* x \in X)(\forall^* t \in T) \;\; \varphi(g) = \hat{b}\big(g * \langle x, t\rangle\big)\, \hat{\alpha}\big(g, \langle x, t\rangle\big)\, \hat{b}\big(\langle x, t\rangle\big)^{-1}.$$

Replacing g with tgt^{-1} and then restricting the first quantifier to range over $\Gamma \leq G$, we obtain

$$(\forall \gamma \in \Gamma)(\forall^* x \in X)(\forall^* t \in T) \;\; \varphi(t\gamma t^{-1}) = \hat{b}\big(t\gamma t^{-1} * \langle x, t\rangle\big)\, \hat{\alpha}\big(t\gamma t^{-1}, \langle x, t\rangle\big)\, \hat{b}\big(\langle x, t\rangle\big)^{-1}.$$

Again applying Fubini-Tonelli (and using the fact that Γ is countable), we may write

$$(\forall^* t \in T)(\forall \gamma \in \Gamma)(\forall^* x \in X) \;\; \varphi(t\gamma t^{-1}) = \hat{b}\big(t\gamma t^{-1} * \langle x, t\rangle\big)\, \hat{\alpha}\big(t\gamma t^{-1}, \langle x, t\rangle\big)\, \hat{b}\big(\langle x, t\rangle\big)^{-1}.$$

Now fix such a $t \in T$ for which the above holds. Then simplifying we have

$$(\forall \gamma \in \Gamma)(\forall^* x \in X) \;\; \varphi(t\gamma t^{-1}) = \hat{b}\big(\langle \gamma x, t\rangle\big)\, \alpha(\gamma, x)\, \hat{b}\big(\langle x, t\rangle\big)^{-1},$$

and hence

$$(\forall \gamma \in \Gamma)(\forall^* x \in X) \;\; \varphi(\gamma) = \varphi(t)^{-1}\hat{b}\big(\langle \gamma x, t\rangle\big)\, \alpha(\gamma, x)\, \hat{b}(\langle x, t\rangle)^{-1}\varphi(t).$$

Now if we define the Borel function $b : X \to H$ by

$$b(x) = \varphi(t)^{-1}\hat{b}(\langle x, t\rangle)$$

for all $x \in X$, then we have for all $\gamma \in \Gamma$ and for μ-a.e. $x \in X$,

$$\varphi(\gamma) = b(\gamma x)\, \alpha(\gamma, x)\, b(x)^{-1},$$

as desired. □

5.4 Zimmer superrigidity

We are now ready to state the special case of Zimmer's Cocycle Superrigidity Theorem upon which our proof of Theorem 2.4.1 is based. The most general statement, together with a proof, may be found in Zimmer [58].

Theorem 5.4.1 (Zimmer [58, 5.2.5])**.** *Suppose G is a connected semisimple algebraic $\mathbb{R}$-group of $\mathbb{R}$-rank at least 2, and assume that $G^0_{\mathbb{R}}$ has no compact factors. Suppose (S, μ) is an irreducible ergodic $G^0_{\mathbb{R}}$-space with invariant probability measure μ. Let H be a simple, connected, algebraic $\mathbb{R}$-group with $H(\mathbb{R})$ not compact, and suppose that $\alpha : G^0_{\mathbb{R}} \times S \to H(\mathbb{R})$ is a cocycle that is not equivalent to a cocycle taking values in a proper algebraic subgroup of $H(\mathbb{R})$. Then there is a rational $\mathbb{R}$-homomorphism $\psi : G \to H$ such that α is equivalent to the cocycle $(g, s) \mapsto \psi(g)$.*

We can now sketch the beginning of the proof of Theorem 2.4.1, up to the application of Zimmer's Theorem. For the remainder of this chapter, including in Section 5 below, we assume the setting of Theorem 2.4.1. Thus let $\mathbb{K}$ and $\mathbb{F}$ be totally real number fields properly extending $\mathbb{Q}$, let $\mathcal{O}_{\mathbb{K}}$ and $\mathcal{O}_{\mathbb{F}}$ be their integer rings, and let $\Lambda_{\mathbb{K}} = PSL_2(\mathcal{O}_{\mathbb{K}})$

and $\Lambda_{\mathbb{F}} = PSL_2(\mathcal{O}_{\mathbb{F}})$. Suppose X is a standard Borel $\Lambda_{\mathbb{K}}$-space with invariant ergodic probability measure μ_1, and let Y be a free standard Borel $\Lambda_{\mathbb{F}}$-space with invariant ergodic probability measure μ_2. Let E_1 and E_2 denote the orbit equivalence relations arising from the actions of $\Lambda_{\mathbb{K}}$ and $\Lambda_{\mathbb{F}}$ on X and Y, respectively, and assume that E_1 is E_0-ergodic. Suppose $f : X \to Y$ is a Borel reduction from E_1 to E_2, and let $\alpha : \Lambda_{\mathbb{K}} \times X \to \Lambda_{\mathbb{F}}$ be the cocycle corresponding to f.

Naïvely, we would like to apply Zimmer's theorem to the cocycle α, which would allow us to adjust the Borel reduction f together with its corresponding cocycle α so as to obtain, μ-a.e., a permutation group homomorphism $(\tilde{f}, \psi)$ from $E^X_{\Lambda_{\mathbb{K}}}$ to $E^Y_{\Lambda_{\mathbb{F}}}$. This would then, hopefully, (and with a little more knowledge about ψ), yield a contradiction due to the incompatibility between $\Lambda_{\mathbb{K}}$ and $\Lambda_{\mathbb{F}}$ unless $\mathbb{K} = \mathbb{F}$.

Of course, there are a number of rather stringent hypotheses on the groups involved that must be satisfied before we can apply Zimmer's theorem. For instance, the domain group of the cocycle should arise as the connected component of the identity (in the Hausdorff topology) of the $\mathbb{R}$-points of a connected, semisimple algebraic $\mathbb{R}$-group of $\mathbb{R}$-rank at least 2. Thus to satisfy this hypothesis we pass from $\Lambda_{\mathbb{K}} \curvearrowright (X, \mu)$ to the induced action $H_{\mathbb{K}} \curvearrowright (\hat{X}, \hat{\mu}_1)$ of the ambient connected group $H_{\mathbb{K}}$ in which $\Lambda_{\mathbb{K}}$ is a lattice, with associated induced cocycle $\hat{\alpha} : H_{\mathbb{K}} \times \hat{X} \to \Lambda_{\mathbb{F}}$. (Recall that $PSL_2(\mathbb{R})$, and hence $H_{\mathbb{K}}$, is connected in the Hausdorff topology). Furthermore, the target group of the cocycle should arise as the $\mathbb{R}$-points of a simple, connected, noncompact algebraic $\mathbb{R}$-group. Thus to satisfy this hypothesis, we first view $\hat{\alpha} : H_{\mathbb{K}} \times \hat{X} \to \Lambda_{\mathbb{F}}$ as taking values in the larger group $H_{\mathbb{F}} \supseteq \Lambda_{\mathbb{F}}$, and then we consider the various projections onto the factors $PSL_2(\mathbb{R})$ in $H_{\mathbb{F}}$. Specifically, for each embedding $\theta_j \in \mathfrak{R}_{\mathbb{F}}$ of $\mathbb{F}$ into $\mathbb{C}$, we define the cocycles

$$\hat{\alpha}_j := \pi^{\mathbb{F}}_j \circ \sigma^{\mathbb{F}} \circ \hat{\alpha} : H_{\mathbb{K}} \times \hat{X} \to PSL_2(\mathbb{R}).$$

This puts us in a position to apply Zimmer's theorem to the cocycles $\hat{\alpha}_j$, provided only that we can show that each of them does not reduce to a cocycle taking values in a proper algebraic subgroup of $PSL_2(\mathbb{R})$.

Verifying this last hypothesis will take some work, and will occupy us in the final section of this chapter below. Hereafter we refer to it as the *cocycle hypothesis* of Zimmer's Theorem. Roughly speaking, and following Thomas [52], we will use the E_0-ergodicity of $E_1 = E^X_{\Lambda_{\mathbb{K}}}$ to prove that this hypothesis of Zimmer's Theorem is satisfied by the cocycles $\hat{\alpha}_j$.

5.5 E_0-ergodicity, 1-amenability, and the cocycle hypothesis

Each of the superrigidity results we have seen thus far makes use of the fact that the acting groups have Kazhdan's Property (T). Moreover, many of the techniques of Adams-Kechris [3] exploit the incompatibility of the actions of Kazhdan groups and amenable groups. Unfortunately, as discussed in Chapter 3, the groups $\Lambda_{\mathbb{K}}$ do not have Property (T). However, as Thomas noticed in [52], the incompatibility of Kazhdan and amenable group actions is to a large extent captured by the notion of E_0-ergodicity. In particular, in this section, we will use E_0-ergodicity to show that the cocycles $\hat{\alpha}_j$ satisfy the cocycle hypothesis of Zimmer's theorem. Our argument is essentially identical to that given in Thomas [52].

Let E be a countable Borel equivalence relation on the standard Borel space X. If ν is any Borel probability measure on X, then E is said to be *ν-hyperfinite* if there exists a ν-conull, E-invariant Borel set $X_0 \subseteq X$ such that $E \restriction X_0$ is hyperfinite. (Here a set $X_0 \subseteq X$ is *E-invariant* if X_0 is a union of E-classes). Recall from Section 3.3 that if Γ is a countable amenable group and if X is a standard Borel Γ-space, then E^X_Γ is ν-hyperfinite for every probability measure ν on X. In order to show that the cocycles $\hat{\alpha}_j$ satisfy the cocycle hypothesis of Zimmer's theorem, we shall need a strengthening of this fact that is due to Kechris.

In order to state this strengthening, we will first need to define the notion of *1-amenability*.

Definition 5.5.1 ([26]). *Suppose that E is a countable Borel equivalence relation on the standard Borel space X. Then E is said to be* 1-amenable *if there exists a sequence*

$$\langle f_n : E \to \mathbb{R}^+ \mid n \in \mathbb{N} \rangle$$

of non-negative Borel functions such that, letting $f_n^x(y) = f_n(x,y)$, the following conditions hold for all $x, y \in X$:

(1) $f_n^x \in \ell_1([x]_E)$ *and* $\|f_n^x\|_1 = 1$*;*

(2) if $x\, E\, y$*, then* $\|f_n^x - f_n^y\|_1 \to 0$ *as* $n \to \infty$*.*

The following two lemmas, together with their proofs, appear as Lemmas 4.5 and 4.6 respectively in Thomas [52]. As pointed out by Thomas, it remains open whether every 1-amenable equivalence relation is hyperfinite.

Lemma 5.5.2 (Jackson-Kechris-Louveau [26])**.**

(1) If E is hyperfinite, then E is 1-amenable.

(2) If H is a countable (discrete) amenable group and X is a standard Borel H-space, then E_H^X is 1-amenable.

(3) If E is a 1-amenable equivalence relation on the standard Borel space X, then E is ν-hyperfinite for every probability measure ν on X. □

Lemma 5.5.3 (Thomas [52, 4.6])**.** *Let E, F be countable Borel equivalence relations on the standard Borel spaces X, Y respectively and let μ be an E-invariant probability measure on X. Suppose that (E, μ) is E_0-ergodic and that F is 1-amenable. Then (E, μ) is F-ergodic.* □

If E is a Borel equivalence relation on the standard Borel space X, then a *complete countable section* for E is a Borel set $Z \subseteq X$ such that for each $x \in X$, the intersection $Z \cap [x]_E$ is nonempty and countable. By Kechris [29], if G is a locally compact Polish group and X is a standard Borel G-space, then E_G^X admits a complete countable section $Z \subseteq X$ such that $E_G^X \sim_B E_G^X \restriction Z$. In particular, E_G^X is essentially countable. We are now ready to state Kechris's result, which is proved as Theorem 4.7 in Thomas [52].

Theorem 5.5.4 (Kechris, cf [52, 4.7])**.** *Suppose that H is an amenable locally compact Polish group and that X is a standard Borel H-space. Let X_0 be a complete countable Borel section for the corresponding orbit equivalence relation E_H^X. Then the countable Borel equivalence relation $E = E_H^X \restriction X_0$ is 1-amenable.* □

It follows from Lemma 5.5.3 and Theorem 5.5.4 that if E is an E_0-ergodic countable Borel equivalence relation and F is an orbit equivalence relation arising from a Borel action of an amenable locally compact Polish group, then E is F-ergodic.

We will now use 5.5.2, 5.5.3, and 5.5.4 to show that the cocycles $\hat{\alpha}_j$ satisfy the cocycle hypothesis of Zimmer's theorem. Recall that we are assuming here, as in the previous section, the notation and hypotheses of Theorem 2.4.1. In particular, assume that (X, μ_1), (Y, μ_2), $\Lambda_{\mathbb{K}} \leq H_{\mathbb{K}}$, $\Lambda_{\mathbb{F}} \leq H_{\mathbb{F}}$, $E_1 = E^X_{\Lambda_{\mathbb{K}}}$, $E_2 = E^Y_{\Lambda_{\mathbb{F}}}$, $f : X \to Y$, and $\alpha : \Lambda_{\mathbb{K}} \times X \to \Lambda_{\mathbb{F}}$ are as in the statement of Theorem 2.4.1, and assume that E_1 is E_0-ergodic. Our argument follows exactly that given in Thomas [52].

Lemma 5.5.5. *α is not equivalent to a cocycle taking values in an amenable subgroup of $\Lambda_{\mathbb{F}}$.*

Proof. Suppose for contradiction that α is equivalent to a cocycle β taking values in an amenable subgroup $A \leq \Lambda_{\mathbb{F}}$. Then there exists a Borel function $b : X \to \Lambda_{\mathbb{F}}$ and a $\Lambda_{\mathbb{K}}$-invariant, μ_1-conull Borel subset $X_0 \subseteq X$ such that for all $\lambda \in \Lambda_{\mathbb{K}}$ and for all $x \in X_0$,

$$\beta(\lambda, x) = b(\lambda x)\, \alpha(\lambda, x)\, b(x)^{-1}.$$

Notice that since E_1 is E_0-ergodic, it follows that $E_1 \upharpoonright X_0$ is also E_0-ergodic.

Let $f' = b \cdot f : X \to Y$ be the adjusted Borel reduction to which β (weakly) corresponds. Note that since f' is a reduction from E_1 to E_2, there does not exist a μ_1-conull subset of X that f' maps into a single E_2-class. Also note that for all $\lambda \in \Lambda_{\mathbb{K}}$ and for all $x \in X_0$, we have

$$\beta(\lambda, x) \cdot f'(x) = f'(\lambda \cdot x).$$

As β takes values in A, it follows that $f' \upharpoonright X_0$ is a Borel reduction from $E_1 \upharpoonright X_0$ to the orbit equivalence relation F induced by the action of the amenable group A on Y. By Lemma 5.5.2, F is 1-amenable. Since $E_1 \upharpoonright X_0$ is E_0-ergodic, Lemma 5.5.3 implies that $E_1 \upharpoonright X_0$ is also F-ergodic, and hence there is a μ_1-conull Borel subset $X_1 \subseteq X_0$ such that f' maps X_1 into a single F-class. Since $F \subseteq E_2$, this contradicts the fact that there does not exist a μ_1-conull subset of X that f' maps into a single E_2-class. $\square$

Now let $(\hat{X}, \hat{\mu})$ be the $H_{\mathbb{K}}$-space induced from the action of $\Lambda_{\mathbb{K}}$ on (X, μ), and let $\hat{\alpha} : H_{\mathbb{K}} \times \hat{X} \to \Lambda_{\mathbb{F}}$ be the corresponding cocycle induced from α. Fix an enumeration $\langle \theta_1, \dots, \theta_n \rangle$ of the set $\mathfrak{R}_{\mathbb{F}}$ of embeddings of $\mathbb{F}$ into $\mathbb{R}$, and for each $\theta_j \in \mathfrak{R}_{\mathbb{F}}$, let $\pi_j^{\mathbb{F}} : H_{\mathbb{F}} \to PSL_2(\mathbb{R})$ be the canonical projection onto the factor in $H_{\mathbb{F}}$ corresponding to θ_j. Then for each j, define the cocycle

$$\hat{\alpha}_j := \pi_j^{\mathbb{F}} \circ \sigma^{\mathbb{F}} \circ \hat{\alpha} : H_{\mathbb{K}} \times \hat{X} \to PSL_2(\mathbb{R}).$$

We are now finally ready to prove that the cocycles $\hat{\alpha}_j$ satisfy the hypothesis in Zimmer's theorem. Our proof follows exactly the proof of Thomas [52, 8.5], but we include it here for completeness.

Lemma 5.5.6. *For each j, the cocycle $\hat{\alpha}_j$ is not equivalent to a cocycle taking values in a proper algebraic subgroup of $PSL_2(\mathbb{R})$.*

Proof. Fix $\theta_j \in \mathfrak{R}_{\mathbb{F}}$, and suppose for contradiction that $\hat{\alpha}_j$ is equivalent to a cocycle taking values in a proper algebraic $\mathbb{R}$-subgroup $A(\mathbb{R})$ of $PSL_2(\mathbb{R})$. Let

$$H_0 = A(\mathbb{R}) \times \prod_{\theta_k \in \mathfrak{R}_{\mathbb{F}},\, k \neq j} PSL_2(\mathbb{R}) \leq H_{\mathbb{F}}.$$

Then $\sigma^{\mathbb{F}} \circ \hat{\alpha}$ is equivalent to a cocycle taking values in the closed subgroup H_0 of $H_{\mathbb{F}}$. By Adams-Kechris [3, 2.3], there exists a cocycle

$$\alpha_1 : \Lambda_{\mathbb{K}} \times X \to H_{\mathbb{F}}$$

such that $\sigma^{\mathbb{F}} \circ \alpha$ is equivalent to α_1 and α_1 takes values in H_0.

Claim 5.5.7. *There exists an element $h \in H_{\mathbb{F}}$ and a cocycle*

$$\alpha_2 : \Lambda_{\mathbb{K}} \times X \to \Lambda_{\mathbb{F}}$$

such that $\alpha_2 \sim \alpha$ and α_2 takes values in the subgroup $\Lambda_{\mathbb{F}} \cap hH_0h^{-1}$.

Assuming that 5.5.7 holds, we can complete the proof of Lemma 5.5.6 as follows. Let $\Lambda_0 = \Lambda_{\mathbb{F}} \cap hH_0h^{-1}$. Let h_j be the coordinate of h corresponding to θ_j. Then $\lambda^{\sigma_j^{\mathbb{F}}} \in h_j A(\mathbb{R}) h_j^{-1}$ for all $\lambda \in \Lambda_0$, whence $\lambda \mapsto h_j^{-1} \lambda^{\sigma_j^{\mathbb{F}}} h_j$ is an isomorphism of Λ_0 onto a subgroup of $A(\mathbb{R})$. Since $A(\mathbb{R})$ is a proper algebraic $\mathbb{R}$-subgroup of $PSL_2(\mathbb{R})$, $A(\mathbb{R})$

is solvable-by-finite. This implies that Λ_0 is also solvable-by-finite. But then Λ_0 is an amenable subgroup of $\Lambda_{\mathbb{F}}$, contradicting Lemma 5.5.5.

Thus it only remains to prove Claim 5.5.7 (this proof is closely based on the proof of Adams-Kechris [3, 2.4]). We begin by showing that the orbit equivalence relation E arising from the action of $\Lambda_{\mathbb{F}}$ on $H_{\mathbb{F}}/H_0$ is 1-amenable. To see this, first note that by the proof of [3, 2.4], E is Borel bireducible with the the orbit equivalence relation F arising from the action of H_0 on $H_{\mathbb{F}}/\Lambda_{\mathbb{F}}$. Furthermore, F is Borel bireducible with the orbit equivalence relation F' arising from the action of $A(\mathbb{R})$ on the complete Borel section

$$Z = \{\rho_j(x)\Lambda_{\mathbb{F}} \mid x \in PSL_2(\mathbb{R})\},$$

where $\rho_j : PSL_2(\mathbb{R}) \to H_{\mathbb{F}}$ is the canonical embedding, and where $\Lambda_{\mathbb{F}}$ has been identified with its image under $\sigma^{\mathbb{F}}$ inside $H_{\mathbb{F}}$. Let Z_0 be a complete countable Borel section for F' (cf [29]). Since $A(\mathbb{R})$ is a amenable locally compact Polish group, Theorem 5.5.4 implies that $F' \restriction Z_0$ is 1-amenable. By Jackson-Kechris-Louveau [26, 2.15], since $E \sim_B F' \restriction Z_0$, it follows that E is also 1-amenable.

Now, by Adams-Kechris [3, 2.1], since $\sigma^{\mathbb{F}} \circ \alpha$ is equivalent to a cocycle taking values in the closed subgroup H_0 of $H_{\mathbb{F}}$, there is a Borel function $g : X \to H_{\mathbb{F}}/H_0$ such that for all $\lambda \in \Lambda_{\mathbb{K}}$,

$$\alpha(\lambda, x) \cdot g(x) = g(\lambda \cdot x) \quad \text{ for } \mu_1\text{-a.e. } x \in X.$$

Thus, by intersecting the countably many $\Lambda_{\mathbb{K}}$-translates of the measure one subset of X on which the above relation holds, we get a $\Lambda_{\mathbb{K}}$-invariant, μ_1-measure one subset X_0 of X such that $g \restriction X_0$ is a Borel homomorphism from $E_1 \restriction X_0$ to the orbit equivalence relation E arising from the action of $\Lambda_{\mathbb{F}}$ on H/H_0. Since E_1 is E_0-ergodic and E is 1-amenable, Lemma 5.5.3 implies that there is a μ_1-conull Borel subset $X_1 \subseteq X_0$ such that g maps X_1 into a single E-class, say, $\Lambda_{\mathbb{F}} \cdot hH_0$. Arguing as in the proof of Adams-Kechris [3, 2.2], it follows that α is equivalent to a cocycle taking values in the stabilizer of hH_0 under the action of $\Lambda_{\mathbb{F}}$; ie, α is equivalent to a cocycle taking values in

$$\mathrm{Stab}_{hH_0}(\Lambda_{\mathbb{F}}) = \{\lambda \in \Lambda_{\mathbb{F}} \mid \lambda hH_0 = hH_0\} = \Lambda_{\mathbb{F}} \cap hH_0h^{-1}.$$

This completes the proof of Claim 5.5.7, and hence also of Lemma 5.5.6. □

The application of E_0-ergodicity in this section explains the presence of Hypothesis (1) in the statement of Theorem 2.4.1. Of course, there remains the issue of showing that the particular actions considered in Theorems 2.2.1 and 2.3.1 actually satisfy Hypothesis (1) of Thereom 2.4.1. That is, we must show that the $\Lambda_{\mathbb{K}}$-Bernoulli actions of Theorem 2.3.1 and the profinite linear actions of Theorem 2.2.1 are actually E_0-ergodic. This will be dealt with in detail in Chapter 7, but we remark here that one of the most important insights of Thomas in [52] is the fact that orbit equivalence relations arising from dense embeddings of finitely generated groups with Property (τ) into compact profinite groups are E_0-ergodic. As discussed in Chapter 3, while the finitely generated groups $\Lambda_{\mathbb{K}}$ do not have Property (T), they do have the weaker Property (τ). And as the groups $\Lambda_{\mathbb{K}}$ embed densely by the Strong Approximation Theory [42, 7.12] into the compact profinite groups $K(J)$, it will follow from Thomas [52, 5.7] that $E_1 = E^X_{\Lambda_{\mathbb{K}}}$ is E_0-ergodic. An entirely different approach making use of the notion of almost invariant sets will be needed in order to show that Bernoulli Λ_S-actions are also E_0-ergodic. We will consider these issues in more detail in Chapter 7.

Chapter 6

Proof of Theorem 2.4.1

Recall that $\Lambda_{\mathbb{K}} = PSL_2(\mathcal{O}_{\mathbb{K}})$ and $\Lambda_{\mathbb{F}} = PSL_2(\mathcal{O}_{\mathbb{F}})$ are irreducible lattices in the connected, semisimple Lie groups

$$H_{\mathbb{K}} = \prod_{\vartheta \in \mathfrak{R}_{\mathbb{K}}} PSL_2(\mathbb{R}) \quad \text{and} \quad H_{\mathbb{F}} = \prod_{\theta \in \mathfrak{R}_{\mathbb{F}}} PSL_2(\mathbb{R}).$$

In what follows it will be convenient to have some notation to refer to the factors $PSL_2(\mathbb{R})$ in $H_{\mathbb{K}}$ and $H_{\mathbb{F}}$. We set $m = [\mathbb{K} : \mathbb{Q}]$ and $n = [\mathbb{F} : \mathbb{Q}]$, and for convenience fix enumerations $\mathfrak{R}_{\mathbb{K}} = \{\vartheta_1, \ldots, \vartheta_m\}$ and $\mathfrak{R}_{\mathbb{F}} = \{\theta_1, \ldots, \theta_n\}$. By convention we agree to let ϑ and θ denote arbitrary elements of $\mathfrak{R}_{\mathbb{K}}$ and $\mathfrak{R}_{\mathbb{F}}$, respectively, and we agree to let i range over the set $\{1, \ldots, m\}$, and j over the set $\{1, \ldots, n\}$. For each $\vartheta = \vartheta_i \in \mathfrak{R}_{\mathbb{K}}$ we define

$$\pi^{\mathbb{K}}_{\vartheta_i} = \pi^{\mathbb{K}}_i : H_{\mathbb{K}} \to PSL_2(\mathbb{R})$$

to be the canonical projection of $H_{\mathbb{K}}$ onto the factor corresponding to $\vartheta_i \in \mathfrak{R}_{\mathbb{K}}$; and similarly we define the projections

$$\pi^{\mathbb{F}}_{\theta_j} = \pi^{\mathbb{F}}_j : H_{\mathbb{F}} \to PSL_2(\mathbb{R})$$

for $\theta_j \in \mathfrak{R}_{\mathbb{F}}$. Finally, for $1 \leq i \leq m$ and $1 \leq j \leq n$, we will use the shorthand notation

$$(H_{\mathbb{K}})_i := \pi^{\mathbb{K}}_i(H_{\mathbb{K}}) \cong PSL_2(\mathbb{R}) \quad \text{and} \quad (H_{\mathbb{F}})_j := \pi^{\mathbb{F}}_j(H_{\mathbb{F}}) \cong PSL_2(\mathbb{R}).$$

Naturally, we identify $(H_{\mathbb{K}})_i$ and $(H_{\mathbb{F}})_j$ with $PSL_2(\mathbb{R})$ whenever convenient.

In this chapter we will prove the following theorem.

Theorem 2.4.1 *Suppose that $\mathbb{K}$ and $\mathbb{F}$ are totally real number fields properly extending $\mathbb{Q}$, let $\mathcal{O}_{\mathbb{K}}$ and $\mathcal{O}_{\mathbb{F}}$ be their integer rings, and let $\Lambda_{\mathbb{K}} = PSL_2(\mathcal{O}_{\mathbb{K}})$ and $\Lambda_{\mathbb{F}} = PSL_2(\mathcal{O}_{\mathbb{F}})$. Let X be a standard Borel $\Lambda_{\mathbb{K}}$-space with invariant ergodic probability measure μ_1, and* *let Y be a free standard Borel $\Lambda_{\mathbb{F}}$-space with invariant ergodic probability measure μ_2. Let E_1 and E_2 denote the orbit equivalence relations arising from the actions of $\Lambda_{\mathbb{K}}$ and $\Lambda_{\mathbb{F}}$ on X and Y, respectively. Further suppose that the following conditions are satisfied:*

(1) E_1 is E_0-ergodic.

(2) The induced $H_{\mathbb{K}}$-space $\hat{X} = X \times H_{\mathbb{K}}/\Lambda_{\mathbb{K}}$ is irreducible.

Suppose that $E_1 \leq_B E_2$. Then there exist

- *a $\Lambda_{\mathbb{K}}$-invariant Borel set $X_0 \subseteq X$ such that $\mu_1(X_0) = 1$,*
- *a Borel function $\tilde{f} : X \to \hat{Y}$, where $\hat{Y}$ is the $H_{\mathbb{F}}$-space induced from the action of $\Lambda_{\mathbb{F}}$ on Y, and*
- *an injective rational $\mathbb{R}$-homomorphism $\varphi : H_{\mathbb{K}} \to H_{\mathbb{F}}$*

such that

- *$\tilde{f}$ is a Borel reduction from E_1 to $E^{\hat{Y}}_{H_{\mathbb{F}}}$,*
- *$\tilde{f}(\lambda x) = \varphi(\sigma^{\mathbb{K}}(\lambda))\tilde{f}(x)$ for all $x \in X_0$ and for all $\lambda \in \Lambda_{\mathbb{K}}$, and*
- *for all $\lambda \in \Lambda_{\mathbb{K}}$, each component $\varphi_j(\sigma^{\mathbb{K}}(\lambda))$ of $\varphi(\sigma^{\mathbb{K}}(\lambda))$ is either λ^{ς}, or λ^{ς} with main diagonal scaled by -1, for some Galois automorphism $\varsigma \in Gal(n.c.(\mathbb{K})/\mathbb{Q})$.*

The remainder of this chapter will be devoted to proving Theorem 2.4.1.

6.1 An application of Zimmer cocycle superrigidity

Assuming the hypotheses of the theorem, let $f : X \to Y$ be a Borel reduction from E_1 to E_2, and let $\alpha : \Lambda_{\mathbb{K}} \times X \to \Lambda_{\mathbb{F}}$ be the strict Borel cocycle defined by

$$\alpha(\lambda, x) \cdot f(x) = f(\lambda x)$$

for all $\lambda \in \Lambda_{\mathbb{K}}$ and for all $x \in X$, so that α is the cocycle corresponding to f, as illustrated by Figure 6.1.

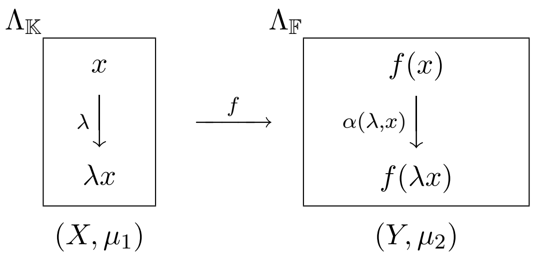


Figure 6.1: The correspondence of α with f.

Now let $\nu_{\mathbb{K}}$ be the Haar probability measure on $H_{\mathbb{K}}/\Lambda_{\mathbb{K}}$, define the measure $\hat{\mu}_1 = \mu_1 \times \nu_{\mathbb{K}}$ on $\hat{X} = X \times H_{\mathbb{K}}/\Lambda_{\mathbb{K}}$, and consider the induced action of $H_{\mathbb{K}}$ on $(\hat{X}, \hat{\mu}_1)$. Let

$$\hat{\alpha} : H_{\mathbb{K}} \times \hat{X} \to \Lambda_{\mathbb{F}}$$

be the corresponding cocycle induced from α, so that $\hat{\alpha}$ corresponds to the Borel reduction $\hat{f} : \hat{X} \to Y$ defined by $\hat{f}(\langle x, t\rangle) = f(x)$, as illustrated in Figure 6.2.

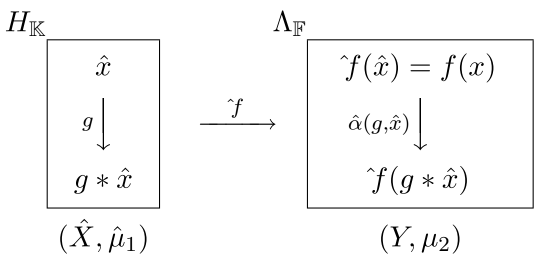


Figure 6.2: The correspondence of $\hat{\alpha}$ with $\hat{f}$.

Now, let $\sigma^{\mathbb{F}} : \Lambda_{\mathbb{F}} \hookrightarrow H_{\mathbb{F}}$ be the embedding that realizes $\Lambda_{\mathbb{F}}$ as an irreducible lattice in $H_{\mathbb{F}}$, and for each embedding $\theta_j \in \mathfrak{R}_{\mathbb{F}}$ of $\mathbb{F}$ into $\mathbb{R}$, define the cocycles

$$\hat{\alpha}_j := \pi_j^{\mathbb{F}} \circ \sigma^{\mathbb{F}} \circ \hat{\alpha} : H_{\mathbb{K}} \times \hat{X} \to (H_{\mathbb{F}})_j \cong PSL_2(\mathbb{R}).$$

By Lemma 5.5.6, any cocycle equivalent to some $\hat{\alpha}_j$ has Zariski dense range in $PSL_2(\mathbb{R})$. By hypothesis, $\hat{X}$ is an *irreducible* ergodic $H_{\mathbb{K}}$-space. The remaining hypotheses of Zimmer's Superrigidity Theorem 5.4.1 are easily verified, and hence we may apply it to the cocycles $\hat{\alpha}_j$ to obtain, for each $\theta_j \in \mathfrak{R}_{\mathbb{F}}$, a rational $\mathbb{R}$-homomorphism

$$\psi_j : H_{\mathbb{K}} \to PSL_2(\mathbb{R})$$

such that $\hat{\alpha}_j$ is equivalent to the cocycle

$$\hat{\beta}_j : H_{\mathbb{K}} \times \hat{X} \to PSL_2(\mathbb{R})$$

defined by

$$\hat{\beta}_j(g, \hat{x}) = \psi_j(g) \quad \text{ for all } \quad g \in H_{\mathbb{K}},\, \hat{x} \in \hat{X}.$$

For each $\theta_j \in \mathfrak{R}_{\mathbb{F}}$, let the Borel map $\hat{h}_j : \hat{X} \to PSL_2(\mathbb{R})$ witness the equivalence of $\hat{\alpha}_j$ with $\hat{\beta}_j$, so that we have, for all $g \in H_{\mathbb{K}}$ and for $\hat{\mu}_1$-a.e. $\hat{x} \in \hat{X}$,

$$\psi_j(g) = \hat{\beta}_j(g, \hat{x}) = \hat{h}_j(g * \hat{x}) \cdot \hat{\alpha}_j(g, \hat{x}) \cdot \hat{h}_j(\hat{x})^{-1}.$$

Define

$$\psi : H_{\mathbb{K}} \to H_{\mathbb{F}}$$

by

$$\psi(g) = \langle \psi_j(g) \rangle_{\theta_j \in \mathfrak{R}_{\mathbb{F}}},$$

and define

$$\hat{h} : \hat{X} \to H_{\mathbb{F}}$$

by

$$\hat{h}(\hat{x}) = \langle \hat{h}_j(\hat{x}) \rangle_{\theta_j \in \mathfrak{R}_{\mathbb{F}}}.$$

Then we have, for all $g \in H_{\mathbb{K}}$ and for $\hat{\mu}_1$-a.e. $\hat{x} \in \hat{X}$,

$$\psi(g) = \hat{\beta}(g, \hat{x}) := \hat{h}(g * \hat{x}) \cdot (\sigma^{\mathbb{F}} \circ \hat{\alpha})(g, \hat{x}) \cdot \hat{h}(\hat{x})^{-1},$$

so that $\sigma^{\mathbb{F}} \circ \hat{\alpha}$ is equivalent to $\hat{\beta}$ with witness $\hat{h} : \hat{X} \to H_{\mathbb{F}}$.

Notice that if we define the Borel function $\hat{}f\hat{} : \hat{X} \to \hat{Y}$ by

$$\hat{}f\hat{}(\langle x, t \rangle) = \langle f(x), 1 \rangle,$$

then $\hat{}f\hat{}$ is a Borel reduction from $E^{\hat{X}}_{H_{\mathbb{K}}}$ to $E^{\hat{Y}}_{H_{\mathbb{F}}}$, and that, furthermore, the cocycle $\sigma^{\mathbb{F}} \circ \hat{\alpha}$ corresponds to $\hat{}f\hat{}$. Hence if we define $\widetilde{f} : \hat{X} \to \hat{Y}$ to be the adjusted Borel reduction $\widetilde{f} = \hat{h} * \hat{}f\hat{}$, then the cocycle $\hat{\beta}(g, \hat{x}) = \psi(g)$ (weakly) corresponds to $\widetilde{f}$; and thus we have, for all $g \in H_{\mathbb{K}}$ and for $\hat{\mu}_1$-a.e. $\hat{x} \in \hat{X}$,

$$\psi(g) * \widetilde{f}(\hat{x}) = \widetilde{f}(g * \hat{x}).$$

We have therefore obtained, from the original Borel reduction $f : X \to Y$ from E_1 to E_2, a permutation group homomorphism $(\widetilde{f}, \psi) : (\hat{X}, H_{\mathbb{K}}) \to (\hat{Y}, H_{\mathbb{F}})$ from $(\hat{X}, H_{\mathbb{K}})$

to $(\hat{Y}, H_{\mathbb{F}})$, as illustrated below in Figure 6.3. Our next goal will be to show that $\psi : H_{\mathbb{K}} \to H_{\mathbb{F}}$ is injective, a fact that will allow us to replace ψ with a simpler homomorphism $\varphi : H_{\mathbb{K}} \to H_{\mathbb{F}}$ that is conjugate to ψ.

6.2 Adjusting a permutation group homomorphism

In this section we will prove that ψ is injective, and use this fact to adjust

$$(\widetilde{f}, \psi) : (\hat{X}, H_{\mathbb{K}}) \to (\hat{Y}, H_{\mathbb{F}})$$

so as to obtain a simpler permutation group homomorphism,

$$(\widetilde{f_1}, \varphi) : (\hat{X}, H_{\mathbb{K}}) \to (\hat{Y}, H_{\mathbb{F}}).$$

Figure 6.3 may be helpful in understanding the proof of the following lemma.

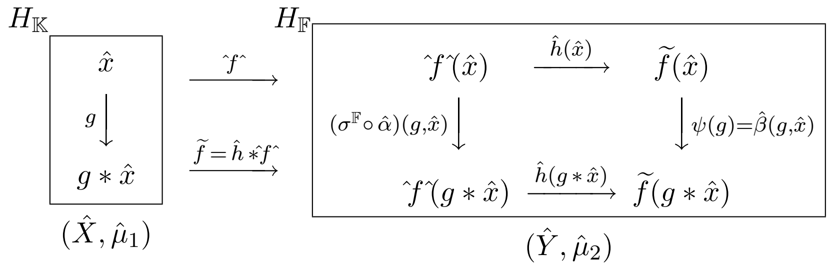


Figure 6.3: The adjustment from $(\hat{}f\hat{},\ \sigma^{\mathbb{F}} \circ \hat{\alpha})$ to $(\widetilde{f}, \psi)$.

Lemma 6.2.1. *$\psi : H_{\mathbb{K}} \to H_{\mathbb{F}}$ is injective, and $m \leq n$.*

Proof. Fix $j \in \{1, \ldots, n\}$, and notice that ψ_j maps $H_{\mathbb{K}}$, a finite product of at least two copies of $PSL_2(\mathbb{R})$, into $PSL_2(\mathbb{R})$. Since $PSL_2(\mathbb{R})$ is simple, the only normal subgroups of

$$H_{\mathbb{K}} = PSL_2(\mathbb{R}) \times \cdots \times PSL_2(\mathbb{R})$$

are products of its factors, ie, subgroups of the form

$$N_1 \times \cdots \times N_m,$$

where each N_i is either $PSL_2(\mathbb{R})$ or the trivial group. As ψ_j is a rational homomorphism of algebraic groups, we have

$$3m = \dim\, H_{\mathbb{K}} = \dim\, \ker\, \psi_j + \dim\, \psi_j(H_{\mathbb{K}}) \leq \dim\, \ker\, \psi_j + 3.$$

In particular dim ker $\psi_j \geq 3m-3$, so if we write ker $\psi = N_1 \times \cdots \times N_m$, then we must have $N_i = PSL_2(\mathbb{R})$ for all but at most one i. But by Lemma 5.5.6, ψ_j is nontrivial; hence it follows that there is $i_0 \in \{1, \ldots, m\}$ such that ker $\psi_j = N_1 \times \cdots \times N_m$, where N_{i_0} is trivial and $N_i = PSL_2(\mathbb{R})$ for all $i \neq i_0$. In other words, each homomorphism

$$\psi_j : PSL_2(\mathbb{R}) \times \cdots \times PSL_2(\mathbb{R}) \to PSL_2(\mathbb{R}), \quad \theta_j \in \mathfrak{R}_{\mathbb{F}},$$

is really a function of just *one* of the factors of $PSL_2(\mathbb{R})$ in $H_{\mathbb{K}}$. For each $1 \leq j \leq n$, Let $(H_{\mathbb{K}})_{l(j)}$ be the factor in $H_{\mathbb{K}}$ on which ψ_j is nontrivial, so that ψ_j is really only a function of $\pi^{\mathbb{K}}_{l(j)}(H_{\mathbb{K}}) \cong PSL_2(\mathbb{R}) \leq H_{\mathbb{K}}$. We claim that the restriction

$$\psi_j \restriction (H_{\mathbb{K}})_{l(j)}$$

of ψ_j to this factor is a rational isomorphism. Since $PSL_2(\mathbb{R})$ is simple, $\psi_j \restriction (H_{\mathbb{K}})_{l(j)}$ is injective, and hence by Proposition 3.2.1, dim im $\psi_j \restriction (H_{\mathbb{K}})_{l(j)} = 3$. But then again by Proposition 3.2.1, im $\psi_j \restriction (H_{\mathbb{K}})_{l(j)}$ is closed in $PSL_2(\mathbb{R})$, and $\psi_j \restriction (H_{\mathbb{K}})_{l(j)}$ is surjective. Hence $\psi_j \restriction (H_{\mathbb{K}})_{l(j)}$ is an isomorphism of groups, and consequently an isomorphism of algebraic groups by [50, 21.2.6].

Thus to each ψ_j, $1 \leq j \leq n$, we can associate a single factor $(H_{\mathbb{K}})_{l(j)}$ in $H_{\mathbb{K}}$ on which the restriction of ψ_j is an automorphism of $PSL_2(\mathbb{R})$. We now show that for *each* factor in $H_{\mathbb{K}}$ there is some ψ_j which is non-trivial on that factor; ie, that the association $l : \{1, \ldots, n\} \to \{1, \ldots, m\}$ is a surjection.

Suppose not; that is, suppose there is some factor $(H_{\mathbb{K}})_i \leq H_{\mathbb{K}}$ that lies in the kernel of *each* ψ_j corresponding to $\theta_j \in \mathfrak{R}_{\mathbb{F}}$. We view this factor, call it N, as a normal subgroup of $H_{\mathbb{K}}$, and consider its restricted action on $\hat{X}$, remembering that $\psi(g) = 1$ for all $g \in N$. Recall that $\hat{X}$ is an irreducible $H_{\mathbb{K}}$-space, which means that N acts ergodically on $(\hat{X}, \hat{\mu}_1)$.

Since $\psi(g) = 1$ for all $g \in N$, the adjusted Borel function $\widetilde{f} : \hat{X} \to \hat{Y}$ is essentially N-invariant; ie, for each $g \in N$, $\widetilde{f}(g * \hat{x}) = \widetilde{f}(\hat{x})$ for $\hat{\mu}_1$-a.e. $\hat{x} \in \hat{X}$. Hence by Zimmer [58, 2.2.18], it follows from the ergodicity of N on $(\hat{X}, \hat{\mu}_1)$ that $\widetilde{f}$ is $\hat{\mu}_1$-a.e. constant on $\hat{X}$. In particular, $\widetilde{f}$ maps a $\hat{\mu}_1$-conull subset of $\hat{X}$ into a single $E^{\hat{Y}}_{H_{\mathbb{F}}}$-class, contradicting the fact that $\widetilde{f}$ is $\hat{\mu}_1$-a.e. a reduction from $E^{\hat{X}}_{H_{\mathbb{K}}}$ to $E^{\hat{Y}}_{H_{\mathbb{F}}}$. Thus each factor $(H_{\mathbb{K}})_i$ in $H_{\mathbb{K}}$

is realized as $(H_{\mathbb{K}})_{l(j)}$ for some $1 \le j \le n$, where again the restriction of ψ_j to $(H_{\mathbb{K}})_{l(j)}$ is an isomorphism. It follows easily that ψ is injective and that $m \le n$. $\square$

In the above proof we saw that each ψ_j can be viewed as an isomorphism of $PSL_2(\mathbb{R}) \cong (H_{\mathbb{K}})_{l(j)} \le H_{\mathbb{K}}$ onto $PSL_2(\mathbb{R}) \cong (H_{\mathbb{F}})_j \le H_{\mathbb{F}}$. That is, each ψ_j is an automorphism of $PSL_2(\mathbb{R})$. But by [24], every automorphism of $PSL_2(\mathbb{R})$ is simply conjugation by an element of $PGL_2(\mathbb{R})$. We shall now use this fact to simplify ψ.

Fix $\theta_j \in \mathfrak{R}_{\mathbb{F}}$, and let $i = l(j)$, so that

$$\psi_j \restriction (H_{\mathbb{K}})_{l(j)} : (H_{\mathbb{K}})_i \to (H_{\mathbb{F}})_j$$

is an automorphism of $PSL_2(\mathbb{R})$. By the previous remark, there is some element of $PGL_2(\mathbb{R})$, say g_j, such that for all $y = \langle y_1, \ldots, y_m \rangle \in H_{\mathbb{K}}$,

$$\psi_j(y) = g_j\, y_i\, g_j^{-1},$$

where $y_i = \pi_i^{\mathbb{K}}(y)$. We now conjugate away as much of g_j as possible. Specifically, we define a new injective rational $\mathbb{R}$-homomorphism $\varphi_j : H_{\mathbb{K}} \to (H_{\mathbb{F}})_j$, a conjugate of ψ_j, as follows:

If $r = \det g_j > 0$, define $\varphi_j : H_{\mathbb{K}} \to (H_{\mathbb{F}})_j$ by

$$\varphi_j(y) = \begin{bmatrix} r^{1/2} & 0 \\ 0 & r^{1/2} \end{bmatrix} g_j^{-1}\, \psi_j(y)\, g_j \begin{bmatrix} r^{-1/2} & 0 \\ 0 & r^{-1/2} \end{bmatrix}.$$

In this case notice that φ_j is a conjugate of ψ_j by an element of $PSL_2(\mathbb{R})$, and that $\varphi_j(y) = \pi_{l(j)}^{\mathbb{K}}(y)$ for all $y \in H_{\mathbb{K}}$.

If, on the other hand, $r = \det g_j < 0$, define $\varphi_j : H_{\mathbb{K}} \to (H_{\mathbb{F}})_j$ by

$$\varphi_j(y) = \begin{bmatrix} -(-r)^{1/2} & 0 \\ 0 & (-r)^{1/2} \end{bmatrix} g_j^{-1}\, \psi_j(y)\, g_j \begin{bmatrix} (-r)^{-1/2} & 0 \\ 0 & -(-r)^{-1/2} \end{bmatrix}.$$

In this case φ_j is again a conjugate of ψ_j by an element of $PSL_2(\mathbb{R})$, and if

$$\pi_{l(j)}^{\mathbb{K}}(y) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{for } y \in H_{\mathbb{K}},$$

then we have

$$\varphi_j(y) = \begin{bmatrix} -a & b \\ c & -d \end{bmatrix}.$$

Write

$$\tilde{g}_j = \begin{bmatrix} r^{1/2} & 0 \\ 0 & r^{1/2} \end{bmatrix} g_j^{-1} \quad \text{or} \quad \begin{bmatrix} -(-r)^{1/2} & 0 \\ 0 & (-r)^{1/2} \end{bmatrix} g_j^{-1},$$

depending, as above, on whether det g_j is positive or negative. Then define

$$\varphi_j = \tilde{g}_j \, \psi_j \, \tilde{g}_j^{-1},$$

noting that $\tilde{g}_j \in PSL_2(\mathbb{R})$. Supposing we have defined $\tilde{g}_j$ in this manner for each $j \in \{1, \ldots, n\}$, we now let

$$\tilde{g} = \langle \tilde{g}_j \rangle_{\theta_j \in \mathfrak{R}_{\mathbb{F}}} \in H_{\mathbb{F}},$$

and define

$$\varphi = \tilde{g} \, \psi \, \tilde{g}^{-1} : H_{\mathbb{K}} \to H_{\mathbb{F}}.$$

Then φ is an injective rational $\mathbb{R}$-homomorphism from $H_{\mathbb{K}}$ to $H_{\mathbb{F}}$ such that for all $\lambda \in \Lambda_{\mathbb{K}}$, each component $\varphi_j(\sigma^{\mathbb{K}}(\lambda))$ of $\varphi(\sigma^{\mathbb{K}}(\lambda))$ is either λ^{ς}, or λ^{ς} with main diagonal scaled by -1, for some Galois automorphism $\varsigma \in \mathrm{Gal}(\text{n.c.}(\mathbb{K})/\mathbb{Q})$. In particular, for $l(j) = 1$, $\vartheta_1 : \mathbb{K} \to \mathbb{R}$ the identity embedding, we have that $\varphi_j(\lambda)$ is either λ, or λ with diagonal entries scaled by -1.

Of course, we must now make the necessary adjustment to $\widetilde{f}$, as well. Define

$$\hat{b} : \hat{X} \to H_{\mathbb{F}}$$

by

$$\hat{b}(\hat{x}) = \tilde{g}\hat{h}(\hat{x}),$$

so that $(\sigma^{\mathbb{F}} \circ \hat{\alpha})$ is equivalent, via $\hat{b}$, to the cocycle $\hat{\beta}_1 : H_{\mathbb{K}} \times \hat{X} \to H_{\mathbb{F}}$ defined by

$$\hat{\beta}_1(g, \hat{x}) = \varphi(g).$$

Thus we have for all $g \in H_{\mathbb{K}}$ and for $\hat{\mu}_1$-a.e. $\hat{x} \in \hat{X}$,

$$\varphi(g) = \hat{\beta}_1(g, \hat{x}) = \hat{b}(g * \hat{x}) \cdot (\sigma^{\mathbb{F}} \circ \hat{\alpha})(g, \hat{x}) \cdot \hat{b}(\hat{x})^{-1}.$$

Then define

$$\widetilde{f_1} : \hat{X} \to \hat{Y}$$

to be the adjusted Borel reduction

$$\widetilde{f_1} = \hat{b} * \hat{}f\hat{},$$

so that $\hat{\beta}_1$ corresponds to $\widetilde{f_1}$, as illustrated in Figure 6.4 below.

$$\begin{tikzcd}
H_{\mathbb{K}} & & H_{\mathbb{F}} & \\
\hat{x} \arrow[d, "g"'] \arrow[r, "\hat{}f\hat{}"] & {} & \hat{}f\hat{}(\hat{x}) \arrow[r, "\hat{b}(\hat{x})"] \arrow[d, "(\sigma^{\mathbb{F}} \circ \hat{\alpha})(g,\hat{x})"'] & \widetilde{f_1}(\hat{x}) \arrow[d, "\varphi(g) = \hat{\beta}_1(g,\hat{x})"] \\
g * \hat{x} \arrow[r, "\widetilde{f_1} = \hat{b} * \hat{}f\hat{}"] & {} & \hat{}f\hat{}(g * \hat{x}) \arrow[r, "\hat{b}(g*\hat{x})"] & \widetilde{f_1}(g * \hat{x}) \\
(\hat{X}, \hat{\mu}_1) & & (\hat{Y}, \hat{\mu}_2) &
\end{tikzcd}$$

Figure 6.4: The adjustment from $(\widetilde{f}, \psi)$ to $(\widetilde{f_1}, \varphi)$.

6.3 Coming down on the left

We have now succeeded in adjusting the permutation group homomorphism

$$(\widetilde{f}, \psi) : (\hat{X}, H_{\mathbb{K}}) \to (\hat{Y}, H_{\mathbb{F}})$$

so as to obtain a simpler permutation group homomorphism,

$$(\widetilde{f_1}, \varphi) : (\hat{X}, H_{\mathbb{K}}) \to (\hat{Y}, H_{\mathbb{F}}).$$

In this section we further adjust $(\widetilde{f_1}, \varphi)$ so as to obtain a permutation group homomorphism

$$(\tilde{f}, \varphi) : (X, \Lambda_{\mathbb{K}}) \to (\hat{Y}, H_{\mathbb{F}}).$$

Specifically we follow Adams [2, 5.4], and come back down on the left side from the induced $H_{\mathbb{K}}$-space $\hat{X}$ to the original $\Lambda_{\mathbb{K}}$-space X. Indeed, by Lemma 5.3.1 the cocycle $\sigma^{\mathbb{F}} \circ \alpha : \Lambda_{\mathbb{K}} \times X \to H_{\mathbb{F}}$ is equivalent to the cocycle

$$\beta : \Lambda_{\mathbb{K}} \times X \to H_{\mathbb{F}}$$

defined by

$$\beta(\lambda, x) = \varphi(\lambda),$$

where here we are identifying the group $\Lambda_{\mathbb{K}}$ with its image under $\sigma^{\mathbb{K}}$ inside $H_{\mathbb{K}}$. We may use this cocycle equivalence to obtain an adjusted Borel reduction from X to $\hat{Y}$ that will be a permutation group homomorphism from $(X, \Lambda_{\mathbb{K}})$ to $(\hat{Y}, H_{\mathbb{F}})$ when paired with the group homomorphism

$$\varphi \restriction \Lambda_{\mathbb{K}} : \Lambda_{\mathbb{K}} \to H_{\mathbb{F}}.$$

Specifically, we first define $\iota : Y \hookrightarrow \hat{Y}$ by $\iota(y) = \langle y, 1 \rangle$, so that ι is $\Lambda_{\mathbb{F}}$-equivariant. Then define

$$f\hat{\ } = \iota \circ f : X \to \hat{Y},$$

so that $\sigma^{\mathbb{F}} \circ \alpha$ corresponds to $f\hat{\ }$. Further let $b : X \to H_{\mathbb{F}}$ witness the equivalence of $\sigma^{\mathbb{F}} \circ \alpha$ with β, as in the proof of 5.3.1, so that for all $\lambda \in \Lambda_{\mathbb{K}}$ and μ_1-a.e. $x \in X$,

$$\beta(\lambda, x) = b(\lambda x) \cdot (\sigma^{\mathbb{F}} \circ \alpha)(\lambda, x) \cdot b(x)^{-1}.$$

Finally, define

$$\tilde{f} = b * f\hat{\ },$$

so that β corresponds to $\tilde{f}$ and hence for all $\lambda \in \Lambda_{\mathbb{K}}$ and for μ_1-a.e. $x \in X$, we have

$$\tilde{f}(\lambda x) = \beta(\lambda, x) = \varphi(\lambda)\tilde{f}(x). \tag{5.2}$$

Our progress is now summarized in Figure 6.5.

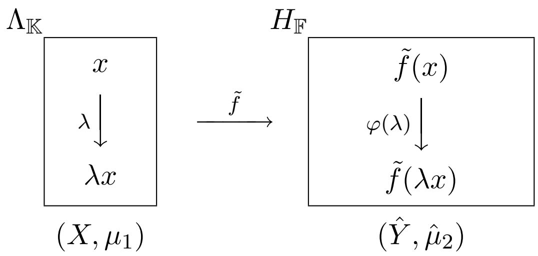


Figure 6.5: The permutation group homomorphism $(\tilde{f},\ \varphi \restriction \Lambda_{\mathbb{K}})$.

From the original Borel reduction $f : X \to Y$ from E_1 to E_2 with corresponding cocycle $\alpha : \Lambda_{\mathbb{K}} \times X \to \Lambda_{\mathbb{F}}$, we have obtained a new Borel reduction $\tilde{f} : X \to \hat{Y}$ from E_1

to $E^{\hat{Y}}_{H_{\mathbb{F}}}$, where this reduction is given μ_1-a.e. by the permutation group homomorphism

$$(\tilde{f}, \varphi) : (X, \Lambda_{\mathbb{K}}) \to (\hat{Y}, H_{\mathbb{F}}).$$

Of course, at this point φ is strictly speaking a homomorphism from $H_{\mathbb{K}}$ to $H_{\mathbb{F}}$, and in Equation (5.2) above we are identifying $\Lambda_{\mathbb{K}}$ with its image under $\sigma^{\mathbb{K}}$ inside $H_{\mathbb{K}}$. To be absolutely precise, we have for all $\lambda \in \Lambda_{\mathbb{K}}$ and for μ_1-a.e. $x \in X$,

$$\tilde{f}(\lambda x) = \varphi(\sigma^{\mathbb{K}}(\lambda))\tilde{f}(x).$$

This completes the proof of Theorem 2.4.1.

Chapter 7

Proofs of Theorems 2.2.1 and 2.3.1

In this chapter, we continue the argument presented in Chapter 6, making use of Theorem 2.4.1 to prove Theorems 2.2.1 and 2.3.1. Much of the work that will go into proving these theorems has already been accomplished in Chapter 6. Aside from verifying the hypotheses of Theorem 2.4.1, essentially all that remains are some arguments involving the possible factors of the dynamical systems appearing in the statements of 2.2.1 and 2.3.1, together with some computations involving the groups $\Lambda_S = PSL_2(\mathcal{O}_S)$. We carry out these computations in the first section below, and then we complete the proofs of Theorems 2.3.1 and 2.2.1 in Sections 2 and 3 respectively. Finally, in Section 4 we discuss potential directions for further research, including some possible generalizations of Theorems 2.2.1 and 2.3.1.

Throughout this chapter, $S = \{p_1, \ldots, p_s\}$ and $T = \{q_1, \ldots, q_t\}$ will denote finite, nonempty sets of primes, and $\mathcal{O}_S$, $\mathcal{O}_T$ the rings of integers in the multi-quadratic number fields $k_S = \mathbb{Q}(\sqrt{p_1}, \ldots, \sqrt{p_s})$ and $k_T = \mathbb{Q}(\sqrt{q_1}, \ldots, \sqrt{q_t})$, respectively. As indicated in Section 2.5, we write

$$\Lambda_S = PSL_2(\mathcal{O}_S), \qquad \Lambda_T = PSL_2(\mathcal{O}_T),$$

$$H_S = \prod_{2^S} PSL_2(\mathbb{R}), \qquad H_T = \prod_{2^T} PSL_2(\mathbb{R}),$$

$$\sigma^S : \Lambda_S \hookrightarrow H_S, \quad \text{and} \quad \sigma^T : \Lambda_T \hookrightarrow H_T.$$

Essentially, we view the arguments presented in this chapter as a continuation of those given in the previous chapter, with the multi-quadratic number fields k_S and k_T replacing the fields $\mathbb{K}$ and $\mathbb{F}$, respectively, as special cases of them.

7.1 Some computations involving subgroups of $PSL_2(\mathbb{R})$

In this section, we prove a pair of computational results involving the groups Λ_S that will be needed in Sections 2 and 3 to finish off the proofs of Theorems 2.3.1 and 2.2.1. It should be noted that this is the only point in the proofs of these theorems in which the specific properties of the groups Λ_S come into play; ie, this is the only point at which it is necessary to work with the groups Λ_S in place of the more general groups $\Lambda_{\mathbb{K}}$.

By a slight abuse of notation, in what follows we shall denote elements of $PSL_2(\mathbb{R})$ by 2×2 matrices (a_{ij}), which we remember to identify with $(-a_{ij})$.

Lemma 7.1.1. *Suppose that*

$$v = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in PSL_2(\mathbb{R})$$

and that there exists a finite index subgroup Λ_S^0 of Λ_S such that $v\Lambda_S^0 v^{-1} \leq \Lambda_T$. Then there exists a positive integer k such that ka^2, kb^2, kc^2, kd^2, kab, kac, kad, kbc, kbd, $kcd \in \mathcal{O}_T$.

Proof. Let Λ_S^0 be a finite index subgroup of Λ_S such that $v\Lambda_S^0 v^{-1} \leq \Lambda_T$. Since $[\Lambda_S : \Lambda_S^0] < \infty$, there exist positive integers k_0, k_1, and k_2 such that

$$\begin{bmatrix} 1 & k_0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ k_1 & 1 \end{bmatrix}, \begin{bmatrix} 1-k_2 & -k_2 \\ k_2 & 1+k_2 \end{bmatrix} \in \Lambda_S^0.$$

Let $k' = k_0 k_1 k_2$, so that

$$\begin{bmatrix} 1 & k' \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ k' & 1 \end{bmatrix}, \begin{bmatrix} 1-k' & -k' \\ k' & 1+k' \end{bmatrix} \in \Lambda_S^0.$$

Then from

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & k' \\ 0 & 1 \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} 1-k'ac & k'a^2 \\ -k'c^2 & 1+k'ac \end{bmatrix} \in \Lambda_T$$

and

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ k' & 1 \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} 1+k'bd & -k'b^2 \\ k'd^2 & 1-k'bd \end{bmatrix} \in \Lambda_T,$$

we obtain

$$k'a^2, k'b^2, k'c^2, k'd^2, k'ac, k'bd \in \mathcal{O}_T.$$

Next consider

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1-k' & -k' \\ k' & 1+k' \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \begin{bmatrix} (1-k')ad + k'ac + k'bd - (1+k')bc & -(1-k')ab - k'a^2 - k'b^2 + (1+k')ab \\ (1-k')cd + k'c^2 + k'd^2 - (1+k')cd & -(1-k')bc - k'ac - k'bd + (1+k')ad \end{bmatrix}.$$

From the upper left, and remembering that $k'ac, k'bd \in \mathcal{O}_T$, we obtain

$$-k'(ad+bc) \in \mathcal{O}_T.$$

Of course, since $ad - bc = 1$, we also have

$$-k'(ad-bc) \in \mathcal{O}_T,$$

and then combining these equations gives

$$2k'ad, 2k'bc \in \mathcal{O}_T.$$

From the lower left, and remembering that $k'c^2, k'd^2 \in \mathcal{O}_T$, we get

$$2k'cd \in \mathcal{O}_T,$$

and similarly, from the upper right,

$$2k'ab \in \mathcal{O}_T.$$

Thus letting $k = 2k'$ completes the proof. □

Remark 7.1.2. We shall use Lemma 7.1.1 to prove that if there exists an element v in $PSL_2(\mathbb{R})$ that conjugates a finite index subgroup of Λ_S into Λ_T, then $S \subseteq T$. In order to prove this, it will suffice to know that if

$$v = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in PSL_2(\mathbb{R})$$

is such an element, then $kcd \in \mathcal{O}_T$ for some $k \in \mathbb{Z}^+$. Hence the full statement of Lemma 7.1.1 will not be needed. We include the full statement in part because of the symmetry of its conclusions, but also because of the fact that from its full statement, the reader may check by direct computation that if $v \in PSL_2(\mathbb{R})$ conjugates a finite index subgroup of Λ_S into Λ_T, then v normalizes $PSL_2(k_T)$. Since $PSL_2(k_T)$ is its own normalizer in $PSL_2(\mathbb{R})$, this implies that $v \in PSL_2(k_T)$. However, we shall not need these results in what follows.

We now deduce the following corollary.

Corollary 7.1.3. *Suppose there exists an element $v \in PSL_2(\mathbb{R})$ and a finite index subgroup Λ_S^0 of Λ_S such that $v\Lambda_S^0 v^{-1} \leq \Lambda_T$. Then $S \subseteq T$.*

Proof. Let $p \in S$. Notice that

$$\begin{bmatrix} m + n\sqrt{p} & 0 \\ 0 & m - n\sqrt{p} \end{bmatrix} \in \Lambda_S \qquad \Longleftrightarrow \qquad m^2 - n^2 p = 1.$$

By the theory of the Pell Equation (for instance, see [5]), there exist infinitely many solutions to the diophantine equation $x^2 - py^2 = 1$. In fact, there exists a fundamental solution $(m_1, n_1) \in \mathbb{N}^2$ such that all other integer solutions $(\pm m_k, \pm n_k)$ are given by

$$m_k + n_k\sqrt{p} = (m_1 + n_1\sqrt{p})^k.$$

This gives an infinite cyclic subgroup of Λ_S consisting of matrices of the above form. Since Λ_S^0 has finite index in Λ_S, we therefore obtain infinitely many such elements in Λ_S^0. Thus fix integers m, n such that

$$\begin{bmatrix} m + n\sqrt{p} & 0 \\ 0 & m - n\sqrt{p} \end{bmatrix} \in \Lambda_S^0.$$

Again writing $v = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we have that

$$v \begin{bmatrix} m + n\sqrt{p} & 0 \\ 0 & m - n\sqrt{p} \end{bmatrix} v^{-1} = \begin{bmatrix} m + n(ad+bc)\sqrt{p} & -2nab\sqrt{p} \\ 2ncd\sqrt{p} & m - n(ad+bc)\sqrt{p} \end{bmatrix} \in \Lambda_T.$$

This implies that $2ncd\sqrt{p} \in \mathcal{O}_T$. Using Lemma 7.1.1, fix a positive integer k such that $kcd \in \mathcal{O}_T$. Then $2nkcd \in \mathcal{O}_T$, and $2nkcd\sqrt{p} \in \mathcal{O}_T$. From this it follows that $\sqrt{p} \in k_T$, which implies that $p \in T$. $\square$

7.2 Proof of Theorem 2.3.1

In this section, we will complete the proof of Theorem 2.3.1, which we recall now for convenience.

Theorem 2.3.1. *For $i = 1, 2$, let (Y_i, ν_i) be a non-trivial standard Borel probability space, and let S_i be a finite, nonempty set of primes. Let $X_i \subseteq Y_i^{\Lambda_{S_i}}$ be the subset of $Y_i^{\Lambda_{S_i}}$ on which Λ_{S_i} acts freely as a group of Bernoulli shifts. Let E_i be the orbit equivalence relation arising from this free Bernoulli action of Λ_{S_i} on X_i. Then $E_1 \leq_B E_2$ implies $S_1 \subseteq S_2$. In particular, if each of $S_1 \setminus S_2$ and $S_2 \setminus S_1$ is nonempty, then E_1 and E_2 are incomparable with respect to Borel reducibility.*

In order to improve notation, throughout this section we shall write S, T in place of S_1, S_2, and $\mu_i = \nu_i^{\Lambda_{S_i}} \restriction X_i$ for each $i = 1, 2$.

Assume the hypotheses of Theorem 2.3.1. We will begin by verifying that Bernoulli actions of the groups Λ_S, Λ_T fit into the context of Theorem 2.4.1. Indeed, all of the hypotheses of this theorem are easily verified, with the exception of hypotheses (1) and (2). We verify hypothesis (1) in 7.2.1, and hypothesis (2) in 7.2.2. Then in 7.2.3 we will establish a result concerning factors of Bernoulli systems, and finally in 7.2.4 we will complete the proof of Theorem 2.3.1.

7.2.1 E_0-ergodicity of orbit equivalence relations arising from Bernoulli actions

In order to verify hypothesis (1) of Theorem 2.4.1, we must check that the free Bernoulli action $\Lambda_S \curvearrowright X_1$ is E_0-ergodic. The fact that Bernoulli actions of non-amenable groups are E_0-ergodic is already well-known for Bernoulli actions with finite state space (for instance, see [23, A4]). We shall use the notion of *almost invariant sets* introduced in Chapter 4 to prove it in the general case. Specifically, we prove the following:

Lemma 7.2.1. *Let (Y, ν) be a nontrivial standard Borel probability space, and let Γ be any nonamenable countable discrete group. Let $X \subseteq Y^\Gamma$ be the (invariant, conull) subset on which Γ acts freely as a group of Bernoulli shifts. Then $\Gamma \curvearrowright X$ is E_0-ergodic.*

Proof. Write F for the orbit equivalence relation arising from the Bernoulli action $\Gamma \curvearrowright (Y,\nu)^\Gamma$. Since this action is ergodic (in fact strongly mixing [31, 2.3]), it follows from Proposition 4.2.5 that F is E_0-ergodic if and only if $\Gamma \curvearrowright (Y,\nu)^\Gamma$ does not have almost invariant sets. By [31, 1.2], a Bernoulli action of a countably infinite group G has almost invariant sets if and only if G is amenable. Hence (F, ν^Γ) is E_0-ergodic. As any Borel homomorphism $\phi : X \to 2^{\mathbb{N}}$ from E_Γ^X to E_0 can be trivially extended to a Borel homomorphism $\tilde{\phi} : Y^\Gamma \to 2^{\mathbb{N}}$ from F to E_0, the desired result follows from the fact that $\nu^\Gamma(X) = 1$. □

Since the discrete group Λ_S contains a nonabelian free subgroup and hence is not amenable, it follows that free Bernoulli actions of Λ_S are E_0-ergodic, and therefore satisfy hypothesis (1) of Theorem 2.4.1.

7.2.2 Irreducibility of spaces induced from Bernoulli actions

In order to verify hypothesis (2) of Theorem 2.4.1, we must show that $(\hat{X}_1, \hat{\mu}_1)$ is an irreducible H_S-space, ie, that nontrivial normal subgroups of H_S still act ergodically on $(\hat{X}_1, \hat{\mu}_1)$. The following argument is based in part on the proof of [57, 2.4].

Lemma 7.2.2. *$(\hat{X}_1, \hat{\mu}_1)$ is an irreducible H_S-space.*

Proof. To simplify notation in this proof, we drop the subscripts on $\hat{X}_1$ and $\hat{\mu}_1$. Recall that $PSL_2(\mathbb{R})$ is simple, and so the only normal subgroups of H_S are products of full factors $PSL_2(\mathbb{R})$ in H_S. Thus since ergodicity of a subgroup passes upwards, it will suffice to prove that a single factor $PSL_2(\mathbb{R}) \cong N \leq H_S$ acts ergodically on $(\hat{X}, \hat{\mu})$.

So we begin by fixing a single factor $PSL_2(\mathbb{R}) \cong N \leq H_S$, and note that by the irreducibility of Λ_S in H_S, the action (by translations) of Λ_S on H_S/N is ergodic. Furthermore, the action of Λ_S on X is strongly mixing [31, 2.3], and hence mildly mixing. Thus by [20, 1.1], it follows that the product action of Λ_S on $X \times H_S/N$ is ergodic. We show now that this implies the ergodicity of N on $(\hat{X}, \hat{\mu})$.

Let ϖ be an H_S-invariant Borel measure on H_S/N, and consider the (product) H_S-space

$$(\hat{X} \times H_S/N,\ \hat{\mu} \times \varpi) = ((X \times H_S/\Lambda_S) \times H_S/N,\ \hat{\mu} \times \varpi).$$

By Zimmer [58, 2.2.2], the ergodicity of N on $(\hat{X}, \hat{\mu})$ is equivalent to the ergodicity of H_S on $(\hat{X} \times H_S/N,\, \hat{\mu} \times \varpi)$. Let Z be any standard Borel probability space, and let

$$F : (X \times H_S/\Lambda_S) \times H_S/N \to Z$$

be an H_S-invariant Borel function. We will show that F is $\hat{\mu} \times \varpi$-a.e. constant. Let

$$\pi : (X \times H_S/\Lambda_S) \times H_S/N \to H_S/\Lambda_S$$

be the canonical projection onto $H_S/\Lambda_S = T$. Then for each $t \in T$, $\pi^{-1}(t)$ is a $t\Lambda_S t^{-1}$-invariant subset of $\hat{X} \times H_S/N$ on which $t\Lambda_S t^{-1}$ acts ergodically. (Each $t\Lambda_S t^{-1}$-action on $\pi^{-1}(t)$ is isomorphic to the ergodic Λ_S-action on $\pi^{-1}(1)$). By Proposition 4.1.1 and the H_S-invariance of F, it follows that F is $(\mu \times \varpi)$-a.e. constant on each of the fibers $\pi^{-1}(t)$. It then follows from the transitivity of H_S on H_S/Λ_S that the H_S-invariant function F is $(\hat{\mu} \times \varpi)$-a.e. constant on $\hat{X} \times H_S/N$. Again using Proposition 4.1.1, we conclude that H_S is ergodic on $\hat{X} \times H_S/N$, and the desired result follows. $\square$

We have now verified that the free Bernoulli actions of Theorem 2.3.1 fit into the context of 2.4.1. Applying 2.4.1, we therefore obtain:

- a Λ_S-invariant, μ_1-conull Borel set $X_0 \subseteq X_1$,
- a Borel reduction $\tilde{f} : X_1 \to \hat{X}_2$ from E_1 to $E_{H_T}^{\hat{X}_2}$, and
- an injective rational $\mathbb{R}$-homomorphism $\varphi : H_S \to H_T$

such that

- $\tilde{f}(\lambda x) = \varphi(\sigma^S(\lambda))\tilde{f}(x)$ for all $x \in X_0$ and for all $\lambda \in \Lambda_S$, and
- for all $\lambda \in \Lambda_S$, each component $\varphi_j(\sigma^S(\lambda))$ of $\varphi(\sigma^S(\lambda))$ is either λ^ς, or λ^ς with main diagonal scaled by -1, for some Galois automorphism $\varsigma \in \mathrm{Gal}(k_S/\mathbb{Q})$.

Before picking up the argument where we left off at the end of Chapter 6, we shall first need one more preliminary result concerning factors of Bernoulli systems.

7.2.3 Entropy and factors of Bernoulli shifts

In this section, we will show that free Bernoulli actions of Λ_S do not admit algebraic factors of a particular form. This result will play an important role in showing that the adjusted Borel reduction $\tilde{f}$ obtained from the application of Theorem 2.4.1 takes values, μ_1-a.e., in a single copy of X_2 lying inside $\hat{X}_2$. In order to show this, we will use the material on entropy developed in Section 4.5. Indeed, most of the work has already been done in Lemma 4.5.2; we shall need only the following simple corollary.

Corollary 7.2.3. *Suppose $C \leq H_T$ is a closed subgroup of H_T that contains $\varphi(\Lambda_S)$, and suppose that $M \leq C$ is a closed subgroup of C such that C/M admits a Haar probability measure m. If $(C/M, \varphi(\Lambda_S), m)$ is a factor of (X_1, Λ_S, μ_1), then C/M is trivial, ie, $C = M$.*

Proof. Suppose for the sake of contradiction that $(C/M, \varphi(\Lambda_S), m)$ is a nontrivial factor of (X_1, Λ_S, μ_1). Then $(C/M, \varphi(\Lambda_S), m)$ is also a factor of $(Y_1^{\Lambda_S}, \Lambda_S, \nu_1^{\Lambda_S})$. Fix a torsion-free unipotent element $\lambda \in \Lambda_S$. By Lemma 4.5.2, $\varphi(\lambda)$ acts on $(C/M, m)$ with positive entropy. But by Dani [9, Appendix] and the unipotence of $\varphi(\lambda)$, the translation action of $\varphi(\lambda)$ on $(C/M, m)$ has zero entropy, a contradiction. □

7.2.4 Studying the image of $\tilde{f}$ in $\hat{Y}$

We are now ready to finish the proof of Theorem 2.3.1. Indeed, our only remaining problem at this point is that $\tilde{f}$ takes values in $\hat{X}_2$ instead of in X_2, and φ takes values in H_T instead of in Λ_T. Recall that $\hat{X}_2$ is a product of various twisted copies of X_2 indexed by the cosets of Λ_T in H_T. Our goal in this section will be to show that, μ_1-a.e., $\tilde{f}$ takes values in a single copy of X_2 lying inside $\hat{X}_2$. As in Adams [2] and Thomas [52], we shall accomplish this by projecting $\tilde{f}(X_1)$ onto the second coordinate of $\hat{X}_2 = X_2 \times H_T/\Lambda_T$ and then applying Lemma 3.6.2 to the resulting image ω of μ_1.

Thus let $\eta : \hat{X}_2 \to H_T/\Lambda_T$ be the projection onto H_T/Λ_T, and define

$$\omega = (\eta \circ \tilde{f})_* \mu_1.$$

Then by 4.1.4, ω is a $\varphi(\Lambda_S)$-invariant, $\varphi(\Lambda_S)$-ergodic probability measure on the homogeneous H_T-space H_T/Λ_T. We proceed now to show that ω is supported on a singleton.

Lemma 7.2.4. *There exists $u\Lambda_T \in H_T/\Lambda_T$ such that ω is supported on $\{u\Lambda_T\}$.*

Proof. Let $C = \mathrm{Stab}_{H_T}(\omega) = \{h \in H_T \mid \omega \text{ is } h\text{-invariant}\}$. Then $C \leq H_T$ is a closed subgroup of H_T in the Hausdorff topology, and $\varphi(\Lambda_S) \leq C$. Applying Lemma 3.6.2, it follows that ω is algebraic. Hence ω is supported on a single C-orbit, say $\Omega = C \cdot x_0$, where $x_0 \in H_T/\Lambda_T$. Since C is transitive on Ω, there is a closed subgroup $M \leq C$ such that Ω and C/M are isomorphic as C-spaces. Pushing ω through this isomorphism, we obtain a C-invariant probability measure m on C/M, so that (Ω, ω) and $(C/M, m)$ are isomorphic as C-spaces. Since $\varphi(\Lambda_S) \leq C$, this implies that (Ω, ω) and $(C/M, m)$ are isomorphic as $\varphi(\Lambda_S)$-spaces. In particular, the $\varphi(\Lambda_S)$-space $(C/M, m)$ is a factor of the Λ_S-space (X_1, μ_1). But then Corollary 7.2.3 implies that $M = C$, and hence Ω is a singleton. □

7.2.5 Untwisting $(\tilde{f}, \varphi)$

We have now shown that, μ_1-a.e., the permutation group homomorphism

$$(\tilde{f}, \varphi) : (X_1, \Lambda_S) \to (\hat{X}_2, H_T)$$

takes values in the single copy $X_2 \times \{u\Lambda_T\}$ of X_2 lying inside $\hat{X}_2 = X_2 \times H_T/\Lambda_T$. Since $\varphi(\Lambda_S)$ preserves ω, we have that $\varphi(\lambda) \cdot u\Lambda_T = u\Lambda_T$ for all $\lambda \in \Lambda_S$. Hence if we conjugate φ by u^{-1} to obtain the group homomorphism

$$\tilde{\varphi}(g) = u^{-1}\varphi(g)u,$$

then $\tilde{\varphi}(\Lambda_S)$ fixes $1\Lambda_T \in H_T/\Lambda_T$, and so we must have $\tilde{\varphi}(\lambda) \in \Lambda_T$ for all $\lambda \in \Lambda_S$. But recall that here we are treating $\Lambda_S \leq H_S$ and $\Lambda_T \leq H_T$ as σ^S- and σ^T-diagonal subgroups, respectively, and hence this implies that there exists a fixed homomorphism

$$\overline{\varphi} : \Lambda_S \to \Lambda_T$$

such that for all $\theta_j \in \mathfrak{R}_T$ and for all $\lambda \in \Lambda_S$,

$$\overline{\varphi}(\lambda)^{\vartheta_{l(j)}} \quad = \quad (\pi_j^T \circ \tilde{\varphi} \circ \sigma^S)(\lambda) \quad = \quad \tilde{\varphi}(\sigma^S(\lambda))^{\vartheta_{l(j)}} \quad = \quad \pi_j^T(u)^{-1}\, \varphi_j(\sigma^S(\lambda))\, \pi_j^T(u).$$

To be absolutely precise, we define $\overline{\varphi} : \Lambda_S \to \Lambda_T$ as follows. Recall that $\vartheta_1 : k_S \to \mathbb{R}$ is the identity embedding, and recall that there is some $j \in \{1, \ldots, n\}$ such that $l(j) = 1$. Fix j_0 such that $l(j_0) = 1$, and define

$$\overline{\varphi}(\lambda) = (\pi^T_{j_0} \circ \tilde{\varphi} \circ \sigma^S)(\lambda) \quad \text{for all } \lambda \in \Lambda_S.$$

Further define

$$v = \pi^T_{j_0}(u^{-1}) \in PSL_2(\mathbb{R}).$$

Then for all $\lambda \in \Lambda_S$, we have

$$\overline{\varphi}(\lambda) = v\, \varphi_{j_0}(\sigma^S(\lambda))\, v^{-1} \in \Lambda_T,$$

where $\varphi_{j_0}(\sigma^S(\lambda))$ is either λ or λ with main diagonal scaled by -1. This shows that the element $v \in PSL_2(\mathbb{R})$ conjugates Λ_S into Λ_T. It follows immediately from Corollary 7.1.3 that $S \subseteq T$, completing the proof of Theorem 2.3.1.

7.3 Proof of Theorem 2.2.1

In this section, we prove Theorem 2.2.1, which we restate now for convenience.

Theorem 2.2.1. *Suppose that S_1, S_2 are finite nonempty sets of primes and that J_1, J_2 are (possibly infinite) nonempty sets of primes. Suppose that $|S_1| = |S_2|$. For $i = 1, 2$, suppose that:*

- $\sqrt{p} \in \mathbb{Z}_q$ *for each* $p \in S_i$ *and* $q \in J_i$;
- $L_i \leq K(J_i)$ *is closed, contains* $Z(\Gamma_{S_i})$, *and satisfies* $\mu_{J_i}(F_{S_i}(J_i, L_i)) = 1$, *where* μ_{J_i} *is Haar probability measure on* $K(J_i)/L_i$, *and* $F_{S_i}(J_i, L_i)$ *is the subset of* $K(J_i)/L_i$ *on which* Λ_{S_i} *acts freely;*
- X_i *is a* μ_{J_i}*-conull,* Λ_{S_i}*-invariant Borel subset of* $F_{S_i}(J_i, L_i)$; *and*
- E_i *is the* Λ_{S_i}*-orbit equivalence relation on* X_i.

Suppose that $f : X_1 \to X_2$ is a Borel reduction from E_1 to E_2. Then

(1) $S_1 = S_2$, *and*

(2) $(K(J_1)/L_1, \Lambda_{S_1}, \mu_{J_1})$ *and* $(K(J_2)/L_2, \Lambda_{S_2}, \mu_{J_2})$ *are virtually isomorphic.*

In order to improve notation for the proof of this theorem, throughout this section we write S and T in place of S_1 and S_2, X and Y in place of X_1 and X_2, and μ_i in place of μ_{J_i} for $i = 1, 2$. Furthermore, we shall recycle certain notation introduced in Section 7.2 for use in the present context, with new (but analogous) meanings. Such notation will be reintroduced in a clear manner at the appropriate time, and should not cause any confusion.

Assume the hypotheses of the theorem. As in the proof of Theorem 2.3.1, we will begin by verifying that the actions $\Lambda_S \curvearrowright X$ and $\Lambda_T \curvearrowright Y$ fit into the context of Theorem 2.4.1. By Zimmer [57, 2.4], the induced H_S-space $\hat{X}$ is irreducible. Each of the remaining hypotheses of 2.4.1 is clearly satisfied except for hypothesis (1), which we discuss in the next section.

7.3.1 Property (τ) and E_0-ergodicity

In order to show that the action of Λ_S on $X \subseteq K(J_1)/L_1$ is E_0-ergodic, we use the technology developed by Thomas in [52]. Recall from Section 3.4 that the groups Λ_S have Property (τ). Furthermore, it follows from the Strong Approximation Theorem [42, 7.12] that the finitely generated groups

$$\Gamma_S = SL_2(\mathcal{O}_S) \supseteq SL_2(\mathbb{Z})$$

embed densely (via the diagonal embedding) into the compact, profinite groups

$$K(J) = \prod_{p \in J} SL_2(\mathbb{Z}_p).$$

The E_0-ergodicity of $\Lambda_S \curvearrowright (X, \mu_1)$ therefore follows immediately from the following theorem proved by Thomas in [52]:

Theorem 7.3.1 ([52, Theorem 5.7])**.** *Let K be a compact second countable group, let $L \leq K$ be a closed subgroup, and let μ be the Haar probability measure on $X = K/L$. Let Γ be a finitely generated dense subgroup of K. Suppose that*

(1) K is a profinite group; and

(2) Γ has Property (τ).

Then the action of Γ on (X, μ) is E_0-ergodic. $\square$

We have now verified that the actions $\Lambda_S \curvearrowright X$ and $\Lambda_T \curvearrowright Y$ of Theorem 2.3.1 fit into the context of 2.4.1. Notice that in light of Lemma 6.2.1, we also obtain Theorem 2.2.2.

Applying 2.4.1, we obtain:

- a Λ_S-invariant, μ_1-conull Borel set $X_0 \subseteq X$,
- a Borel reduction $\tilde{f} : X \to \hat{Y}$ from E_1 to $E^{\hat{Y}}_{H_T}$, and
- an injective rational $\mathbb{R}$-homomorphism $\varphi : H_S \to H_T$

such that

- $\tilde{f}(\lambda x) = \varphi(\sigma^S(\lambda))\tilde{f}(x)$ for all $x \in X_0$ and for all $\lambda \in \Lambda_S$, and
- for all $\lambda \in \Lambda_S$, each component $\varphi_j(\sigma^S(\lambda))$ of $\varphi(\sigma^S(\lambda))$ is either λ^ς, or λ^ς with main diagonal scaled by -1, for some Galois automorphism $\varsigma \in \mathrm{Gal}(k_S/\mathbb{Q})$.

Since we are now assuming $|S| = |T|$, it follows that $H_S = H_T$, that $\varphi : H_S \to H_T$ is surjective, and that $\varphi(\Lambda_S)$ is an irreducible lattice in H_T.

Just as in the proof of Theorem 2.3.1 in the previous section, we will now need to prove a suitable result concerning the factors of the dynamical system (X, Λ_S, μ_1). This will be the point at which the proofs of Theorems 2.3.1 and 2.2.1 diverge, for we will control the factors of (X, Λ_S, μ_1) in a different way than we did the factors of Bernoulli actions. More precisely, we shall use the notion of strong mixing instead of entropy, and we will not be able to obtain as general a result as Corollary 7.2.3.

7.3.2 Controlling the factors of $\Lambda_S \curvearrowright K(J)/L$

The following argument is essentially the same as that of Thomas [52, 6.4]. Recall that $\Gamma_S = SL_2(\mathcal{O}_S)$. We remark that any quotient of $(X,\, \Lambda_S,\, \mu_1)$ is clearly also a quotient of $(K(J_1)/L_1,\, \Gamma_S,\, \mu_1)$.

Lemma 7.3.2. *Suppose that H is a finite product of copies of the group $PSL_2(\mathbb{R})$, and suppose that $M \leq H$ is a proper closed subgroup of H such that H/M admits a Haar probability measure m. Let Δ be the image in H of $PSL_2(\mathbb{Z})$ under the diagonal embedding, and suppose that Δ^+ is a countable group such that $\Delta \leq \Delta^+ \leq H$. Then $(H/M, \Delta^+, m)$ is not a quotient of $(K(J_1)/L_1, \Gamma_S, \mu_1)$.*

Proof. Let A be a finite set that we can use as an index set for the product group H, so that

$$H = \prod_{a \in A} PSL_2(\mathbb{R}).$$

For each nonempty subset $D \subseteq A$, let

$$H_D = \prod_{a \in D} PSL_2(\mathbb{R}) \leq H,$$

and let

$$\pi_D : H \to H/H_{A \setminus D}$$

be the canonical surjection, so that we may identify $\pi_D(H)$ with H_D. By Margulis [36, II.6.2], since M has finite covolume in H, there exists a nonempty subset $B \subseteq A$ and a lattice Θ in H_B such that

$$M = \Theta \times \prod_{a \in A \setminus B} PSL_2(\mathbb{R}).$$

Let $C \subseteq B$ be a minimal nonempty subset such that $\Theta_C = \pi_C(\Theta)$ is a lattice in H_C, and let m_C be the Haar probability measure on H_C/Θ_C. Notice that Θ_C is an irreducible lattice in H_C, that $(H_C/\Theta_C, \pi_C(\Delta^+), m_C)$ is a quotient of $(H/M, \Delta^+, m)$, and that $\pi_C(\Delta)$ is simply the image of $PSL_2(\mathbb{Z})$ under the diagonal embedding in H_C. Since $\pi_C(\Delta)$ is a closed, noncompact subgroup of H_C, there exists a sequence $\langle \delta_n \mid n \geq 0 \rangle$ of elements of $\pi_C(\Delta)$ that does not contain a subsequence which converges in H_C. Furthermore, by Adams [2, 6.3] together with Zimmer [58, 2.2.20], the action of H_C on $(H_C/\Theta_C, m_C)$ is strongly mixing. Therefore, if $\langle \delta_n \mid n \geq 0 \rangle$ is any such sequence and if Z is any Borel subset of H_C/Θ_C, then

$$m_C(\delta_n(Z) \cap Z) \to m_C(Z)^2 \quad \text{as } n \to \infty.$$

Now suppose that $(H/M,\, \Delta^+,\, m)$ is a quotient of $(K(J_1)/L_1,\, \Gamma_S,\, \mu_{J_1})$. Then clearly $(H_C/\Theta_C,\, \pi_C(\Delta^+),\, m_C)$ is also a quotient of $(K(J_1)/L_1,\, \Gamma_S,\, \mu_{J_1})$. Hence there exist

(1) a surjective Borel group homomorphism $\phi : \Gamma_S \to \pi_C(\Delta^+)$ and

(2) a Borel function $F : K(J_1)/L_1 \to H_C/\Theta_C$

such that the following conditions are satisfied:

(a) $F_*\mu_1 = m_C$, and

(b) for all $\gamma \in \Gamma_S$, $F(\gamma \cdot x) = \phi(\gamma) \cdot F(x)$ for μ_1-a.e. $x \in K(J_1)/L_1$.

Let $\langle \delta_n \mid n \geq 0 \rangle$ be a sequence of elements of $\pi_C(\Delta)$ that has no subsequence which converges in H_C. Let Z be a Borel subset of H_C/Θ_C such that $m_C(Z) = 1/2$, and let $W = g^{-1}(Z) \subseteq K(J_1)/L_1$. Arguing as in the proof of Adams [2, 7.3], we see that there exists an open neighborhood U of 1 in $K(J_1)$ such that for all $k \in U$,

$$|\mu_1(k(W) \cap W) - \mu_1(W)| < 1/8.$$

For each n, fix $\gamma_n \in \Gamma_S$ such that $\phi(\gamma_n) = \delta_n$. By compactness of $K(J_1)$, after passing to a suitable subsequence if necessary, we can suppose that there exists $k_0 \in K(J_1)$ such that $\gamma_n \to k_0$ as $n \to \infty$. Furthermore, since Γ_S is dense in $K(J_1)$, after replacing each γ_n by $\xi\gamma_n$ and each δ_n by $\phi(\xi)\delta_n$ for a suitably chosen element $\xi \in \Gamma_S$, we can suppose that $k_0 \in U$. Note that $\mu_1(W) = m_C(Z) = 1/2$, and that

$$\mu_1(\gamma_n(W) \cap W) = m_C(\delta_n(Z) \cap Z)$$

for all n. In particular, if $\gamma_n \in U$, then

$$|m_C(\delta_n(Z) \cap Z) - 1/2| = |\mu_1(\gamma_n(W) \cap W) - \mu_1(W)| < 1/8.$$

But this contradicts that fact that

$$m_C(\delta_n(Z) \cap Z) \to m_C(Z)^2 = 1/4 \quad \text{as } n \to \infty.$$

This completes the proof of Lemma 7.3.2. $\square$

7.3.3 Studying the image of $\tilde{f}$ in $\hat{Y}$

We are now ready to finish the proof of Theorem 2.2.1. As in Section 7.2.4, we are faced with the problem that $\tilde{f}$ takes values in $\hat{Y}$ instead of in Y, and that φ takes values in H_T instead of in Λ_T. In Section 7.2.4, we were able to show that in fact, $\tilde{f}$ takes values a.e. in a single twisted copy of X_2 in the induced space. We will not be able to obtain quite as strong a result here; however, we will be able to show that, μ_1-a.e., $\tilde{f}$ takes values in only finitely many of the copies of Y lying inside $\hat{Y}$. As before, we shall accomplish this by projecting $\tilde{f}(X)$ onto the second coordinate of $\hat{Y}$ and then applying 3.6.2 to the resulting image of μ_1, which we again denote in this new context by ω.

Thus let $\eta : \hat{Y} \to H_T/\Lambda_T$ be the projection onto the second coordinate, and let

$$\omega = (\eta \circ \tilde{f})_* \mu_1.$$

Then by Proposition 4.1.4, ω is a $\varphi(\Lambda_S)$-invariant, $\varphi(\Lambda_S)$-ergodic probability measure on the homogeneous H_T-space H_T/Λ_T.

We now proceed to show that ω is finitely supported. Our argument is based on Thomas [52, 8.7], which is based in turn on an unpublished argument of Witte Morris.

Lemma 7.3.3. *ω is supported on a finite set $\Omega_0 \subseteq H_T/\Lambda_T$.*

Proof. Let $C = \mathrm{Stab}_{H_T}(\omega) = \{h \in H_T \mid \omega \text{ is } h\text{-invariant}\}$, so that $C \leq H_T$ is closed with $\varphi(\Lambda_S) \leq C$. By Lemma 3.6.2, ω is supported on a single C-orbit Ω. Since C contains the lattice $\varphi(\Lambda_S)$ in H_T, it follows that H_T/C has finite volume. Hence, by Margulis [36, II.6.2], since $\varphi(\Lambda_S)$ is an irreducible lattice in H_T, one of the following two possibilities holds:

- $C = H_T$; or
- C is a lattice in H_T.

First suppose that $C = H_T$. So ω is supported on the single H_T-orbit Ω. Then since H_T is transitive on Ω, there exists a proper closed subgroup M of H_T such that (Ω, ω) and $(H_T/M, m)$ are isomorphic as H_T-spaces, where m is the Haar probability measure on H_T/M. Of course, this implies that they are also isomorphic as $\varphi(\Lambda_S)$-spaces. In particular, the $\varphi(\Lambda_S)$-space $(H_T/M, m)$ is a quotient of the Λ_S-space (X, μ_1). But this contradicts Lemma 7.3.2, and hence C must be a lattice in H_T. In particular, C is countable and so ω is supported on a countable set $\Omega_0 \subseteq \Omega$. Since ω is a C-invariant probability measure, this implies that Ω_0 is actually finite. $\square$

Now, ω is $\varphi(\Lambda_S)$-invariant, which clearly implies that Ω_0 is $\varphi(\Lambda_S)$-invariant, and hence we may consider the restricted action of $\varphi(\Lambda_S)$ on Ω_0. Since Ω_0 is finite, the kernel of this action must have finite index in $\varphi(\Lambda_S)$; and since φ is injective, the pullback of this kernel will have finite index in Λ_S. Hence there exists a finite index subgroup of Λ_S, call it Λ_S^0, whose image under φ acts trivially on Ω_0.

7.3.4 Obtaining the virtual isomorphism

In order to understand the next stage in our proof, it may be helpful to refer to Figure 7.1. Intuitively, we would like to adjust φ so that it sends Λ_S^0 into Λ_T. Thus we fix

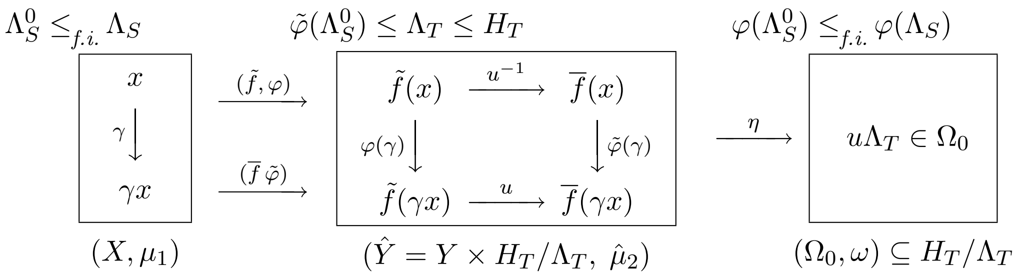


Figure 7.1:

$u\Lambda_T \in \Omega_0$, and then define $\overline{f} : X \to \hat{Y}$ by

$$\overline{f}(x) = u^{-1} * \tilde{f}(x),$$

and $\tilde{\varphi} : H_S \to H_T$ by

$$\tilde{\varphi}(g) = u^{-1}\varphi(g)u,$$

so that for all $\lambda \in \Lambda_S$ and for all $x \in X_0$,

$$\tilde{\varphi}(\lambda) \cdot \overline{f}(x) = \overline{f}(\lambda x).$$

Then $\tilde{\varphi}(\Lambda_S^0)$ fixes $\Lambda_T \in H_T/\Lambda_T$, so we must have $\tilde{\varphi}(\lambda) \in \Lambda_T$ for all $\lambda \in \Lambda_S^0$. But here we are treating $\Lambda_S^0 \leq \Lambda_S \leq H_S$ and $\Lambda_T \leq H_T$ as σ^S- and σ^T-diagonal subgroups, respectively, and hence this implies that there exists a fixed homomorphism

$$\overline{\varphi} : \Lambda_S^0 \to \Lambda_T$$

such that for all $\theta_j \in \mathfrak{R}_{k_T}$ and for all $\lambda \in \Lambda_S^0$,

$$\overline{\varphi}(\lambda)^{\vartheta_{l(j)}} \quad = \quad \tilde{\varphi}(\sigma^S(\lambda))^{\vartheta_{l(j)}} \quad = \quad \pi_j^T(u^{-1})\, \varphi_j(\sigma^S(\lambda))\, \pi_j^T(u).$$

As before, to be absolutely precise, we define $\overline{\varphi} : \Lambda_S^0 \to \Lambda_T$ as follows. There is some $j \in \{1, \ldots, n\}$, say $j = j_0$, such that $l(j_0) = 1$, where $\vartheta_1 : k_S \to \mathbb{R}$ is the identity embedding. We therefore define

$$\overline{\varphi}(\lambda) = (\pi_{j_0}^T \circ \tilde{\varphi} \circ \sigma^S)(\lambda) \quad \text{for all } \lambda \in \Lambda_S^0.$$

Further define, again as in Section 7.2,

$$v = \pi_{j_0}^T(u^{-1}) \in PSL_2(\mathbb{R}).$$

Then for all $\lambda \in \Lambda_S^0$, we have

$$\overline{\varphi}(\lambda) = v\, \varphi_{j_0}(\sigma^S(\lambda))\, v^{-1} \in \Lambda_T,$$

where $\varphi_{j_0}(\sigma^S(\lambda))$ is either λ, or λ with main diagonal scaled by -1.

In particular, the element $v \in PSL_2(\mathbb{R})$ conjugates the finite index subgroup Λ_S^0 of Λ_S into Λ_T. It follows from Corollary 7.1.3 that $S \subseteq T$. As we are assuming $|S| = |T|$, this implies that $S = T$.

Claim 7.3.4. $\overline{\varphi}(\Lambda_S^0)$ *has finite index in* Λ_T.

Proof. Recall that if we identify Λ_S with its image $\sigma^S(\Lambda_S)$ under the diagonal embedding in H_S, then Λ_S is a lattice in H_S. Furthermore, a subgroup $\Delta \leq \Lambda_S$ is a lattice in H_S if and only if $[\Lambda_S : \Delta] < \infty$ (for instance, see [2]). Hence Λ_S^0 is a lattice in H_S. Since conjugation by v (which we identify with $\sigma^S(v)$) is an automorphism of H_S, it follows that $v\Lambda_S^0 v^{-1}$ is also a lattice in H_S. In particular $v\Lambda_S^0 v^{-1}$ must have finite index in $\Lambda_S = \Lambda_T$, as desired. $\square$

Now let $\overline{f}_1 : X \to Y$ be the Borel map obtained by projecting

$$\overline{f}(x) \in \hat{Y} = Y \times (H_T/\Lambda_T)$$

into Y, and define

$$X_u = \{x \in X_0 \mid (\eta \circ \tilde{f})(x) = u\Lambda_T\}.$$

Then X_u is Λ_S-invariant, and for all $\lambda \in \Lambda_S$ and for all $x \in X_u$,

$$\overline{\varphi}(\lambda) \cdot \overline{f}_1(x) = \overline{f}_1(\lambda x).$$

Moreover, $\mu_1(X_u) = \omega(\{u\Lambda_T\}) = 1/|\Omega_0| > 0$. Therefore, since the only μ_1-positive measure, Λ_S^0-invariant subsets of X are unions of ergodic components for the action of Λ_S^0 on X, there exists an ergodic component $X_u' \subseteq X_u$ for the action of Λ_S^0 on X. Since Γ_T is dense in $K(J_2)$, the action of Λ_T on $Y \subseteq K(J_2)/L_2$ is uniquely ergodic by [52, 2.2(a)]. Hence Y decomposes into finitely many ergodic components for the action of $\overline{\varphi}(\Lambda_S^0)$ on Y, and the action of $\overline{\varphi}(\Lambda_S^0)$ on each of these components is also uniquely ergodic. By the ergodicity of the action of Λ_S^0 on X_u', there exists an ergodic component $Y_0 \subseteq Y$ for the action of $\overline{\varphi}(\Lambda_S^0)$ on Y such that $\overline{f}_1(X_u') \subseteq Y_0$. Since Λ_S^0 preserves the probability measure $(\mu_1)_{X_u'}$ on X_u', it follows that $\overline{\varphi}(\Lambda_S^0)$ preserves the probability measure $(\overline{f}_1)_*(\mu_1)_{X_u'}$ on Y_0, as the latter is just the image of $(\mu_1)_{X_u'}$ through a homomorphism of permutation groups. Since the action of $\overline{\varphi}(\Lambda_S^0)$ on Y_0 is uniquely ergodic, this implies that $(\overline{f}_1)_*(\mu_1)_{X_u'} = (\mu_2)_{Y_0}$. Hence, after deleting a μ_2-null subset of Y_0 if necessary, we may assume that $\overline{f}_1(X_0') = Y_0$. It is now evident that the standard Borel systems $(X_u', \Lambda_S^0, (\mu_1)_{X_u'})$ and $(Y_0, \overline{\varphi}(\Lambda_S^0), (\mu_2)_{Y_0})$ are isomorphic. It follows that $(K(J_1)/L_1, \Lambda_S, \mu_1)$ and $(K(J_2)/L_2, \Lambda_T, \mu_2)$ are virtually isomorphic, thus completing the proof of Theorem 2.2.1.

7.4 Suggestions for future research

In this concluding section, we outline some ideas for continued research in this area.

- As suggested in Remark 2.1.11, it seems likely that Theorem 2.2.1, and consequently Theorems 2.1.5 and 2.1.6, should still be true without the assumption that $|S_1| = |S_2|$. At present, this assumption is needed in the proof of Lemma 7.3.3; however, it would be desirable to find a way around this.

- It would be interesting to prove Theorem 2.3.1 with the groups $\Gamma_S = SL_2(\mathcal{O}_S)$ in place of their projectivized versions Λ_S. Surprisingly, given how similar Γ_S and Λ_S are to each other, this does not yet appear to be a trivial task.

- As noted in Section 7.1, most of the proof of Theorem 2.3.1, and also of Theorem 2.2.1, goes through in the more general setting of the groups $PSL_2(\mathcal{O}_{\mathbb{K}})$, where $\mathbb{K}$ is an arbitrary totally real number field and not necessarily one of the multiquadratic number fields k_S. It would be interesting to attempt to prove these theorems for the groups $PSL_2(\mathcal{O}_{\mathbb{K}})$, or to find some natural class of number fields for which this is possible. Of course, this question naturally raises the following purely algebraic question: given the totally real number fields $\mathbb{K}$ and $\mathbb{F}$, when does the inclusion $\mathcal{O}_{\mathbb{K}} \subseteq \mathcal{O}_{\mathbb{F}}$ follow from the existence of an element $v \in PSL_2(\mathbb{R})$ and a finite index subgroup $\Lambda^0_{\mathbb{K}} \leq \Lambda_{\mathbb{K}}$ such that

$$v\,\Lambda^0_{\mathbb{K}}\,v^{-1} \leq \Lambda_{\mathbb{F}}?$$

- Finally, we point out that Theorem 2.3.1 should also hold for the groups

$$\Delta_S = PSL_2(\mathbb{Z}[S]),$$

where $S = \{p_1, \ldots, p_s\}$ and $\mathbb{Z}[S] = \mathbb{Z}[1/p_1, \ldots, 1/p_s]$. In fact, this almost follows from Thomas [52, Theorem 8.2], together with Lemma 7.2.1, Lemma 7.2.2, and Corollary 7.2.3 of this thesis. The only point left to check is that Dani's theorem [9, Appendix] concerning the entropy of an affine transformation also holds for products of real and p-adic Lie groups, rather than simply real Lie groups.

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Vita

Scott Schneider

2009	Ph. D. in Mathematics, Rutgers University
1998-03	B. A. in Mathematics from the University of Wisconsin-Madison
1998	Graduated from Wauwatosa East High School.
2003-2009	Graduate fellow, Teaching assistant, Graduate assistant; Department of Mathematics, Rutgers University