Common Prior Type Spaces In Which Payoff Types and Belief Types Are Independent

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Abstract

Common prior type spaces in which for each agent the agent's payoff type and the agent's belief type are independent deserve attention as the polar opposites of common prior type spaces in which agents' beliefs determine their preferences - a class of type spaces whose special properties are much studied. We find a necessary and sufficient condition for the independence of each agent's payoff type and belief type. Different agents' payoff types must be independent. Agents may hold payoff irrelevant information. The payoff irrelevant signals that agents receive may be correlated with each other, but they must be jointly independent of all agents' payoff types. We conclude that type spaces with independent payoff types, as commonly used in game theory and mechanism design, constitute, up to payoff irrelevant information, the class of *all* type spaces in which payoff types and belief types are independent for each agent.

1 Introduction

The notion of a type space is central to the analysis of games with incomplete information (Harsanyi, 1967-68) and to mechanism design (e.g. Myerson, 1981, Bergemann and Morris, 2005). Types describe agents' payoff relevant as well as other, payoff irrelevant information, and also agents' beliefs about other agents' types, and agents' hierarchies of beliefs about other agents' beliefs, agents' beliefs about other agents' beliefs about other agents' beliefs, etc. Bayesian Nash equilibria, or, for example, correlated equilibria of games are defined with respect to a given

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type space. Type spaces are flexible modeling devices that can describe complex belief structures.

Applied game theory often focuses on "naive" type spaces, that is, common prior type spaces in which all information that an agent receives is payoff relevant. If we call the payoff relevant agent of an agent that agent's "payoff type," [1] then naive type spaces are characterized by the fact that types and payoff types are the same for each agent. Two special classes of naive payoff type spaces have received special attention. One such class consists of the naive type spaces in which different agents' types are independent (e.g. Myerson 1981). An assumption embedded in this construction is that agents' first order beliefs about other agents' types are the same, irrespective of their own type. This implies that agents' first order beliefs about other agents' types are common knowledge among the agents.

A second special class of naive type spaces that are frequently studied in the literature are type spaces in which no agent has two distinct types with identical hierarchies of beliefs. Referring to an agent's hierarchy of beliefs about another agents' types as the agent's "belief type," these type spaces are characterized by the property that "belief types determine payoff types." Implicit in this construction is the assumption that the function mapping belief types into payoff types is common knowledge among agents. In mechanism design these types spaces often allow the construction of mechanisms that elicit agents' beliefs about other agents, and by doing so also elicit agents' payoff types. Agents then earn no information rents, and the mechanism designer can "extract the full surplus" (Crémer and McLean, 1985, 1988, Neeman, 2004). A recent line of work has examined whether the sets of type spaces that have the "belief types determine payoff types" property, or that allow "full surplus extraction," are generic (Heifetz and Neeman, 2006, Chen and Xiong, 2011a, 2011b, Gizatulina and Hellwig, 2011).

The polar opposite of the condition that belief types determine payoff types is the condition that belief types and payoff types are stochastically independent for every agent, so that knowing the belief type of an agent does not allow any inferences at all about that agent's payoff type.[2] In this paper we investigate the class of type spaces for which this opposite condition, to which we shall refer as the "independence property," is true. We are interested in type spaces satisfying this strong condition because an analysis of games or mechanism design problems for such type spaces allows the modeler to exclude all effects due to correlation between payoff and belief types. Moreover, it will turn out that large portions of the existing game theoretic and mechanism design literature can be re-interpreted as being concerned with exactly the class of type spaces that satisfy the independence condition.

[1] We borrow the expression "payoff type" from Bergemann and Morris (2005).

[2] Note that the condition that we investigate is in an informal sense the *opposite*, but importantly by no means the *negation* of the "beliefs determine preferences" condition. The negation encompasses the condition that we study in this paper, but is far more general. "Environment 2""in Neeman (2004) is an example of a common prior type space in which one agent's belief types don't determine that agent's payoff types, but in which this agent's belief and payoff types are not stochastically independent either.

We restrict attention to type spaces in which agents' beliefs are derived from a common prior. We allow type spaces that are not naive, that is, in which an agent's type includes payoff-irrelevant information. Naive type spaces in which types are independent obviously have the independence property because in such type spaces all types of a given agent have the same belief types so that belief types are constant, and constant random variables are stochastically independent of any other random variable. Our interest is in the question whether there are other type spaces with the independence property. We answer this question positively, and we characterize all type spaces with the independence property. All such type spaces can be interpreted as follows: Agents have independent payoff types. They also receive further information that is potentially not independent among agents, but that is independent of all agents' payoff types. Therefore, all types of a given agent have the same belief about other agents' payoff types, as is the case in naive type spaces with independent types, but different types of the same agent may hold different beliefs about other agents' payoff irrelevant information. Thus, the class of type spaces with the independence property is a generalization of the class of naive type spaces with independent types. A simple, and not surprising, implication of our result is that common priors for which belief and payoff types are *not* independent are generic in the senses considered in the literature on the genericity of the "beliefs determine preferences" property which we mentioned earlier.[3]

What is remarkable about our characterization is that we begin with an independence assumption that refers to each agent separately: each agent's payoff type and belief type are independent, and we show that this is equivalent to a form of independence across agents: different agents' payoff types are independent, and payoff irrelevant information is independent of all agents' payoff types. Figuratively speaking, independence propagates from each agent separately to the group of agents as a whole.

Using the language of the recent literature, type spaces with the independence property differ from naive type spaces with independent types only through the introduction of "redundant types," that is, multiple types that have the same payoff types, and the same hierarchies of beliefs regarding the underlying payoff relevant uncertainty. There is thus a connection between our main result and Theorems 1 and 2 in Liu (2011), who characterizes for general common prior type spaces the connection between type spaces with redundant types, and the same type spaces without redundant types. He shows for common prior type spaces that the type space with redundant type is obtained from the corresponding type space without redundant types by adding a common prior correlation device where the correlation is conditional on the vector of agents' payoff types. Our result shows in a common prior context that the independence property holds if and only if different players' payoff types are independent of each other, and the payoff irrelevant information is independent of all players' payoff types.

Our analysis is subtly related to Aumann and Brandenburger (1995). Seeking

[3] As this is straightforward to see, but tedious to state formally, we have not included this observation in the main body of the paper.

an epistemic foundation for Nash equilibrium, they infer in their Theorem B from the assumption that beliefs are common knowledge that beliefs must be product measures. Although their model and their motivation are entirely different from ours, the proof of our main result includes an important step that is also included in Aumann and Brandenburger's proof of their Theorem B. At the end of Section 3, we shall comment further on the relation between Aumann and Brandenburger's result and ours.[4]

In the last two sections of the paper we describe the implications of our analysis for game theory and mechanism design. In game theory an exploration of the Bayesian equilibria of a strategic form game using a type space with the independence property is equivalent to the exploration of the "strategic form correlated equilibria" (Cotter, 1991, Forges, 1993) of the game with the type space in which the payoff irrelevant information is omitted. This result is closely related to Lemma 2 in Liu (2011). However, Liu studies general type spaces, and therefore his result refers to a more general version of correlated equilibrium than ours. In his version of correlated equilibrium, before suggesting strategies to agents, the "mediator" observes the agents' types. By contrast, in "strategic form correlated equilibrium" the "mediator" does not observe agents' types before recommending strategies.

In mechanism design we show for a wide variety of possible objectives of the mechanism designer, that mechanisms that are optimal for a type space with the independence property are essentially the same as the mechanisms that are optimal for the corresponding type space in which no payoff irrelevant information is provided.

2 Framework

There are $n \geq 2$ agents. We write N for the set of agents. For each agent $i \in N$ there is a finite set P_i of possible "payoff types" p_i of agent i. We borrow the expression "payoff type" from Bergemann and Morris (2005), where payoff types are the possible realizations of a signal that agent i observes, and whose realizations potentially affect i's own or other agents' payoffs in a game. The payoff type is the only signal that i observes that may affect payoffs. Agent i may make other observations, but these don't affect payoffs. In this and the next section, payoff types are in fact completely abstract. In these sections it is irrelevant whether there is an underlying game. In Sections 4 and 5, the interpretation of the elements of P_i as payoff types will, by contrast, be important. For concreteness, we shall even in Sections 2 and 3 occasionally interpret payoff types as payoff relevant information, and the reader may have this interpretation in mind throughout.

Throughout the paper, we use notations such as $p \in P \equiv \prod_{i \in N} P_i$, and $p_{-i} \in P_{-i} \equiv \prod_{j \neq i} P_j$. Also, for any non-empty, finite set X, we denote by $\Delta(X)$ the set of all probability distributions on X.

[4]We are very grateful to Qingmin Liu for pointing out the relation between our result and Aumann and Brandenburger's result.

We use type spaces to describe the agents' beliefs about their own and others' payoff types, their beliefs about these beliefs, etc. The modeling device of type spaces is due to Harsanyi (1967-68). The focus of this paper is on type spaces with a common prior. The analysis does not apply to type spaces with subjective priors.To keep our analysis straightforward, we restrict attention to finite type spaces where the common prior has full support.

Definition 1. *A* type space *is a list* $((T_i)_{i\in N}, (\hat{p}_i)_{i\in N}, \mu)$ *such that:*

1. *for every* $i \in N$, T_i *is a non-empty, finite set;*
2. *for every* $i \in N$, $\hat{p}_i$ *is a function of the form:* $\hat{p}_i : T_i \to P_i$*;*
3. $\mu \in \Delta(T)$ *where* $T \equiv \prod_{i\in N} T_i$*;*
4. $\mu(t) > 0$ *for all* $t \in T$.

Here, a (standard) implicit assumption is that the type space is common knowledge, and that each agent i observes her own type t_i, but not other agents' types t_{-i}.

Without loss of generality, we assume that the range of $\hat{p}_i$ is P_i. Writing $\mu(t_{-i}|t_i)$ for the conditional probability of t_{-i} where we condition on i's type being t_i, we define next:

Definition 2. *For a given type space* $((T_i)_{i\in N}, (\hat{p}_i)_{i\in N}, \mu)$*, agent* i*'s belief function*

$$\hat{b}_i : T_i \to \Delta(T_{-i}) \tag{1}$$

is defined by:

$$\hat{b}_i(t_i)(t_{-i}) = \mu(t_{-i}|t_i) \tag{2}$$

for every $t_i \in T_i, t_{-i} \in T_{-i}, i \in N$.

Thus, $\hat{b}_i(t_i)$ is the belief about other players' types that agent i holds if her type is t_i. This belief is derived from the prior μ by conditioning on t_i. We shall refer to $\hat{b}_i(t_i)$ also as agent i's "belief type." We write B_i for the range of $\hat{b}_i$. B_i is thus the set of all belief types. We shall write $\mu(p_i, b_i)$ for the probability assigned by μ to the set of all type vectors such that agent i's preference is p_i and agent i's belief is b_i, and similarly use notation such as $\mu(p_i)$, $\mu(b_i)$, etc.

We make throughout the following assumption which says in words that there are no "duplicate types:"

Assumption 1. *For every* $i \in N$*, if* $t_i, t_i' \in T_i$ *and* $t_i \neq t_i'$*, then* $\hat{p}_i(t_i) \neq \hat{p}_i(t_i')$ *or* $\hat{b}_i(t_i) \neq \hat{b}_i(t_i')$.

Duplicate types, that we rule out, are thus types with identical payoff types and with identical beliefs.[5] To apply our main result to type spaces in which duplicate types

[5]Our Assumption 1 is the same as Assumption 1 in Liu (2011).

exist, one has to successively "merge" duplicate types into a single type.Assumption 1 means that every type t_i is uniquely identified by t_i's payoff type $\hat{p}_i(t_i)$ and t_i's belief type $\hat{b}_i(t_i)$. Without loss of generality we therefore relabel the type space from now on as follows:

$$T_i = P_i \times B_i \text{ for all } i \in N. \tag{3}$$

Note that Assumption 1 does not rule out what the literature refers to as "redundant types," that is, multiple types with identical payoff types and hierarchies of beliefs about other players' payoff types. This is because a players' type may encode more information than just the player's payoff type and the players' beliefs about other players' payoff types. This point is crucial for our paper. The potential importance of redundant types for the analysis of incomplete information games has been emphasized by Forges (1993, pp. 284/5). The following discussion of the role of redundant types is taken from Liu (2009, p. 2117):

> "..., if the analyst knows only the payoff structures - he is unaware of (or unable to specify) some other variables that the players know, ... , but he is aware of his unawareness (or misspecification) - then a redundant type structure is a "safe" modeling choice: the players "reason" within a redundant structure *as if* they were reasoning about some parameters unknown to the analyst. In other words, the analyst should not make use of a redundant structure unless he is not sure of the players' space of basic uncertainties."

Liu (2009, 2011) provides formal results that support this interpretation of redundant types, and that apply to our model. When allowing redundant types in our model it is Liu's interpretation that we have in mind, and thus we allow that the type space is constructed by an analyst who is aware that he is unaware of some variables that players may have beliefs about.

The property of type spaces in which we are interested in this paper is the following:

Definition 3. *A type space* $((T_i)_{i\in N}, (\hat{p}_i)_{i\in N}, \mu)$ *has the* independence property *if for every* $i \in N$ *the random variables* $\hat{p}_i$ *and* $\hat{b}_i$ *are independent.*

As explained in the Introduction, we view this property as the polar opposite of the "beliefs determine payoff types" property. In type spaces with the independence property, knowing an agent's beliefs provides no information about that agent's preferences.

3 Result

Before stating our result, we give an example that illustrates the result. We observed already in the Introduction that every naive type space with independent types has the independence property trivially because each agent's beliefs are constants.

Type spaces with independent types, however, embed a very restrictive common knowledge assumption: each agent's first order beliefs are common knowledge. We therefore give an example in which agents beliefs about the other agents' types are not constant, and the agents' first order beliefs are not common knowledge.

Example 1. $N = \{1, 2\}$. *For every* $i \in N$*, the set of payoff types is* $P_i = \{p_i^1, p_i^2\}$*, and the set of types is* $T_i = \{t_i^k : k = 1, 2, 3, 4\}$*. Payoff types are given by* $\hat{p}_i(t_i^1) = \hat{p}_i(t_i^2) = p_i^1$ *and* $\hat{p}_i(t_i^3) = \hat{p}_i(t_i^4) = p_i^2$ *for* $i = 1, 2$*. The common prior* μ *is described in Figure 2. Conditional on agent 1's payoff type being* p_1^k*, his beliefs about agent 2's types are* $(1/6, 2/6, 1/6, 2/6)$ *with probability* 0.5*, and* $(2/6, 1/6, 2/6, 1/6)$ *with probability* 0.5*. This probability does not depend on* k*. Therefore, for agent 1, beliefs and payoff types are independent. A similar calculation shows that also for agent 2 beliefs and payoff types are independent.*

μ	t_2^1	t_2^2	t_2^3	t_2^4
t_1^1	$\frac{1}{24}$	$\frac{2}{24}$	$\frac{1}{24}$	$\frac{2}{24}$
t_1^2	$\frac{2}{24}$	$\frac{1}{24}$	$\frac{2}{24}$	$\frac{1}{24}$
t_1^3	$\frac{1}{24}$	$\frac{2}{24}$	$\frac{1}{24}$	$\frac{2}{24}$
t_1^4	$\frac{2}{24}$	$\frac{1}{24}$	$\frac{2}{24}$	$\frac{1}{24}$

Figure 1: The common prior μ in Example 1

There is an equivalent representation of the type space in Example 1. Note that in Example 1 the pair of the agents' payoff types, $(\hat{p}_1, \hat{p}_2)$, is independent of the pair of the agents' belief types, $(\hat{b}_1, \hat{b}_2)$. This is a stronger property than the independence property which only requires independence of payoff types and belief types agent by agent. In Example 1 one can then imagine types being determined by two independent draws: one draw determines (p_1, p_2), and another draw determines (b_1, b_2). We describe these draws in Figure 2, where the left square represents the common prior for the draw of (p_1, p_2), and the right square represents the common prior for the pair (b_1, b_2). We denote the common prior distribution of payoff types by μ', the two possible belief types of each agent by b_i^1 and b_i^2 (in the order that they were listed in the description of Example 1), and the common prior distribution of belief types by μ''.

Now note a further independence: the distribution of payoff types is a product distribution, that is, payoff types are independent across agents. This implies that agents' beliefs about other agents' payoff types are in fact constant in the model, and therefore common knowledge, as they are when types are drawn independently. The variation in agents' beliefs stems from the variation in their beliefs about other variables, that are not payoff related. These are captured by the belief distribution in Figure 2. Note that this distribution is not a product distribution.

μ'	p_2^1	p_2^2
p_1^1	$\frac{1}{4}$	$\frac{1}{4}$
p_1^2	$\frac{1}{4}$	$\frac{1}{4}$

μ''	b_2^1	b_2^2
b_1^1	$\frac{1}{6}$	$\frac{2}{6}$
b_1^2	$\frac{2}{6}$	$\frac{1}{6}$

Figure 2: An equivalent representation of the common prior in Example 1: $\mu = \mu' \times \mu''$.

The main result of this paper is that a similar representation as the one in Figure 2 can be given for any type space with the independence property.

Proposition 1. *A type space has the independence property if and only if*

$$\mu(t_1, t_2, \dots t_N) = \mu(p_1)\mu(p_2)\dots\mu(p_n)\mu(b_1, b_2, \dots, b_n) \tag{4}$$

for all $(t_1, t_2, \dots, t_n) \in T$.

Proof. It is immediate that (4) implies that $\hat{p}_i$ and $\hat{b}_i$ are independent for each agent. We prove that (4) is necessary for the independence property in three claims.

Claim 1: For all $i \in N, p_i \in P_i, b_i \in B_i, p_{-i} \in P_{-i}, b_i \in B_{-i}$:

$$\mu(p_{-i}, b_{-i}|p_i, b_i) = \mu(p_{-i}, b_{-i}|b_i) \tag{5}$$

Proof.

$$\begin{aligned}
\mu(p_{-i}, b_{-i}|b_i) &= \frac{\mu(p_{-i}, b_i, b_{-i})}{\mu(b_i)} \\
&= \sum_{p_i' \in P_i} \frac{\mu(p_i', p_{-i}, b_i, b_{-i})}{\mu(b_i)} \\
&= \sum_{p_i' \in P_i} \frac{\mu(p_i', b_i)}{\mu(b_i)} \frac{\mu(p_i', p_{-i}, b_i, b_{-i})}{\mu(p_i', b_i)} \\
&= \sum_{p_i' \in P_i} \mu(p_i'|b_i)\mu(p_{-i}, b_{-i}|p_i', b_i) \\
&= \sum_{p_i' \in P_i} \mu(p_i'|b_i) b_i(p_{-i}, b_{-i}) \\
&= \sum_{p_i' \in P_i} \mu(p_i'|b_i)\mu(p_{-i}, b_{-i}|p_i, b_i) \\
&= \mu(p_{-i}, b_{-i}|p_i, b_i)
\end{aligned} \tag{6}$$

Here, the fifth and sixth line follow from the definition of belief types.

Claim 2: For all $i \in N, p \in P, b \in B$:

$$\mu(p, b) = \mu(p_i)\mu(p_{-i}, b). \tag{7}$$

Proof.

$$\begin{aligned}\mu(p,b) &= \mu(p_i,b_i)\mu(p_{-i},b_{-i}|p_i,b_i)\\ &= \mu(p_i)\mu(b_i)\mu(p_{-i},b_{-i}|p_i,b_i)\\ &= \mu(p_i)\mu(b_i)\mu(p_{-i},b_{-i}|b_i)\\ &= \mu(p_i)\mu(p_{-i},b) \qquad (8)\end{aligned}$$

The second line follows from the independence property, and the third line follows from CLAIM 1.

CLAIM 3:[6] If for all $i \in N, p \in P, b \in B$:

$$\mu(p,b) = \mu(p_i)\mu(p_{-i},b), \qquad (9)$$

then for all $p \in P, b \in B$:

$$\mu(p,b) = \mu(p_1)\mu(p_2)\dots\mu(p_N)\mu(b) \qquad (10)$$

Proof. We prove this by induction over n, beginning with the case $n = 2$. By assumption:

$$\mu(p_1,p_2,b) = \mu(p_2)\mu(p_1,b). \qquad (11)$$

Therefore, we can complete the proof by showing:

$$\mu(p_1,b) = \mu(p_1)\mu(b). \qquad (12)$$

By assumption:

$$\mu(p_1,p_2,b) = \mu(p_1)\mu(p_2,b). \qquad (13)$$

Summing (13) over all p_2, we obtain (12).

Now suppose the claim had been shown for all numbers of agents up to some number $n \geq 2$. We prove the claim for $n+1$. By assumption:

$$\mu(p,b) = \mu(p_{n+1})\mu(p_{-(n+1)},b) \qquad (14)$$

Therefore, we can complete the proof by showing:

$$\mu(p_{-(n+1)},b) = \mu(p_1)\mu(p_2)\dots\mu(p_n)\mu(b) \qquad (15)$$

We prove this using the inductive assumption. For this, it is sufficient to show that the "if-condition" of CLAIM 3 holds for $(p_{-(n+1)},b)$:

$$\mu(p_{-(n+1)},b) = \mu(p_i)\mu(p_{-(i,n+1)},b) \qquad (16)$$

for all $i \neq n+1$. This is implied by the "if-condition" of CLAIM 3 for (p,b):

$$\mu(p,b) = \mu(p_i)\mu(p_{-i},b) \qquad (17)$$

if we sum over all p_{n+1}. □

[6] CLAIM 3 and its proof are identical to Lemma 4.6 and its proof in Aumann and Brandenburger (1995), except that the type space in Aumann and Brandenburger's model does not include a component that is analogous to the component "b" in our type space. We comment further on the relation between our work and that of Aumann and Brandenburger in the paragraph following the proof of Proposition 1.

Proposition 1 is subtly related to Theorem B in Aumann and Brandenburger (1995). In Aumann and Brandenburger's model a type space describes hierarchies of beliefs over strategies, not over payoff types. However, one can reinterpret their model, replacing strategies by payoff times. Aumann and Brandenburger then investigate the assumption that beliefs about other players' payoff types are common knowledge. They infer that beliefs have to be product measures.[7] Their assumption is stronger than ours, as the assumption that beliefs are common knowledge implies that they are independent of payoff types, but it is in another sense weaker, because it only refers to beliefs about payoff types, not to beliefs about types per se. Their conclusion is similar to ours, except that their conclusion does not address the possible existence of redundant types.

We can translate Aumann and Brandenburger's result into our setting. Suppose we say that a fact is common knowledge in our model if it is true for every $t \in T$.[8] In particular, let us say that agent i's beliefs are common knowledge if there is some $b_i \in \Delta(T_{-i})$ such that $\hat{b}_i(t_i) = b_i$ for all $t_i \in T_i$. Then Aumann and Brandenburger's proof of their Theorem B shows:

Remark 1. *All agents' beliefs are common knowledge if and only if for all $t \in T$:*

$$\mu(t) = \mu(t_1)\mu(t_2)\dots\mu(t_n). \tag{18}$$

The proof of this remark is essentially the same as the proof of Proposition 1. In particular, to show that (18) is necessary for beliefs to be common knowledge, one begins with the observation that the constancy of belief types implies: $\mu(t_{-i}|t_i)\mu(t_{-i})$ for all $t \in T$, which is the analog of Claim 1. The proof continues with analogs of Claims 2 and 3, omitting, as in the analog of Claim 1, the conditioning on belief types, as belief types are the same everywhere in the type space.

We mentioned already in the Introduction and at the beginning of this section that type spaces with independent types are important in the literature, yet extremely special. We noted at the beginning of this section that independent types imply that beliefs about others' types are common knowledge. Remark 1 adds to this the observation that the reverse is also true: common knowledge of beliefs implies that types are independent. Remark 1 thus characterizes the most prominent special case of type spaces with the independence property.[9]

[7]Combining this with the assumption of mutual knowledge of rationality, they obtain that beliefs form a Nash equilibrium.

[8]Our assumption of full support beliefs for every type implies that the standard definition of a fact being common knowledge reduces in our model to the condition that the fact is true for all t.

[9]A small caveat is that Remark 1, unlike our earlier comments, covers type spaces in which types and payoff types are not the same.

4 Implications for Game Theory

Now we introduce a game played by the n agents whom we have also considered so far. The (finite) sets of pure actions in this game are: $S_1, S_2, \ldots, S_n$. Also, for each player a utility function $u_i : S_1 \times S_2 \times \ldots \times S_n \times P \to \mathbb{R}$ is given. If we combine a type space with the action sets and utility functions, then we obtain a game of incomplete information. We shall refer to this game as "the incomplete information game generated by the type space." A pure strategy of player i is a mapping: $\sigma_i : T_i \to S_i$. Denote the set of all pure strategies of player i by Σ_i. We define: $\Sigma = \prod_{i \in N} \Sigma_i$.

Our goal is to find a connection between the Bayesian equilibria of a game generated by a type space that has the independence property and the equilibria, for an appropriate equilibrium concept, of the game in which we have dropped the payoff irrelevant component from the type space. If we find such a relation, it will be possible to analyze games with independent payoff and belief types without taking account of the possibility of payoff irrelevant information, and yet at the same time capture the results that an analysis of the Bayesian equilibria of all incomplete information games generated by a type space with independent payoff and belief types would yield.

It turns out that for our purposes the relevant equilibrium concept for the analysis of the game without payoff irrelevant information is a version of correlated equilibrium. Care is needed regarding the precise definition of a correlated equilibrium. Cotters (1991, 1994), Forges (1993, 2006), Liu (2011), and others distinguish different notions of correlated equilibria of incomplete information games. In this paper the appropriate notion is what Forges refers to as "strategic form correlated equilibrium" (Cotters, 1991, and Forges, 1993). A "strategic form correlated equilibrium" is a probability distribution γ on Σ that is a correlated equilibrium in the sense of Aumann (1974, 1987) of the strategic form of the incomplete information game. A Bayesian equilibrium is a strategic form correlated equilibrium γ that is the product of its marginals on the n pure strategy sets Σ_i.[10]

To conduct our analysis formally, we next need to be precise about what it means to drop the sets B_i from a type space, and what it means to re-introduce them. This is done in the following definition.

Definition 4. *(i) For given type space with the independence property* $((T_i)_{i \in N}, (\hat{p}_i)_{i \in N}, \mu)$ *such that:* $T_i = P_i \times B_i$, *the* corresponding reduced type space $((T'_i)_{i \in N}, (\hat{p}'_i)_{i \in N}, \mu')$ *is:* $((T'_i)_{i \in N}, (\hat{p}'_i)_{i \in N}, \mu')$ *where:* $T'_i = P_i$ *for all* $i \in N$, $\hat{p}'_i(p_i) = p_i$ *for all* $i \in N$ *and* $p_i \in P_i$, *and* $\mu'(p_1, p_2, \ldots, p_n) = \mu(p_1)\mu(p_2)\ldots\mu(p_n)$ *for all* $(p_1, p_2, \ldots, p_n) \in P$.

(ii) For given type space with the independence property $((T_i)_{i \in N}, (\hat{p}_i)_{i \in N}, \mu)$ *where* $T_i = P_i$ *for all* $i \in N$, *a* corresponding augmented type space $((T'_i)_{i \in N}, (\hat{p}'_i)_{i \in N}, \mu')$ *is a type space with the independence property such that the corresponding reduced space is* $((T_i)_{i \in N}, (\hat{p}_i)_{i \in N}, \mu)$.

[10] To simplify our notation, we use Milgrom and Weber's (1985) distributional approach to the representation of mixed strategies.

Next, we introduce a correspondence between a vector of mixed strategies for an incomplete information game generated by a type space with the independence property and a correlated strategy for the incomplete information game generated by the type space for which the payoff irrelevant information is dropped.

Definition 5. *Let γ be a product distribution on the set of pure strategy combinations Σ in the incomplete information game generated by a type space with the independence property. Then the* equivalent probability distribution γ' on the set of pure strategy combinations Σ' in the game generated by the corresponding reduced type space *is defined by:*

$$\gamma'(s') = \sum_{s \in S} \left(\gamma(s)\mu\left(\{(b_1, \ldots, b_N) | s_i(p_i, b_i) = s'_i(p_i) \text{ for all } i \in N \text{ and } p_i \in P_i\}\right)\right) \tag{19}$$

for all $s' \in \Sigma'$.

Our result is:

Proposition 2. *(i) Let γ be a Bayesian equilibrium of the incomplete information game generated by a type space with the independence property. Then the equivalent probability distribution γ' on the set of pure strategy combinations Σ' in the incomplete information game generated by the corresponding reduced type space is a strategic form correlated equilibrium of that incomplete information game.*

(ii) Let γ' be a strategic form correlated equilibrium of the incomplete information game generated by a type space with the independence property in which $T_i = P_i$ for all $i \in N$. Then there are a corresponding augmented type space, and a product distribution γ on the space of pure strategies Σ' in the incomplete information game generated by the augmented type space such that γ' is equivalent to γ, and such that γ is a Bayesian equilibrium of that incomplete information game.

Proposition 2 is a re-statement of the revelation principle for our model. We therefore omit a formal proof. Cotter (1991, p. 54) and Forges (1993, p. 289) observed that the revelation principle applies to the strategic form correlated equilibrium. An appropriately phrased version of part (i) of Proposition 2 remains true if one replaces strategic form correlated equilibrium by agent normal form correlated equilibrium, because, roughly speaking, every strategic form correlated equilibrium is also an agent normal form correlated equilibrium (Forgers, p. 290). It is not true, however, that every agent normal form correlated equilibrium is a strategic form correlated equilibrium (see Example 3 in Forges (1993)), and thus part (ii) of Proposition 2 does not hold for agent normal form correlated equilibria.

The question answered by Proposition 2 for Bayesian equilibria can also be asked for other game theoretic solution concepts. An alternative to Bayesian equilibria is in particular the concept of rationalizability. Several notions of rationalizability for incomplete information games have been proposed in the literature. If we employ the concept of "interim correlated rationalizability" as defined by Dekel, Fudenberg and Morris (2007), then the result is simple. According to Proposition 1 in Dekel, Fudenberg and Morris (2007), the set of interim correlated rationalizable strategies of a player only depends on that player's hierarchy of beliefs about payoff relevant information. It is not affected by payoff irrelevant information included in the type space. Therefore, it is without loss of generality in our context, in which we postulate the independence property, to analyze the set of interim correlated rationalizable strategies using the reduced type space in which only payoff types are included.

5 Implications for Mechanism Design

Next, we examine the implications of our analysis for mechanism design. We consider the same n agents as in the previous sections, as well as a mechanism designer. There are a (finite) set of possible outcomes Y, and for every agent i a utility function $u_i : P \times Y \to \mathbb{R}$. The mechanism designer supposes that the agents' information is described by a type space with the independence property. The mechanism designer chooses a game form, consisting of strategy sets for each agent, a mapping of strategies into outcomes, and a Bayesian equilibrium of the incomplete information game defined by the game form, the utility functions, and the type space. We leave the mechanism designer's objective function unspecified except that we assume that it only depends on the implied mapping between agents' payoff types and probability distributions over outcomes. By the revelation principle we can restrict attention to direct game forms $q : T \to \Delta(Y)$ such that truth telling is a Bayesian equilibrium in the corresponding incomplete information game. We refer to such direct game forms as "incentive compatible."

Our objective is to find a correspondence between the direct and incentive compatible mechanisms for a type space with the independence property and the direct and incentive compatible mechanisms for the corresponding reduced type space. Here, we use the terminology for type spaces introduced in the previous section. We shall find such a correspondence if we focus on the mapping between payoff types and probability distribution over outcomes. As we have postulated that the mechanism designer's objective depends only on that mapping, our result implies that mechanisms that are optimal for a type space with the independence property and mechanisms that are optimal for the the corresponding type space in which all payoff irrelevant information has been removed can achieve exactly the same values of the mechanism designer's objective function. It is therefore without loss of generality to study the mechanism designer's maximization problem only for the reduced type space, as the literature has mostly done.

We first define how we relate direct mechanisms for a type space with the independence property to direct mechanisms for the same type space but without payoff irrelevant information.

Definition 6. *(i) Consider a direct mechanism $q : T \to \Delta(Y)$ for a type space with the independence property. The* equivalent direct mechanism for the corresponding reduced type space *is the mechanism* $q' : T' \to \Delta(Y)$ *where for every* $p \in P$ *and* $y \in Y$ *we have:*

$$q'(p, y) = \sum_{b \in B} (q((p, b), y)\mu(b)) . \tag{20}$$

Here $q(t, y)$ *denotes the probability that a direct mechanism assigns to outcome* y *when the vector of types is* t.

(ii) Consider a type space with the independence property where $T_i' = P_i$ *for all* $i \in N$, *and a* corresponding augmented type space. *Let* $q : T' \to \Delta(Y)$ *be a direct mechanism for the first type space. Then the* equivalent direct mechanism for the augmented type space *is the mechanism* $q : T \to \Delta(Y)$ *where for every* $t = (p_i, b_i)_{i \in N}$ *and* $y \in Y$ *we have:*

$$q(t, y) = q'(p, y). \tag{21}$$

Our result is:

Proposition 3. *(i) If a direct mechanism* $q : T \to \Delta(Y)$ *for a type space with the independence property is incentive compatible, then the equivalent direct mechanism for the corresponding reduced type space is incentive compatible.*

(ii) If a direct mechanism $q : T \to \Delta(Y)$ *for a type space with the independence property and with* $T_i' = P_i$ *for all* $i \in N$ *is incentive compatible, then the equivalent direct mechanism for a corresponding augmented type space is incentive compatible.*

Part (ii) is immediate, as in the augmented type space agents simply ignore the payoff irrelevant information B which then is strategically irrelevant as well. Like Proposition 2, part (i) of Proposition 3 is a version of the revelation principle. In particular, suppose the true type space were the reduced type space, but the mechanism designer provided the payoff irrelevant information B to agents as part of an extensive form mechanism. By the standard revelation principle, the mechanism could collapse such an extensive form mechanism into a direct mechanism in which truth-telling is an equilibrium. This is essentially what part (i) of Proposition 3 says. We omit the proof of Proposition 3.

Propositions 2 and 3 together indicate that a mechanism designer's range of possibilities does not expand if the mechanism designer is allowed to suggest a strategic form correlated equilibrium to agents rather than a Bayesian equilibrium.

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