Diversity, Disagreement, and Information Aggregation[*]

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Abstract

Two imperfectly informed experts are hired to advise a decision maker. The experts are assumed to report their private information truthfully. In this paper we compare the informativeness of different joint (conditional on the true state) distributions of the experts' private signals, keeping the conditional marginal distribution of each expert's private signal given and fixed. Our comparisons use Blackwell's (Blackwell, 1951) notion of informativeness. We interpret "diversity" as an absence of perfect correlation among experts' signals. Such diversity manifests itself in a positive probability that the experts disagree on which state of the world is more likely the true state. We find that joint distributions in which experts disagree more frequently often have an advantage over distributions in which disagreement is observed rarely. Disagreement may thus be a manifestation of beneficial diversity.

Keywords: joint distribution, disagreement, Blackwell dominance.

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1. Introduction

Diversity of the membership of decision making committees is often viewed as desirable. There are many possible reasons for the desirability of diversity. In this paper we focus on one particular such reason: diversity may increase the value of committee members' shared information. To focus on this particular reason we consider a committee the members of which have no reason to strategically misrepresent their private information, and therefore report this information truthfully. The committee's information thus is the combination of all members' private information. If committee members' private information is perfectly correlated, then diversity is minimized, and also the value of the committee's combined information is as if there was only a single committee member. Not perfectly correlated private information, and thus some form of diversity, will be more valuable. But which precise form of diversity increases the informativeness of the committee members' combined information?

To formalize this question we focus in this paper on the *joint* distribution of committee members' private signals, conditional on the possible true states of the world, and we take as given and fixed the *marginal* distribution of each member's private signals, conditional on the possible true states of the world. We use Blackwell's (Blackwell, 1951) order of experiments to compare different possible conditional joint distributions of the private signals. Intuitively, we interpret lack of perfect correlation as "diversity," and ask which form, and how much, diversity is desirable in the sense of leading to Blackwell more informative joint signals. Absence of perfect correlation expresses itself in the positive probability of disagreements among experts. We ask, in particular, when lower, or higher, probability of disagreements implies lower, or higher, Blackwell informativeness.

We assume that there are only two possible states of the world. Blackwell comparisons are known to be less complex when there are only two states of the world than when there are more than two states of the world (see e.g. Börgers, 2024). We also focus for simplicity on the case that there are only two experts. Finally, in most parts of the paper, we assume that each expert's private signal has only two realizations. This framework facilitates the definition of "agreement" and "disagreement" among experts. By "agreement" we mean that each expert's signal realization, by itself, makes the same state appear more likely than it is under the prior. The opposite corresponds to the case of "disagreement." It turns out that understanding the case of two realizations is a useful tool for analyzing the case of more than two realizations.

We show how our findings for the two realization case can be used to derive related results about the more general case.

We begin by asking whether the Blackwell worst joint distribution minimizes the probability of disagreement. Posing the question in this way assumes, implicitly, that there is a unique Blackwell worst joint distribution, that is, a joint distribution that is Blackwell dominated by all other distributions. We prove that this is indeed the case. We then show that Blackwell worst joint distributions either minimize the probability of disagreement, or make disagreement uninformative.

We next focus on the case that signals are conditionally independent. This is frequently assumed in the literature. We prove that conditionally independent signals are always Blackwell dominated by other joint distributions which have a higher probability of disagreement among experts.[1] We describe some features of the upper contour set, in the Blackwell order, of conditionally independent signals. Some of these results apply also to joint distributions other than conditional independence.

Finally, we ask which joint signal distributions are not Blackwell dominated. We find that the set of Blackwell undominated joint distributions is a surprisingly small subset of the set of all joint distributions that are compatible with given marginals, but it may contain more than one element. Specifically, we show that Blackwell undominated joint distributions can be found by maximizing certain probabilities of disagreements among experts.

We acknowledge that not all joint distributions may be feasible committee compositions. This is why our focus is not exclusively on Blackwell undominated joint distributions, but also on upper contour sets of given distributions. These sets indicate how improvements can be achieved even if undominated joint distributions are not feasible.

We distinguish two channels by which increased frequency of disagreements among experts may increase Blackwell informativeness. The first channel is that increased probability of disagreements may make agreements more informative. The second channel by contrast makes disagreements themselves informative. Sometimes, the decision maker may find joint distributions desirable because they allow the decision maker to learn from the way in which the experts disagree with each other.

In mathematical terms we are studying in this paper the Blackwell order on a set of pairs of conditional probability distributions fixing the number of signal realizations.

[1] Getting an "independent second opinion" might not always be the best one can do.

Many of our results are a reflection of simple and more general results in this setting. We develop the general theme in Section 2. We also introduce in that section the notion of "strict Blackwell dominance," a mild refinement of Blackwell dominance, which is simply the asymmetric part of the Blackwell dominance relation. Strict Blackwell dominance will actually be the focus of our analysis.

We present our model of diversity and disagreement in Section 3. In Section 4 we consider joint distributions of signals that are worst in the Blackwell order, asking whether these distributions also minimize the probability of disagreement. In Section 5 we consider how a given joint distribution can be improved in the Blackwell order, giving particular attention to potential improvements of conditionally independent signals. In Section 6 we consider joint distributions that are not Blackwell dominated by any other distributions. Section 7 considers generalizations of our results when signals have more than two realizations, and Section 8 concludes.

Related Literature: This paper is related to Ladha (1992) who considers in a common interest setting the likelihood that non-strategic, honest majority voting will produce the correct choice from a set of two alternatives. Unlike traditional formalizations of Condorcet's Jury Theorem Ladha (1992) allows for voters' signals to be positively or negatively correlated, and he argues extensively for the real world plausibility of such correlation. He demonstrates how positive correlation among votes reduces the likelihood of the vote being correct. Majority voting is for a finite number of voters not necessarily the best way of pooling voters' information. By contrast, in this paper we assume that the pooled information is used optimally.[2]

Hong and Page (2024) pursue a related, but different approach to collective decision making by majority voting. They assume that individuals' voting strategies for a binary vote are exogenously given, and describe the likelihood of accurate decisions as a function of the average probability that individual votes are correct, and of a measure of diversity of individuals' votes, where diversity is interpreted as the likelihood of disagreements. Their results say that increased diversity may, but need not, imply increased accuracy.[3]

Another applied context in which the issue of correlation among multiple signals has been raised is the principal-agent problem. This setting differs from ours because the unobserved true state of the world is endogenous - it is the agent's unobserved

[2]See, for example, Pivato (2017) for a summary of subsequent literature on correlated information and democratic decision making.

[3]Related work is Hong and Page (2009).

action. But Rajan and Sarath (1997) obtain results some of which, as they observe, carry over to the setting in which the state is exogenous. On can show that those results in Rajan and Sarath (1997) that extend to the case of an exogenous state are implications of our more general Proposition 5.

Clemen and Winkler (1985) study a model in which state and signals are jointly normally distributed. They show for the case of positive correlation among signals that a smaller correlation makes signals more beneficial. Their model, unlike ours, does not allow the conditional correlation of signals to depend on the true state, and they invoke an additional auxiliary assumption[4] that rules out some effects that we study. They mention speculatively that in some contexts negatively correlated signals may be more valuable than independent signals. This speculation is related to our Proposition 5.

DiTillio et al. (2021) investigate a particular form of dependence among signals, namely that generated by selecting a few maximal signal realizations from a larger independent sample. In comparison to independent sampling, this selection creates not only dependence among signals, but it also modifies the marginal distributions of signals. DiTillio et al. (2021) construct a multi-dimensional extension of the Lehmann (1988) order of experiments and show that sample selection may increase or reduce the Lehmann informativeness of experiments in comparison to independent sampling.

This paper is also related to Börgers et al. (2013) in which complementarity of signals is defined. One can interpret our paper as a study of ways in which the complementarity of two signals can be improved. By contrast, Börgers et al. (2013) defines complementarity as a binary notion: signals are either complementary, or they are not.

Our paper is also related to de Oliveira et al. (2023). They consider a decision maker who has access to two signals and who knows the conditional marginal distributions of these signals, but does not know their joint distribution. Their decision maker uses a maxmin criterion to resolve this uncertainty, and therefore focuses on Blackwell worst distributions. In Section 4 below we also consider Blackwell worst joint distributions. While de Oliveira et al. (2023) already show the existence of such distributions in the case of two states, we also prove their uniqueness and analyze some of their properties.

Brooks et al. (2024) characterize when one signal is preferred to another regardless of the decision problem and regardless of the joint distribution of these signals and

[4]Their inequality (13).

other signals that might be available to the decision maker. They obtain a condition that is significantly more demanding than Blackwell dominance. Their objective is thus to rank marginal distributions regardless of the joint distribution, whereas our paper compares joint distributions for fixed marginal distributions.

2. Strict Blackwell Dominance: General Results

Suppose that there is a finite set of states of the world Ω with a full support prior distribution. We fix a finite set S of possible signal realizations. A signal P is a mapping:

$$P : \Omega \to \Delta(S)$$

where $\Delta(S)$ is the set of all probability distributions over S. P can be identified with a Markov matrix where rows correspond to states, columns correspond to signal realizations, and each entry describes the probability of a signal realization conditional on a state. Let $\mathcal{P}$ be the set of all such matrices. The set $\mathcal{P}$ is obviously convex.

A "finite decision problem" is a pair (A, u) where A is a finite set (the set of actions), and u is a utility function of the form $u : A \times \Omega \to \mathbb{R}$. We write $V((A, u), P)$ for the ex ante expected utility that the decision maker obtains when first observing the realization s of signal P and then choosing the action that maximizes expected utility conditional on s. A signal P Blackwell dominates a signal P' if $V((A, u), P) \geq V((A, u), P')$ for all finite decision problems (A, u) (Blackwell, 1951). In this paper we shall also use the notion of "strict Blackwell dominance."

Definition 1. *Signal P "strictly Blackwell dominates" signal P' if P Blackwell dominates P' but P' does not Blackwell dominate P.*

Thus, P strictly Blackwell dominates P' if $V((A, u), P) \geq V((A, u), P')$ for all finite decision problems (A, u), with strict inequality for at least one finite decision problem (A, u).

We shall be interested in the lower and upper contour sets in the strict Blackwell order of given signals P. A key observation that helps to derive properties of these sets is that for fixed finite decision problem (A, u) the value function $V((A, u), P)$ is convex in P:

Proposition 1. *For all finite decision problems (A, u), all $P, P' \in \mathcal{P}$, and all $\lambda \in [0, 1]$:*

$$V((A, u), \lambda P + (1 - \lambda)P') \leq \lambda V((A, u), P) + (1 - \lambda)V((A, u), P').$$

Proof. $V((A,u), \lambda P + (1-\lambda)P')$ is the expected utility that the decision maker obtains when choosing conditionally optimal actions in (A,u) after observing the signal that follows distribution P with probability λ and distribution P' with probability $(1-\lambda)$, and if the decision maker is *not* told whether the signal was generated by P or by P'. By contrast, $\lambda V((A,u), P) + (1-\lambda) V((A,u), P')$ is the expected utility in the same situation if the decision maker *is* told whether the signal was generated by P or P'. Because the value of information for a decision maker is always non-negative, the latter expression is at least as large as the former expression. □

The lower contour sets of a convex function are convex. The lower contour sets of the strict Blackwell dominance relation are therefore intersections of convex sets, and thus themselves convex. A precise statement of this observation is as follows.

Lemma 1. *If $P \in \mathcal{P}$ strictly Blackwell dominates both $P' \in \mathcal{P}$ and $P^* \in \mathcal{P}$ and if $\lambda \in (0,1)$, then P strictly Blackwell dominates $\lambda P' + (1-\lambda)P^*$.*

Proof. The convexity of V immediately implies that P Blackwell dominates $\lambda P' + (1-\lambda)P^*$. It remains to show that there exists a finite decision problem (A,u) such that:

$$V((A,u), P) > V((A,u), \lambda P' + (1-\lambda)P^*).$$

Because P strictly Blackwell dominates P' we can find a decision problem (A,u) such that $V((A,u), P) > V((A,u), P')$. For this decision problem we have:

$$V((A,u), P) > \lambda V((A,u), P') + (1-\lambda) V((A,u), P^*) \geq V((A,u), \lambda P' + (1-\lambda)P^*),$$

The first inequality follows from the choice of (A,u) and the fact that P Blackwell dominates P^*, and the second inequality follows from the fact that V is convex. This sequence of two inequalities implies what we had to show. □

A slightly different, but closely related result about lower contour sets is the following. We omit the proof.

Lemma 2. *Let $P, P' \in \mathcal{P}$ and suppose P strictly Blackwell dominates P'. Then for all $\lambda \in [0,1)$, P strictly Blackwell dominates $\lambda P + (1-\lambda) P'$.*

The upper contour sets of convex functions are the complements of convex sets. The upper contour sets of the strict Blackwell order are intersections of such sets. The following lemma states a property of the complements of convex sets that is preserved under intersection. If signal P' is in the lower contour set of a signal P then the line segment pointing from P into the opposite direction of P' is in the upper contour set of P.

Lemma 3. *Let $P, P' \in \mathcal{P}$ and suppose P strictly Blackwell dominates P'. If $\lambda > 0$ and $P + \lambda(P - P') \in \mathcal{P}$ then $P + \lambda(P - P')$ strictly Blackwell dominates P.*

Proof. Define $P^* \equiv P + \lambda(P - P')$. Note that:

$$P = \frac{\lambda}{1+\lambda}P' + \frac{1}{1+\lambda}P^*.$$

Because V is convex, for every finite decision problem (A, u):

$$V\left(\left(A,u\right),P\right) \leq \frac{\lambda}{1+\lambda}V\left(\left(A,u\right),P'\right) + \frac{1}{1+\lambda}V\left(\left(A,u\right),P^*\right).$$

Because P Blackwell dominates P', this implies:

$$V\left(\left(A,u\right),P\right) \leq \frac{\lambda}{1+\lambda}V\left(\left(A,u\right),P\right) + \frac{1}{1+\lambda}V\left(\left(A,u\right),P^*\right).$$

This is equivalent to:

$$V\left(\left(A,u\right),P\right) \leq V\left(\left(A,u\right),P^*\right).$$

Therefore, P^* Blackwell dominates P. It is easy to see that the strict Blackwell dominance of P over P' implies that there exists a finite decision problem in which the last inequality is strict, and thus P^* strictly Blackwell dominates P. □

A special case of Lemma 3 arises when there is a subset $\widehat{\mathcal{P}}$ of $\mathcal{P}$ such that one distribution in $\widehat{\mathcal{P}}$ is strictly Blackwell dominated by *all* other distributions in $\widehat{\mathcal{P}}$. In this case *every* distribution in the interior of $\widehat{\mathcal{P}}$ can be strictly Blackwell improved upon by moving away from the "Blackwell worst" distribution on a straight line that points into the opposite direction of the worst distribution while staying within $\widehat{\mathcal{P}}$.[5] This is the case with which Proposition 5 below deals.

We can extend Lemma 3 slightly to obtain the following monotonicity result.

Lemma 4. *Let $P, P' \in \mathcal{P}$ and suppose P strictly Blackwell dominates P'. If $\lambda > \lambda' > 0$ and $P + \lambda(P - P') \in \mathcal{P}$, then $P + \lambda(P - P')$ strictly Blackwell dominates $P + \lambda'(P - P')$.*

[5]If the distribution itself is the "Blackwell worst" one, any movement away from it is obviously a Blackwell improvement.

Proof. Define $P^* \equiv P + \lambda(P - P')$ and $P^\circ \equiv P + \lambda'(P - P')$. By Lemma 3 P° strictly Blackwell dominates P. Moreover:

$$P^* = P^\circ + \frac{\lambda - \lambda'}{\lambda'}(P^\circ - P).$$

Therefore, applying Lemma 3 again, P^* strictly Blackwell dominates P°. □

In addition to the lower and upper contour sets of the strict Blackwell order we shall also be interested in distributions that are not strictly Blackwell dominated. There is often a close connection between choices that are not dominated and choices that are optimal in some circumstances.[6] Therefore it is worthwhile to record the following observation about potential maximizers of the value function $V((A, u), P)$ for given and fixed finite decision problem (A, u). The result is known as the *Bauer Maximum Principle*, and we omit its proof.

Lemma 5. *Let $\widehat{\mathcal{P}}$ be a compact and convex subset of $\mathcal{P}$. Let (A, u) be a finite decision problem. Then there exists an extreme point $\hat{P}$ of $\widehat{\mathcal{P}}$ such that:*

$$V((A, u), \hat{P}) \geq V((A, u), P) \quad \text{for all } P \in \widehat{\mathcal{P}}.$$

Extreme points of a set of admissible signal distributions will play a key role in our characterization of signals that are not strictly Blackwell dominated.

Next, we shall show a simple decomposability property of Blackwell dominance. This is relevant in the context of our paper because we focus on signals with a small number of realizations. Decomposability of Blackwell dominance means that results about Blackwell dominance when there is a small number of signal realizations imply results about Blackwell dominance when there is a large number of signal realizations.

Let $P \in \mathcal{P}$. For any $\hat{S} \subseteq S$ and $\omega \in \Omega$ we denote by $P(\omega)(\hat{S})$ the probability that P assigns in state ω to signal realizations in $\hat{S}$. If $P(\omega)(\hat{S}) > 0$ for all $\omega \in \Omega$ then we denote by $P|_{\hat{S}}$ the signal that has realizations in $\hat{S}$ and where for each $\omega \in \Omega$ the distribution $P|_{\hat{S}}(\omega)$ is the conditional distribution of $P(\omega)$ on $\hat{S}$.

Proposition 2. *Suppose $P, \hat{P} \in \mathcal{P}$, and let $\{S_1, \cdots, S_n\}$ be a partition of S. If for all $i \in \{1, \cdots, n\}$:*

$$P(\omega)(S_i) = \hat{P}(\omega)(S_i) > 0 \text{ for all } \omega \in \Omega,$$

[6]See, for example, Theorem 2 in Cheng ⓡ Börgers (2024). That result does not exactly apply, however, to our setting because the set of signal distributions is infinite, and because the set of all finite decision problems is infinite dimensional.

and if for all $i \in \{1, \cdots, n\}$:

$$\hat{P}|_{S_i} \text{ Blackwell dominates } P|_{S_i},$$

then $\hat{P}$ Blackwell dominates P. If, in addition, there exists a $j \in \{1, \cdots, n\}$ such that $P(\omega)(S_j) > 0$ for some $\omega \in \Omega$ and $\hat{P}|_{S_j}$ strictly Blackwell dominates $P|_{S_j}$, then $\hat{P}$ strictly Blackwell dominates P.

Proof. Consider an arbitrary finite decision problem (A, u), and denote by $V((A, u), P|s)$ the conditional expected utility that the decision maker obtains when choosing optimally after observing signal realization s when the signal is distributed according to P. If we denote by $P(s)$ the probability of observing signal realization s we have:

$$V\left(\left(A,u\right),P\right)=\sum_{i=1}^{n}\sum_{s\in S_i}P(s)V\left(\left(A,u\right),P|s\right).$$

If we define for every $i = 1, 2, \ldots, n$ the probability $P(S_i) = \sum_{s \in S_i} P(s)$ we can write:

$$V\left(\left(A,u\right),P\right)=\sum_{i=1}^{n}P(S_i)\sum_{s\in S_i}\frac{P(s)}{P(S_i)}V\left(\left(A,u\right),P|s\right).$$

Using the analogous notation for signal $\hat{P}$, we can write:

$$V((A,u),\hat{P})=\sum_{i=1}^{n}\hat{P}(S_i)\sum_{s\in S_i}\frac{\hat{P}(s)}{\hat{P}(S_i)}V((A,u),\hat{P}|s).$$

The assumptions of Proposition 2 imply that for every $i \in \{1, 2, \ldots, n\}$:

$$\hat{P}(S_i) = P(S_i),$$

and

$$\sum_{s\in S_i}\frac{\hat{P}(s)}{\hat{P}(S_i)}V\left(\left(A,u\right),\hat{P}|s\right)\geq\sum_{s\in S_i}\frac{P(s)}{P(S_i)}V\left(\left(A,u\right),P|s\right).$$

We thus have:

$$V(\left(A,u\right),\hat{P})\geq V\left(\left(A,u\right),P\right),$$

which proves the first part of Proposition 2.

To prove the second part we note that the additional condition implies that there exists a decision problem $(\hat{A}, \hat{u})$ such that:

$$\sum_{s\in S_j}\frac{\hat{P}(s)}{\hat{P}(S_j)}V((\hat{A},\hat{u}),\hat{P}|s)>\sum_{s\in S_j}\frac{P(s)}{P(S_j)}V((\hat{A},\hat{u}),P|s)).$$

Combining this with the argument in the first part of this proof we get:

$$V((\hat{A},\hat{u}),\hat{P})>V((\hat{A},\hat{u}),P),$$

which proves the second part of Proposition 2.

□

Suppose we wanted to modify a given signal P so that the modified signal strictly Blackwell dominated the original signal. In the next sections of this paper we describe how this can be done if there is only a small number of signal realizations. Proposition 2 shows one can obtain strict Blackwell improvements for signals with a larger number of realizations by focusing on a small subset of all signal realizations, modifying only the conditional probabilities of signal realizations in that subset, and leaving the conditional probabilities of all other signal realizations unchanged. Further strict Blackwell improvements can be obtained by iterating this approach. We shall return to this idea in Section 7.

3. A Simple Model of Diversity and Disagreement

We consider a decision maker who consults two experts, $i \in \{A, B\}$, to gather information about the state of the world. There are two possible states of the world: ω_g and ω_b, where for concreteness we refer to ω_g as the "good state" and to ω_b as the "bad state of the world." Both states have prior probability 1/2. Each expert i observes a signal s^i. Proposition 2 indicates that results for the case that the number of signal realizations is small can be used to derive results for the case of many signal realizations. Here, we assume that each signal s^i has just two possible realizations: s^i_g and s^i_b. The probability that expert A's signal equals s^A_g in state ω_g is α_g, and the probability that expert A's signal equals s^A_b in state ω_b is α_b. The analogous probabilities for expert B are denoted by β_g and β_b. We assume that all these probabilities are strictly between 0 and 1. We also assume that both signals are not completely uninformative. It is then without loss of generality to assume that for each expert i Bayesian updating after observing signal realization s^i_g leads to an updated probability of state ω_g larger than 1/2, and observing signal realization s^i_b leads to an updated probability of state ω_b larger than 1/2. This assumption holds if and only if:

$$\frac{\alpha_g}{1-\alpha_b} > 1 \quad \text{and} \quad \frac{\beta_g}{1-\beta_b} > 1, \tag{1}$$

which can also be written as:

$$\alpha_g + \alpha_b > 1 \quad \text{and} \quad \beta_g + \beta_b > 1. \tag{2}$$

With this assumption we can refer to signal realization s_g^i as "good news," and to signal realization s_b^i as "bad news."

We assume that each expert reports the realization of their signal truthfully to the decision maker. To be able to use Bayes' law to update her beliefs about the state after receiving both reports, the decision maker must know the joint distribution of the two signals conditional on each state. The conditional joint distributions that are compatible with the assumptions introduced so far are displayed in Table 1.

ω_g:

	s_g^B	s_b^B
s_g^A	$\alpha_g - d_g$	d_g
s_b^A	$(\beta_g - \alpha_g) + d_g$	$(1 - \beta_g) - d_g$

ω_b:

	s_g^B	s_b^B
s_g^A	$(1 - \beta_b) - d_b$	$(\beta_b - \alpha_b) + d_b$
s_b^A	d_b	$\alpha_b - d_b$

Table 1. A generic joint signal distribution.

Table 1 shows a pair of conditional joint distributions with two parameters: d_g and d_b. These two parameters are the probabilities of particular types of disagreements in states ω_g and ω_b. The particular type of disagreement is the one that results if expert A observes the "correct" signal realization (for example: s_g^A if the state is good), and expert B observes the "incorrect" signal realization. The probability of the reverse type of disagreement is equal to d_g resp. d_b plus a constant that depends on the parameters of the marginal distributions. To ensure that all probabilities in Table 1 are non-negative, the two parameters have to satisfy:

$$\begin{aligned} &\max\{0, \alpha_g - \beta_g\} \leq d_g \leq \min\{\alpha_g, 1 - \beta_g\}, \text{ and} \\ &\max\{0, \alpha_b - \beta_b\} \leq d_b \leq \min\{\alpha_b, 1 - \beta_b\}\,. \end{aligned} \tag{3}$$

We denote the left hand side of the first inequality by $\underline{d}_g$, and the right hand side of the first inequality by $\bar{d}_g$. We denote the left hand side of the second inequality by $\underline{d}_b$, and the right hand side of the second inequality by $\bar{d}_b$.

4. Minimizing the Probability of Disagreement and Minimizing Informativeness

The benefit that the decision maker gets from observing the reports of both experts rather than just one is obviously minimized when the experts' reports are perfectly correlated with each other, and thus the probability that the decision makers disagree with each other is zero. But this is a feasible joint distribution only if the signals have identical marginal distributions: $\alpha_g = \beta_g$ and $\alpha_b = \beta_b$. In this section we ask whether minimizing the probability of disagreement minimizes the decision maker's potential benefit from observing the two signals also if the marginal distributions are not identical.

One might think that the joint distribution that minimizes the decision maker's benefit will depend on the decision problem she faces. However, this is not the case. There is a unique joint distribution that is strictly Blackwell dominated by all other distributions. We call such a joint distribution "Blackwell worst."[7]

That a Blackwell worst joint distribution exists is easiest to see when one signal, if it were the only observed signal, strictly Blackwell dominates the other signal, if that signal were the only signal. Without loss of generality let's assume that expert A's signal strictly Blackwell dominates expert B's signal. This is the case if and only if observing s_b^A implies a smaller posterior probability of a good state than observing s_b^B, and observing s_g^A implies a larger posterior probability of a good state than observing s_g^B, i.e.:

$$\frac{1-\alpha_g}{1-\alpha_g+\alpha_b} \leq \frac{1-\beta_g}{1-\beta_g+\beta_b} \quad \text{and} \quad \frac{\beta_g}{\beta_g+1-\beta_b} \leq \frac{\alpha_g}{\alpha_g+1-\alpha_b} \tag{4}$$

and at least one inequality is strict.

Note that, of course, it may be useful to consult expert B in addition to expert A even if A's signal strictly Blackwell dominates B's signal. Whether B's signal is useful in such a situation depends on the joint distribution of the two signals, not just on their marginal distributions.

Proposition 3. *If signal A strictly Blackwell dominates signal B, then there is a unique joint distribution that is strictly Blackwell dominated by all other joint distributions. The probability of disagreement is* not *minimized by this Blackwell worst distribution if and only if both inequalities in* (4) *are strict.*

Proof. To prove the first sentence of the proposition we first display a joint distribution and show that all other joint distributions Blackwell dominate this distribution. Then we show that it is the only joint distribution with this property. This then implies that all other joint distributions strictly Blackwell dominate this distribution.

[7] de Oliveira et al. (2023), Lemma 3, prove the existence of a joint distribution that is Blackwell dominated by all other joint distributions for the case that there are only two states. We show below that there is a unique joint distribution with this feature and characterize it.

Because signal A strictly Blackwell dominates signal B, there is a garbling of signal A that has the same marginal distribution conditional on each state as signal B. Suppose that according to this garbling s_g^A is transformed into s_b^B with probability $1-\xi_g$ and s_b^A is transformed into s_g^B with probability $1-\xi_b$. Now consider the joint distribution displayed in Table 2. When signals follow this joint distribution, the

ω_g:

	s_g^B	s_b^B
s_g^A	$\alpha_g\xi_g$	$\alpha_g(1-\xi_g)$
s_b^A	$(1-\alpha_g)(1-\xi_b)$	$(1-\alpha_g)\xi_b$

ω_b:

	s_g^B	s_b^B
s_g^A	$(1-\alpha_b)\xi_g$	$(1-\alpha_b)(1-\xi_g)$
s_b^A	$\alpha_b(1-\xi_b)$	$\alpha_b\xi_b$

Table 2. The Blackwell worst joint distribution when signal A strictly Blackwell dominates signal B.

distribution of B conditional on the realization of A is state independent. Therefore, the decision maker does not gain anything from observing the realization of B once she sees the realization of A. Clearly, this joint distribution is Blackwell dominated by all other joint distributions.

We now argue that this is the unique Blackwell distribution that is Blackwell dominated by all other distributions. Every such distribution must imply that the decision maker does not learn anything new from observing s^B. Therefore, the joint distribution must be derived from a garbling of signal s^A that creates signal s^B in the way shown in Table 2. It therefore suffices to prove that there is only one garbling of signal s^A that creates signal s^B. Using the same notation as before for garblings, this means that we need to prove that there are unique $\xi_g, \xi_b \in (0,1)$ such that:

$$\begin{pmatrix} \beta_g & 1-\beta_g \\ 1-\beta_b & \beta_b \end{pmatrix} = \begin{pmatrix} \alpha_g & 1-\alpha_g \\ 1-\alpha_b & \alpha_b \end{pmatrix} \begin{pmatrix} \xi_g & 1-\xi_g \\ 1-\xi_b & \xi_b \end{pmatrix}. \tag{5}$$

Observe that the determinant of the first matrix on the right hand side is $\alpha_b + \alpha_g - 1$ which by assumption is strictly positive. Therefore this matrix is invertible, and therefore ξ_b and ξ_g are unique.

We prove the second sentence of Proposition 3 in Appendix A.1, but we illustrate it in the two paragraphs following this proof. □

The second sentence of Proposition 3 says that, when both inequalities in (4) are strict, the Blackwell worst joint distribution and the distribution that minimizes the probability of disagreement are not the same. In this case, in the Blackwell worst joint distribution disagreement is uninformative. The decision maker always only learns from signal A. But when the probability of disagreement is minimized, disagreement is informative. As an example, we consider the case that $\alpha_g > \beta_g$ and $\alpha_b > \beta_b$ hold, which implies (but is not equivalent to) strict inequalities in (4). In this case disagreement is minimized when $d_g = \alpha_g - \beta_g$, and $d_b = \alpha_b - \beta_b$. This corresponds to the distribution shown in Table 3.

ω_g	s_g^B	s_b^B
s_g^A	β_g	$\alpha_g - \beta_g$
s_b^A	0	$1 - \alpha_g$

ω_b	s_g^B	s_b^B
s_g^A	$1 - \alpha_b$	0
s_b^A	$\alpha_b - \beta_b$	β_b

TABLE 3. Minimizing the probability of disagreement when $\alpha_g > \beta_g$ and $\alpha_b > \beta_b$ and therefore signal A strictly Blackwell dominates signal B.

Intuitively, the joint distribution in Table 3 corresponds to the case in which expert B disagrees with expert A if and only if expert A is right. By contrast, the joint distribution in Table 2 corresponds to the case in which expert B's disagreement with expert A is independent of the true state. Paradoxically, if expert B disagrees with expert A if and only if expert A is right, then expert B adds information: when the two experts disagree, the true state is revealed. When they agree, some uncertainty about the true state remains.

It remains to consider the situation in which neither of the two signals strictly Blackwell dominates the other. The case in which both signals are identical is easy to deal with. If signals are not identical, then it is without loss of generality to consider the case that:

$$\frac{1-\beta_g}{1-\beta_g+\beta_b} < \frac{1-\alpha_g}{1-\alpha_g+\alpha_b} \quad \text{and} \quad \frac{\beta_g}{\beta_g+1-\beta_b} < \frac{\alpha_g}{\alpha_g+1-\alpha_b}. \tag{6}$$

Equivalently we can write this condition in likelihood ratios:

$$\frac{1-\beta_g}{\beta_b} < \frac{1-\alpha_g}{\alpha_b} \quad \text{and} \quad \frac{\beta_g}{1-\beta_b} < \frac{\alpha_g}{1-\alpha_b}. \tag{7}$$

Intuitively, the above inequalities say that good news from expert A is stronger evidence that the state is good than good news from expert B, in the sense that it moves the belief further away from the prior, and on the other hand bad news from expert

B is stronger evidence that the state is bad than bad news from expert A. Expert A is specialized in good news, and expert B is specialized in bad news.

We show in Appendix A.2 that the above inequality implies that $\alpha_g < \beta_g$ and $\alpha_b > \beta_b$. This implies that the lower bound of d_g is 0 and the lower bound of d_b is $\alpha_b - \beta_b$. Thus, the joint distribution that minimizes the probability of disagreement is the one displayed in Table 4.

ω_g	s_g^B	s_b^B
s_g^A	α_g	0
s_b^A	$\beta_g - \alpha_g$	$1 - \beta_g$

ω_b	s_g^B	s_b^B
s_g^A	$1 - \alpha_b$	0
s_b^A	$\alpha_b - \beta_b$	β_b

Table 4. Minimizing the probability of disagreement when neither signal strictly Blackwell dominates the other.

In this joint distribution, it is never the case that expert B suggests the state is bad when expert A suggests it is good, and expert A never suggests that the state is good when expert B suggests it is bad. Intuitively, no expert ever disagrees with the other on the state on which the other can obtain the stronger evidence. Our next result implies that the joint distribution displayed in Table 4 is also the Blackwell worst joint distribution.

Proposition 4. *If neither of the two signals strictly Blackwell dominates the other, then there is a unique Blackwell worst joint distribution. It minimizes the probability of disagreement.*

The proof of Proposition 4 is in Appendix A.3. To obtain some intuition note that Table 4 shares with Table 2 the feature that the lowest and the highest posteriors that may be derived from observing the realization of only one of the two signals are the same as the lowest and the highest posteriors that may be derived from observing the realizations of both signals. Observing the joint signal can never move the extreme posteriors further to the extremes if the joint distribution is one of those shown in Tables 2 and 4.

5. Blackwell Improvements Of Conditional Independence

In this section we derive results on Blackwell improvements of given joint distributions. Informally speaking, our first result shows that for any given joint distribution a strict Blackwell improvement is achieved by adjusting it "into the opposite direction" from the worst joint distribution, if such an adjustment is possible. The result is then an immediate consequence of Lemma 3.

Proposition 5. *Denote by* $\left(d_g^*, d_b^*\right)$ *the parameters of the Blackwell-worst joint distribution. For any joint signal distribution corresponding to parameters* $(d_g, d_b) \neq \left(d_g^*, d_b^*\right)$*, and for all* $\varepsilon > 0$ *such that the parameters*

$$(d_g', d_b') \equiv \left(d_g + \varepsilon \left(d_g - d_g^*\right), d_b + \varepsilon \left(d_b - d_b^*\right)\right)$$

satisfy inequality (3), the joint distribution corresponding to (d_g', d_b') *strictly Blackwell dominates the joint distribution corresponding to* (d_g, d_b)*.*

An obvious but important corollary to Proposition 5 is this:

Corollary 1. *Every joint distribution that corresponds to disagreement probabilities* $(d_g, d_b) \in (\underline{d}_g, \bar{d}_g) \times (\underline{d}_b, \bar{d}_b)$ *is strictly Blackwell dominated.*

The argument for this result is simple: If a joint distribution corresponds to an interior pair (d_g, d_b), and does *not* correspond to the Blackwell worst joint distribution, then it is strictly Blackwell dominated by a distribution as described in Proposition 5 if one chooses ε in that proposition small enough. If a joint distribution that corresponds to an interior pair (d_g, d_b) is, on the other hand, the Blackwell worst joint distribution then it is strictly Blackwell dominated because it is the only Blackwell worst joint distribution.

Corollary 1 is mathematically obvious, following Proposition 5, but it is also surprising because the Blackwell order is, in general, regarded as "very incomplete." Yet, in our setting, almost everything is Blackwell dominated.

To illustrate Proposition 5 we consider the case that (d_g, d_b) corresponds to conditionally independent signals. The joint distribution of signals under conditional independence is shown in Table 5. Note that in all cases the conditionally independent distribution is not Blackwell worst.

Consider first the case that signal A strictly Blackwell dominates signal B, so that the Blackwell worst distribution is given by the garbling distribution in Table 2. To understand Blackwell improvements of the joint distribution under conditional independence, we begin by comparing the joint distribution under conditional independence in Table 5 to the garbling distribution in Table 2. We claim that agreement of the two signals on the state is *less* likely under conditional independence, but that such agreement provides *stronger evidence*, in the sense that it moves the decision

	s_g^B	s_b^B
s_g^A	$\alpha_g\beta_g$	$\alpha_g(1-\beta_g)$
s_b^A	$(1-\alpha_g)\beta_g$	$(1-\alpha_g)(1-\beta_g)$

ω_g

	s_g^B	s_b^B
s_g^A	$(1-\alpha_b)(1-\beta_b)$	$(1-\alpha_b)\beta_b$
s_b^A	$\alpha_b(1-\beta_b)$	$\alpha_b\beta_b$

ω_b

Table 5. Conditionally independent signals.

maker's belief more, than under conditional independence. To show these claims note that the ex ante probability that the signal combination (s_g^A, s_g^B) is observed under the garbling distribution is:

$$\frac{1}{2}\alpha_g\xi_g + \frac{1}{2}(1-\alpha_b)\xi_g.$$

Under conditional independence the probability of observing (s_g^A, s_g^B) is:

$$\frac{1}{2}\alpha_g\beta_g + \frac{1}{2}(1-\alpha_b)(1-\beta_b).$$

An explicit formula for ξ_g is in Appendix A.1. An elementary calculation shows that $\beta_g < \xi_g$. Because by assumption $1-\beta_b < \beta_g$, this implies that $1-\beta_b < \xi_g$. It is thus obvious that with conditionally independent signals it is less likely that the two signals agree on a good state than it is when signal B is obtained by garbling signal A. An analogous observation is true for the probability that the two signals agree on a bad state.

To show that, when signals agree under conditional independence the decision maker's belief is moved by more than when signals agree under garbling we compare the likelihood ratios. We have to show that:

$$\frac{\alpha_g\beta_g}{(1-\alpha_b)(1-\beta_b)} > \frac{\alpha_g\xi_g}{(1-\alpha_b)\xi_g}.$$

But this follows directly from $\beta_g > 1-\beta_b$.

A Blackwell improvement of conditionally independent signals that is based on Proposition 5 will reduce the probability of agreement, but will make such agreement stronger evidence. We illustrate this with Example 1. Suppose $\alpha_g = \alpha_b = 0.8$ and $\beta_g = \beta_b = 0.6$. Then under conditional independence the joint distribution of signals is shown in red in Table 6, and the Blackwell worst joint distribution, i.e. the garbling distribution, is shown in green in Table 6. In Table 7 we then show a Blackwell improvement of the conditionally independent distribution that is constructed as indicated in Proposition 5, setting $\varepsilon = 1$. The improved joint distribution makes agreement an "almost revealing" signal. The improved distribution is shown in blue in Table 7 and the distribution under conditional independence is shown in red.

ω_g:

	s_g^B		s_b^B	
s_g^A	0.48	0.53	0.32	0.27
s_b^A	0.12	0.07	0.08	0.13

ω_b:

	s_g^B		s_b^B	
s_g^A	0.08	0.13	0.12	0.07
s_b^A	0.32	0.27	0.48	0.53

TABLE 6. Conditional independence and garbling in EXAMPLE 1: $\alpha_g = \alpha_b = 0.8$ and $\beta_g = \beta_b = 0.6$.

ω_g:

	s_g^B		s_b^B	
s_g^A	0.43	0.48	0.37	0.32
s_b^A	0.17	0.12	0.03	0.08

ω_b:

	s_g^B		s_b^B	
s_g^A	0.03	0.08	0.17	0.12
s_b^A	0.37	0.32	0.43	0.48

TABLE 7. Blackwell improving on conditional independence in EXAMPLE 1: $\alpha_g = \alpha_b = 0.8$ and $\beta_g = \beta_b = 0.6$.

We turn next to the implications of Proposition 5 in the case that neither of the two signals strictly Blackwell dominates the other. Without loss of generality we focus on the case that the Blackwell worst distribution is the joint distribution in Table 4. We again first compare conditionally independent signals to those following the joint distribution in Table 4. It is obvious that conditionally independent signals will imply a higher probability of disagreement than the joint distribution in Table 4. This is because the joint distribution in Table 4, by Proposition 4, is not only Blackwell worst, but also minimizes the probability of disagreement. It is also easy to see that with conditionally independent signals, the evidence provided when the two experts agree, is stronger than it is in the Blackwell worst case. This is because in the Blackwell worst case, for every form of agreement there is an expert such that the agreement provides as much information as the signal realization observed by this one expert. By contrast, in the conditionally independent case, agreement provides stronger evidence than the signal realization observed by either of the two agents. These observations also imply that, as in the previous case, improvement of conditionally independent signals along the lines of Proposition 5 will imply a further increase in the likelihood of disagreement, and a further increase in the strength of the evidence provided by agreement.

We illustrate using Example 2. Suppose $\alpha_g = 0.6$, $\alpha_b = 0.7$, $\beta_g = 0.7$ and $\beta_b = 0.6$. Then under conditional independence the joint distribution of signals is shown in red in Table 8, and the Blackwell worst joint distribution is shown in green in Table 8. In Table 9 we then show a Blackwell improvement of the conditionally independent distribution that is constructed as indicated in Proposition 5, setting $\varepsilon = 0.5$. The improved distribution is shown in blue and the distribution under conditional independence is shown in red in Table 9. As in Example 1 the improved joint distribution makes agreement an "almost revealing" signal.

	s_g^B	s_b^B
s_g^A	0.42 0.6	0.18 0.0
s_b^A	0.28 0.1	0.12 0.3

ω_g

	s_g^B	s_b^B
s_g^A	0.12 0.3	0.18 0.0
s_b^A	0.28 0.1	0.42 0.6

ω_b

Table 8. Conditionally independent and Blackwell worst joint distributions in Example 2: $\alpha_g = 0.6,\ \alpha_b = 0.7,\ \beta_g = 0.7,\ \beta_b = 0.6$.

	s_g^B	s_b^B
s_g^A	0.33 0.42	0.27 0.18
s_b^A	0.37 0.28	0.03 0.12

ω_g

	s_g^B	s_b^B
s_g^A	0.03 0.12	0.27 0.18
s_b^A	0.37 0.28	0.33 0.42

ω_b

Table 9. Blackwell improving on conditional independence in Example 2: $\alpha_g = 0.6,\ \alpha_b = 0.7,\ \beta_g = 0.7,\ \beta_b = 0.6$.

Proposition 5 describes only a subset of the joint distributions that may strictly Blackwell dominate any given distribution, i.e. of the upper contour set of a given distribution in the strict Blackwell order. To proceed, we display next the upper contour sets for the conditionally independent distribution in Examples 1 and 2. We determined these upper contour sets numerically. In Example 1 the disagreement probabilities (d_g, d_b) must be contained in the square $[0.2, 0.4]^2$. In Figure 1 we show the parameter pair that corresponds to conditional independence as a red dot. The upper contour set for the conditionally independent distribution is indicated in blue.

The Blackwell worst distribution is indicated in green. A green line indicates all those joint distributions that Blackwell dominate conditional independence according to Proposition 5. Figure 2 is the same diagram for Example 2.

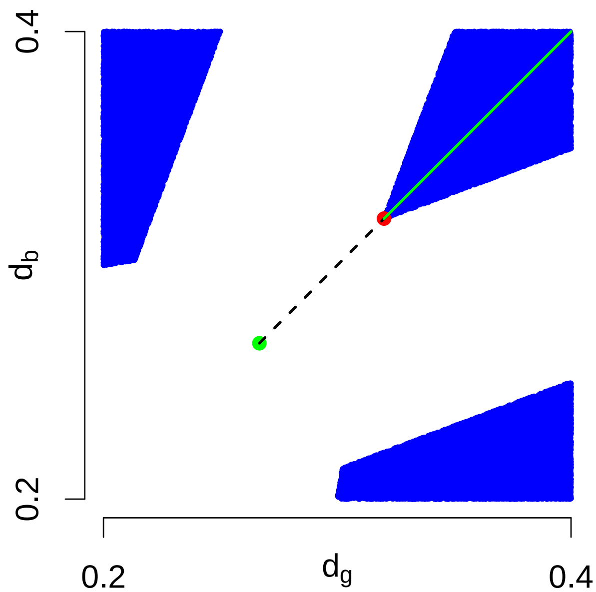

Figure 1. Upper contour set of conditional independence in Example 1.

Figures 1 and 2 indicate that joint distributions characterized by disagreement probabilities in the bottom right and top left corner of the parameter space strictly Blackwell dominate conditionally independent signals. To obtain intuition for this, we now display the joint distribution that corresponds to the bottom right corner when $\alpha_g = \alpha_b = \beta_g = \beta_b = 0.5$, that is, when signals A and B, each by itself, are uninformative.[8] The bottom right corner of the parameter space then corresponds to the disagreement probabilities $d_g = 0.5$ and $d_b = 0$. The corresponding joint distribution of signals is in Table 10.

In Table 10 the joint signal is completely informative, i.e. after observing the realization of the joint signal the decision maker knows the true state with certainty, even though each of the two signals by itself is uninformative. Specifically, if the two

[8]We find this case useful as an illustrative example even though it violates inequality (1).

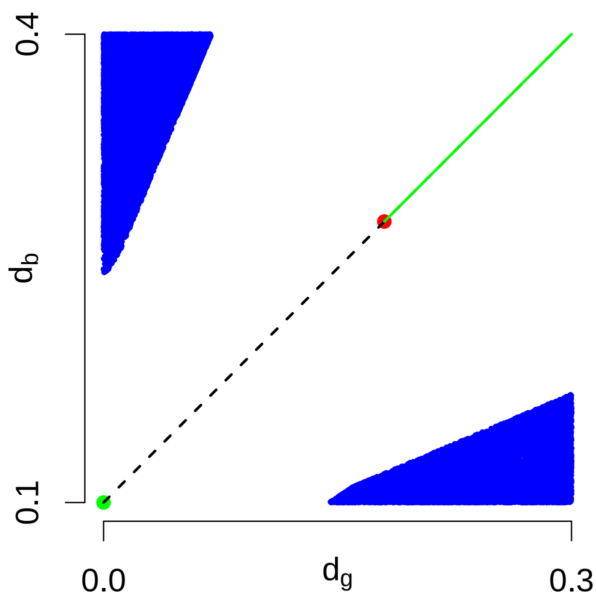

FIGURE 2. Upper contour set of conditional independence in EXAMPLE 2.

	s_g^B	s_b^B
s_g^A	0	0.5
s_b^A	0.5	0

ω_g

	s_g^B	s_b^B
s_g^A	0.5	0
s_b^A	0	0.5

ω_b

TABLE 10. Learning from disagreement.

signals disagree, then the decision maker knows that the state is ω_g whereas if the two signals agree, then the decision maker knows that the state is ω_b.[9]

In the general case in which the marginal signals are informative the joint distributions that correspond to the top left and bottom right corners of the parameter space share with the example in Table 10 that the decision maker arrives at probability one beliefs after observing certain agreements or disagreements among the signals. For example, if d_g has its maximal value, i.e. $d_g = \min\{\alpha_g, 1 - \beta_g\}$, then both types of disagreement have strictly positive probability in state ω_g, and if d_b has its minimal

[9]Table 10 is an example of what is known in cryptography as a "secret sharing algorithm" (see e.g. Shamir, 1979).

value, i.e. $d_b = \max\{0, \alpha_b - \beta_b\}$, then at least one type of disagreement has probability zero in state ω_b. Therefore, at least one type of disagreement leads the decision maker to be certain that the state is ω_g. Similarly, for these parameters, at least one type of agreement leads the decision maker to be certain that the state is ω_b. The case that d_g has its minimal value and that d_b has its maximal value is symmetric.

The reason why joint distributions that correspond to parameter values in the top left and bottom right corners potentially strictly Blackwell dominate conditionally independent signals thus contrasts with the reason why joint distributions referred to in Proposition 5 strictly Blackwell dominate conditionally independent signals: In Proposition 5 frequent disagreement raised Blackwell informativeness because it made agreement induce larger changes in the decision maker's beliefs, but not because disagreement itself carried more information (see EXAMPLES 1 and 2 in Tables 6 - 9). By contrast, when considering joint distributions related to the one in Table 10, we find that strict Blackwell improvements occur because the decision maker learns from whether or not the experts disagree more than the decision maker learns from the content of their advice.

We generalize these observations in the following proposition that says that if signals are by themselves sufficiently uninformative, then the joint signal that corresponds to the case that one disagreement probability is close to its upper bound whereas the other is close to its lower bound strictly Blackwell dominates the signal obtained from conditional independence. For simplicity we state the proposition only for the case that d_g is close to its upper bound and d_b is close to its lower bound. Proposition 6 demonstrates the robustness of the example displayed in Table 10. The examples in Figures 1 and 2 show that the marginal signals don't have to be "extremely uninformative," and the disagreement probabilities don't have to be "extremely" close to their bounds.

Proposition 6. *There exist an $\bar{\varepsilon} > 0$ such that for every ε with $0 < \varepsilon < \bar{\varepsilon}$ there exists a $\bar{\delta}$ such that for every δ with $0 < \delta < \bar{\delta}$ the following is true. If:*

$$(\alpha_g, \alpha_b, \beta_g, \beta_b) \in [0.5 - \varepsilon, 0.5 + \varepsilon]^4,$$

and if:

$$(d_g, d_b) \in (\bar{d}_g - \delta, \bar{d}_g) \times (\underline{d}_b, \underline{d}_b + \delta),$$

then the joint distribution corresponding to (d_g, d_b) strictly Blackwell dominates the joint distribution corresponding to conditional independence.

The proof of this result is in Appendix B.1. The proof uses Proposition 5 in Börgers (2024). According to this result, it is sufficient to prove that there is an interval contained in $[0, 1]$ such that all posterior beliefs that have positive probabilities when signals are conditionally independent are contained in that interval, whereas all posterior beliefs that have positive probabilities when (d_g, d_b) satisfies the condition in Proposition 6 fall outside of this interval. To apply this proposition we show that, if the parameters $\alpha_g, \alpha_b, \beta_g$, and β_b satisfy the assumption of Proposition 6, then posterior beliefs that have positive probabilities under independence are all contained in an interval around 0.5 that becomes small as ε becomes small. On the other hand, when d_g and d_b satisfy the conditions in Proposition 6, and ε and δ are small, then the posterior beliefs that have positive probabilities are either close to 0 or close to 1. These observations allow us to use Proposition 5 in Börgers (2024) to prove Proposition 6.

6. Strictly Blackwell Undominated Joint Distributions

We say that a joint signal distribution is "strictly Blackwell undominated" if it is not strictly Blackwell dominated by any other joint signal distribution. In this section we provide results about joint signal distributions that are strictly Blackwell undominated. Whereas, as we showed in Section 4, there is always a unique Blackwell worst distribution there may be typically multiple strictly Blackwell undominated joint distributions. We give an example illustrating multiplicity in part (ii) of Proposition 9 below.

Corollary 1 implies that only the sides of the rectangle of admissible parameters are candidates for the Blackwell undominated joint signal distributions. This means that the joint conditional distributions of signals have to assign probability zero to at least one pair of signal realizations in at least one state. If this pair of signal realizations has strictly positive probability in the other state then it follows that this pair of signal induces the decision maker to have posterior belief of either 0 or 1. We shall say that such a joint signal realization "reveals the state." The following result gives sufficient conditions for at least one joint signal realization to reveal the state in any joint distribution that is strictly Blackwell undominated.

Proposition 7. *For every strictly Blackwell undominated joint distribution of signals if:*

(i) signal A strictly Blackwell dominates signal B, $\alpha_g > \beta_g$, $\alpha_b > \beta_b$, and $(\alpha_g, \alpha_b, \beta_g, \beta_b) \in (0.5, 1)^4$, or if:

(ii) neither of the two signal strictly Blackwell dominates the other, $\alpha_g + \beta_g \geq 1$, $\alpha_b + \beta_b \geq 1$, *and at least one of these inequalities is strict,*

then there is at least one joint signal realization that reveals the state.

We provide the proof of Proposition 7 only at the end of this section because it builds on some of the results in this section.

We focus first in this section on the question when joint distributions in which the disagreement probabilities are maximal or minimal, and that thus correspond to one of the four corners of the admissible parameter space, are strictly Blackwell undominated. One motivation for the focus on these joint distributions is that they form extreme points of the set of admissible joint distributions, and thus, by Lemma 5, for any given finite decision problem (A, u) one of the four extreme points is optimal if any point in the admissible parameter space can be chosen. This result is even relevant when the decision maker is uncertain about which finite decision problem is the one that she will encounter, because, if the decision maker's belief over possible finite decision problems (A, u) has finite support, it is equivalent to a point belief attaching probability 1 to a larger finite decision problem. Thus, even for a decision maker with finite support beliefs it is without loss of generality to restrict attention to extreme points in the parameter space.

The distribution in which both disagreement probabilities are maximized is not always strictly Blackwell undominated. Consider the extreme example where $\alpha_g = \alpha_b = \beta_g = \beta_b = 0.5$. In this example, if disagreement probabilities are maximized, disagreement occurs with probability 1 in both states, but disagreement is uninformative, because both types of disagreement occur with equal probability in both states. This joint signal probability distribution is clearly strictly Blackwell dominated. This will remain true when each signal by itself is informative, but moves beliefs only by very little. When signals are very informative, however, then the joint distribution which maximizes disagreement probabilities is generically strictly Blackwell undominated. The proof of the following proposition is in Appendix C.1.

Proposition 8. *If*

$$(\alpha_g, \alpha_b, \beta_g, \beta_b) \in (2/3, 1)^4,$$

and

$$(\alpha_g - \beta_g)(\alpha_b - \beta_b) \neq 0,$$

then the joint distribution with parameters $d_g = \bar{d}_g$ *and* $d_b = \bar{d}_b$ *is strictly Blackwell undominated.*

The condition in the proposition implies that $\bar{d}_g = 1-\beta_g$ and $\bar{d}_b = 1-\beta_b$. Therefore, the joint distribution which maximizes the probability of disagreement in both states is the distribution in Table 11. Table 11 shows that agreement reveals the state. This reinforces the theme that we have mentioned earlier: maximizing the probability of disagreement maximizes the informativeness of agreement.

ω_g

	s_g^B	s_b^B
s_g^A	$\alpha_g+\beta_g-1$	$1-\beta_g$
s_b^A	$1-\alpha_g$	0

ω_b

	s_g^B	s_b^B
s_g^A	0	$1-\alpha_b$
s_b^A	$1-\beta_b$	$\alpha_b+\beta_b-1$

Table 11. The joint distribution corresponding to $d_g = \bar{d}_g$ and $d_b = \bar{d}_b$ when $\alpha_g + \beta_g > 1$ and $\alpha_b + \beta_b > 1$.

We consider next joint distributions where one disagreement probability is maximized and the other one is minimized. Part (i) of the following proposition shows that regardless of parameter values at least one of these joint distributions is strictly Blackwell undominated. We also provide in part (ii) of the proposition sufficient conditions for both joint distributions to be strictly Blackwell undominated. We focus for simplicity on the case that the probabilities of "correct" signals are sufficiently high: $\alpha_g, \beta_g, \alpha_b, \beta_b > 0.5$. In this case generically both joint distributions are Blackwell undominated. The proof of the next result is in Appendix C.2.

Proposition 9. *(i) At least one of the two joint signal distributions corresponding to $d_g = \underline{d}_g$ and $d_b = \bar{d}_b$ and corresponding to $d_g = \bar{d}_g$ and $d_b = \underline{d}_b$ is strictly Blackwell undominated.*

(ii) If

$$(\alpha_g, \alpha_b, \beta_g, \beta_b) \in (0.5, 1)^4 ,$$

and

$$(\alpha_g - \beta_g)(\alpha_b - \beta_b) \neq 0,$$

then the two joint signal distributions corresponding to $d_g = \underline{d}_g$ and $d_b = \bar{d}_b$ and corresponding to $d_g = \bar{d}_g$ and $d_b = \underline{d}_b$ are both *strictly Blackwell undominated.*

Finally, we consider the joint distribution where both disagreement probabilities are minimized. When neither of the two signals strictly Blackwell dominates the other, we know from Proposition 4 that this joint distribution is strictly Blackwell dominated. It remains to ask whether it is also strictly Blackwell dominated when one of the two signals, say signal A, strictly Blackwell dominates the other. Proposition 10 shows under some additional assumptions that allow a particularly simple proof that the joint distribution minimizing both disagreement probabilities is also strictly Blackwell dominated in this case. This result is proved in Appendix C.3.

Proposition 10. *If*

$$(\alpha_g, \alpha_b, \beta_g, \beta_b) \in (0.5, 1)^4,$$

and

$$\alpha_g > \beta_g \quad \text{and} \quad \alpha_b > \beta_b,$$

then the joint signal distribution corresponding to $d_g = \underline{d}_g$ *and* $d_b = \underline{d}_b$ *is strictly Blackwell dominated by the joint distribution corresponding to* $d_g = \bar{d}_g$ *and* $d_b = \bar{d}_b$*.*

We conclude this section by providing the proof of Proposition 7.

Proof of Proposition 7. We observed at the beginning of this section that if a joint distribution is not strictly Blackwell dominated, there must be at least one pair of signal realizations that has probability zero in one of the two states. It thus only remains to show that it cannot be that a pair of signal realizations has zero probability in both states. Inspection of Table 1 shows that this requires that either $d_g = \underline{d}_g$ and $d_b = \underline{d}_b$ or that $d_g = \bar{d}_g$ and $d_b = \bar{d}_b$.

Proposition 10 shows that under condition (i) the joint distribution corresponding to $d_g = \underline{d}_g$ and $d_b = \underline{d}_b$ is strictly Blackwell dominated. Proposition 4 shows that under condition (ii) the joint distribution corresponding to $d_g = \underline{d}_g$ and $d_b = \underline{d}_b$ is strictly Blackwell dominated. Both conditions imply that for $d_g = \bar{d}_g$ and $d_b = \bar{d}_b$ the joint signal distribution is the one displayed in Table 11. In this joint distribution agreement on the good state reveals the good state and agreement on the bad state reveals the bad state. □

7. Extensions

We now consider the generalization of our model to the case that every expert $i \in \{A, B\}$ observes a signal s^i with realizations in a finite set S^i with at least two elements. We maintain the assumption that there are only two states of the world: $\Omega = \{\omega_g, \omega_b\}$. The reason why we maintain the assumption that there are only two states is that we want to combine results about the model in Section 3 with Proposition 2 in Section 2 to obtain results about the generalized model, and the model in Section 3 has only two states.

The marginal distributions of each signal conditional on each state are exogenously given. For simplicity we assume that all these marginal conditional distributions have full support. We are again interested in the strict Blackwell order of joint distributions that are compatible with the given marginal distributions. We denote the set of all such joint distributions by $\mathcal{D}$ and we denote its elements by P, in analogy to the notation in Section 2.

The existence of a joint distribution in $\mathcal{D}$ that is Blackwell dominated by all other distributions in $\mathcal{D}$ follows from the proof of Lemma 3 in de Oliveira et al. (2023). The assumption that there are only two states of the world is crucial for this result. If this distribution is unique, then the analog of Proposition 5 follows directly from Lemma 3: moving away from the Blackwell worst distribution on a straight line is strictly Blackwell improving.

However, the uniqueness is not guaranteed in general. In Section 4 we proved for the case that there are only two realizations per signal that uniqueness holds. In Proposition 3 we showed this result for the case that one of the two signals strictly Blackwell dominates the other. Our proof of uniqueness relied on the invertibility of the Markov matrix that described the Blackwell dominating signal, and on the fact that the two signals had equal numbers of realizations. If the straightforward analogs of these assumptions in the case of multiple realizations are not satisfied, the uniqueness is not guaranteed. Similarly, the argument that we used in the proof of Proposition 4 to show the uniqueness when neither of the two signals strictly Blackwell dominates the other does not have a straightforward extension to the case in which at least one signal has more than two signal realizations. If uniqueness does not hold, a perhaps less interesting version of Proposition 5 with strict Blackwell dominance replaced by Blackwell dominance holds.[10]

A straightforward extension of our results to signals with more than two realizations follows, however, from Proposition 2 in Section 2.

Corollary 2. *Suppose $P, \hat{P} \in \mathcal{D}$, and suppose $s_1^A, s_2^A \in S^A$ with $s_1^A \neq s_2^A$ and $s_1^B, s_2^B \in S^B$ with $s_1^B \neq s_2^B$. If*

(i) $P(\omega)(\{s_1^A, s_2^A\} \times \{s_1^B, s_2^B\}) = \hat{P}(\omega)(\{s_1^A, s_2^A\} \times \{s_1^B, s_2^B\}) > 0$ *for all* $\omega \in \Omega$,

(ii) $P(\omega)(s_i^A, s_j^B) = \hat{P}(\omega)(s_i^A, s_j^B)$ *for all* $(s_i^A, s_j^B) \in \big(S^A \times S^B\big) \setminus \big(\{s_1^A, s_2^A\} \times \{s_1^B, s_2^B\}\big)$ *and all* $\omega \in \Omega$,

[10] This can be shown by an obvious modification of the proof of Lemma 3 which is the lemma on which the proof of Proposition 5 relies.

(iii) $\hat{P}|_{\{s_1^A,s_2^A\}\times\{s_1^B,s_2^B\}}$ *strictly Blackwell dominates* $P|_{\{s_1^A,s_2^A\}\times\{s_1^B,s_2^B\}}$,

then $\hat{P}$ *strictly Blackwell dominates* P.

This result says that a strict Blackwell improvement of a given distribution P can be found if we focus on two different elements of S^A and two different elements of S^B, and apply the results of this paper to the conditional joint distributions of the four corresponding joint signal realizations, leaving the conditional distributions of all other signal realizations unchanged.

Building on this observation, and using the results of Sections 4 and 5, we obtain the following result that is an extension of Corollary 1.

Corollary 3. *Every signal* $P \in \mathcal{D}$ *for which there are* $s_1^A, s_2^A \in S^A$ *with* $s_1^A \neq s_2^A$ *and* $s_1^B, s_2^B \in S^B$ *with* $s_1^B \neq s_2^B$ *such that*

$$P(\omega)(s_i^A, s_j^B) > 0 \textit{ for all } i, j \in \{1, 2\} \textit{ and all } \omega \in \Omega$$

is strictly Blackwell dominated.

This result shows that all joint distributions that are not strictly Blackwell dominated must be characterized by "sparse" Markov matrices, that is, Markov matrices for which many joint signal realizations have zero probability conditional on at least one of the two states. If conditional joint distributions of pairs of signal realizations satisfy the assumption of Proposition 7 some of these joint signal realizations induce the decision maker to hold beliefs that attach probability 1 to one of the two states.

8. Conclusion

We have focused on a model with only two states of the world. This setting reduced the technical complexity of our analysis, and it also made it natural to interpret combinations of signal realizations as "agreement" or "disagreement." However, the project of comparing different conditional joint distributions with the same conditional marginal distributions can, of course, be conducted in general. Recall that the methodology that we developed in Section 2 is valid with an arbitrary finite number of states. The methodology would also apply when there are more than two experts. We leave these extensions for future work.

In this paper we have assumed that experts have no reason not to be truthful. If experts cannot be trusted to report their information truthfully, then important new issues arise. For example, while in the context of this paper, experts with perfectly correlated signals were not a good choice of committee members, when incentive issues are important the opposite may be true. Experts with perfectly correlated signals may monitor each others' truth telling. Incentive issues are therefore also an important additional avenue of investigation.

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Appendix A. Proofs for Section 4

A.1. Proof of the Second Sentence of Proposition 3.

To prove the second sentence of the proposition note that the probability of disagreement is *not* minimized by the Blackwell worst joint distribution if and only if at least one of the following two inequalities is strict:

$$\alpha_g(1-\xi_g) \geq \max\{0, \alpha_g - \beta_g\}$$
$$\alpha_b(1-\xi_b) \geq \max\{0, \alpha_b - \beta_b\}.$$

This can be seen from inequality (3) and Table 2. We can re-write these inequalities as:

$$\alpha_g(1-\xi_g) \geq 0$$
$$\alpha_g(1-\xi_g) - (\alpha_g - \beta_g) \geq 0$$
$$\alpha_b(1-\xi_b) \geq 0$$
$$\alpha_b(1-\xi_b) - (\alpha_b - \beta_b) \geq 0$$

The probability of disagreement is *not* minimized if and only the first two inequalities are strict, or the last two inequalities are strict, or both of these statements hold.

To re-write these four inequalities, we use (5) to calculate ξ_g and ξ_b. We obtain:

$$\xi_g = \frac{\alpha_b\beta_g - (1-\alpha_g)(1-\beta_b)}{\alpha_g + \alpha_b - 1}$$
$$\xi_b = \frac{\alpha_g\beta_b - (1-\alpha_b)(1-\beta_g)}{\alpha_g + \alpha_b - 1}.$$

Plugging this in, our four inequalities can be re-written as:

$$\frac{\alpha_g}{\alpha_g + \alpha_b - 1}\left[\alpha_b(1-\beta_g) - (1-\alpha_g)\beta_b\right] \geq 0$$
$$\frac{1-\alpha_g}{\alpha_g + \alpha_b - 1}\left[\alpha_g(1-\beta_b) - (1-\alpha_b)\beta_g\right] \geq 0$$
$$\frac{\alpha_b}{\alpha_g + \alpha_b - 1}\left[\alpha_g(1-\beta_b) - (1-\alpha_b)\beta_g\right] \geq 0$$
$$\frac{1-\alpha_b}{\alpha_g + \alpha_b - 1}\left[\alpha_b(1-\beta_g) - (1-\alpha_g)\beta_b\right] \geq 0.$$

The first two inequalities are strict if and only if:

$$\alpha_g\,(1-\beta_b)-(1-\alpha_b)\,\beta_g>0 \quad \text{and} \quad \alpha_b\,(1-\beta_g)-(1-\alpha_g)\,\beta_b>0$$

Note that this is also necessary and sufficient for the second pair of inequalities to be strict. We can re-write these inequalities as:

$$\frac{1-\alpha_g}{1-\alpha_g+\alpha_b}<\frac{1-\beta_g}{1-\beta_g+\beta_b} \quad \text{and} \quad \frac{\beta_g}{\beta_g+1-\beta_b}<\frac{\alpha_g}{\alpha_g+1-\alpha_b}.$$

Thus, it is necessary and sufficient for minimization of the probability of disagreement in the Blackwell worst joint distribution that both inequalities in (4) hold as strict inequalities.

A.2. Proof that (7) implies $\alpha_g < \beta_g$ and $\alpha_b > \beta_b$.

We repeat (7):

$$\frac{1-\beta_g}{\beta_b}<\frac{1-\alpha_g}{\alpha_b} \quad \text{and} \quad \frac{\beta_g}{1-\beta_b}<\frac{\alpha_g}{1-\alpha_b}.$$

This is equivalent to:

$$\beta_b>\frac{\alpha_b(1-\beta_g)}{1-\alpha_g} \quad \text{and} \quad 1-\beta_b>\frac{\beta_g(1-\alpha_b)}{\alpha_g}.$$

Adding up these two inequalities, and re-arranging, yields:

$$(\alpha_g-\beta_g)(1-\alpha_g-\alpha_b)>0.$$

Now recall that by assumption $\alpha_g+\alpha_b>1$. Therefore, we obtain: $\alpha_g<\beta_g$. But then it follows immediately from the second inequality in (7) that we must have: $\alpha_b>\beta_b$.

A.3. Proof of Proposition 4.

When signals A and B have identical marginal distributions it is obvious that the joint distribution where the two signals are perfectly correlated is Blackwell dominated by all other joint distributions. One can prove that it is the only distribution with this property using the same argument as in the proof of Proposition 3. We therefore focus in this proof on the case that the signals are not identical, and, without loss of generality, that (6) holds.

In this proof we shall use Blackwell's original definition (Blackwell, 1951)[11] according to which one signal Blackwell dominates another if the set of feasible mappings

[11] Blackwell himself attributed this version of the definition to Bohnenblust et al. (1949).

of states into action distributions that the former signal makes possible contains the set of feasible such mappings made possible by the latter signal. We begin by making this definition of Blackwell dominance, which Blackwell showed to be equivalent to several other well-known characterizations, precise in our setting.

For any signal s denote by P the Markov matrix where rows correspond to states, columns correspond to signal realizations, and each entry describes the probability of a signal realization conditional on a state. Now suppose that a finite set of possible actions A is given. It is well-known that, because we are considering a model with two states only, it is sufficient to restrict attention to sets A with two elements, say $A = \{a, b\}$.[12] The decision maker's payoff is determined by the mapping ρ that maps the set of possible states of the world into probability distributions over A. In our setting: $\rho : \{\omega_g, \omega_b\} \to [0, 1]$ where we interpret $\rho(\omega)$ as the probability that action a is taken in state ω. We can obviously re-interpret ρ as the vector: $\rho = (\rho(\omega_g), \rho(\omega_b)) \in [0, 1]^2$. A decision maker who has access to signal s can implement a mapping ρ if and only if there is a strategy $\sigma : S \to [0, 1]$ such that $\rho = P\sigma$. Let Z be the set of all mappings ρ that can be implemented by signal s, and let Z' be the analogous set for a signal s'. Signal s Blackwell dominates signal s' if and only if $Z \supseteq Z'$.

In our setting we denote for any signal s^i by P^i the corresponding Markov matrix, and by Z^i the corresponding set of implementable mappings ρ. Let the analogous matrix that corresponds to the joint distribution in Table 4 be denoted by P^*:

$$P^* = \begin{array}{c} \\ \omega_g \\ \omega_b \end{array} \begin{array}{c} \begin{array}{ccc} s_g^A s_g^B & s_b^A s_g^B & s_b^A s_b^B \end{array} \\ \begin{pmatrix} \alpha_g & \beta_g - \alpha_g & 1 - \beta_g \\ 1 - \alpha_b & \alpha_b - \beta_b & \beta_b \end{pmatrix} \end{array}.$$

We denote by Z^* the set of ρ mappings that can be implemented using P^*.

In Step 1 of the proof of Proposition 4 we shall prove that the joint distribution displayed in Table 4 is Blackwell dominated by all other joint distributions by showing that:

$$Z^* \subseteq co(Z^A \cup Z^B), \tag{8}$$

where "*co*" denotes the convex hull. This is sufficient because the set Z corresponding to *any* joint distribution of s^A and s^B must contain the set $co(Z^A \cup Z^B)$. This is so because, by definition, the sets Z are convex, and because the set for any joint distribution must include Z^A (because the decision maker can ignore the realization

[12]See, for example, Proposition 2 in Börgers (2024).

of s^B) and Z^B (because the decision maker can ignore the realization of s^A). In STEP 2 we will show that for any other joint distribution we will have that:

$$Z \setminus co(Z^A \cup Z^B) \neq \emptyset. \tag{9}$$

STEP 1: We shall use the fact that that Z^* is the convex hull of the set:

$$Z^\circ \equiv \{\rho | \exists \sigma \in \{0,1\}^3 \text{ such that } \rho = P^* \sigma\},$$

that is, the convex hull of the set of all mappings ρ that can be implemented using pure strategies. To prove (8) it is then sufficient to prove:

$$Z^\circ \subseteq co(Z^A \cup Z^B).$$

We enumerate Z°:

$$\begin{aligned} Z^\circ &= \{(0,0),(\alpha_g, 1-\alpha_b),(1-\alpha_g,\alpha_b),(\beta_g,1-\beta_b),(1-\beta_g,\beta_b),(1,1)\} \\ &\cup \quad \{(\beta_g-\alpha_g,\alpha_b-\beta_b),(1-\beta_g+\alpha_g,1-\alpha_b+\beta_b)\}. \end{aligned}$$

It is obvious that $(0,0)$ and $(1,1)$ are elements both of Z^A and Z^B, and it is also relatively obvious that the other four vectors listed in the first line are elements of either Z^A or Z^B. For example, $(\alpha_g, 1-\alpha_b) \in Z^A$ because $(\alpha_g, 1-\alpha_b) = P^A \cdot (1,0)$. It thus remains to show that:

$$(\beta_g-\alpha_g,\alpha_b-\beta_b) \in co(Z^A \cup Z^B) \quad \text{and} \quad (1-\beta_g+\alpha_g,1-\alpha_b+\beta_b) \in co(Z^A \cup Z^B).$$

To show $(\beta_g-\alpha_g,\alpha_b-\beta_b) \in co(Z^A \cup Z^B)$, note that with

$$\begin{aligned} \lambda_1 &= \frac{\alpha_g(1-\beta_b)-\beta_g(1-\alpha_b)}{\alpha_b\beta_g-(1-\alpha_g)(1-\beta_b)} \\ \lambda_2 &= \frac{(1-\alpha_g)\beta_b-\alpha_b(1-\beta_g)}{\alpha_b\beta_g-(1-\alpha_g)(1-\beta_b)} \\ 1-\lambda_1-\lambda_2 &= \frac{\alpha_g\beta_b-(1-\alpha_b)(1-\beta_g)}{\alpha_b\beta_g-(1-\alpha_g)(1-\beta_b)} \end{aligned}$$

we have

$$(\beta_g-\alpha_g,\alpha_b-\beta_b) = \lambda_1(1-\alpha_g,\alpha_b)+\lambda_2(\beta_g,1-\beta_b)+(1-\lambda_1-\lambda_2)(0,0).$$

Given $(1-\alpha_g,\alpha_b) \in Z^A$, $(\beta_g,1-\beta_b) \in Z^B$, and $(0,0) \in Z^A$ (of course also $(0,0) \in Z^B$), we just need to show all the coefficients are non-negative. First note that their denominators are positive:

$$\begin{aligned}\alpha_b\beta_g - (1-\alpha_g)(1-\beta_b) &> 0 \Leftrightarrow \\ \frac{\alpha_b}{1-\alpha_g} &> \frac{1-\beta_b}{\beta_g}\end{aligned}$$

which is true because (2) implies $\frac{\alpha_b}{1-\alpha_g} > 1$ and $\frac{1-\beta_b}{\beta_g} < 1$. Next we show that the numerator of λ_1 is positive:

$$\begin{aligned}\alpha_g(1-\beta_b) - \beta_g(1-\alpha_b) &> 0 \Leftrightarrow \\ \frac{\alpha_g}{1-\alpha_b} &> \frac{\beta_g}{1-\beta_b}\end{aligned}$$

which is assumed in (7). Next we show that the numerator of λ_2 is positive:

$$\begin{aligned}(1-\alpha_g)\beta_b - \alpha_b(1-\beta_g) &> 0 \Leftrightarrow \\ \frac{1-\alpha_g}{\alpha_b} &> \frac{1-\beta_g}{\beta_b}\end{aligned}$$

which is also assumed in (7). Finally we show that the numerator of $1-\lambda_1-\lambda_2$ is positive:

$$\begin{aligned}\alpha_g\beta_b - (1-\alpha_b)(1-\beta_g) &> 0 \Leftrightarrow \\ \frac{\alpha_g}{1-\alpha_b} &> \frac{1-\beta_g}{\beta_b}\end{aligned}$$

which is true because (2) implies $\frac{\alpha_g}{1-\alpha_b} > 1$ and $\frac{1-\beta_g}{\beta_b} < 1$.

To prove $(1-\beta_g+\alpha_g, 1-\alpha_b+\beta_b) \in co(Z^A \cup Z^B)$ note that:

$$\begin{aligned}& (1-\beta_g+\alpha_g, 1-\alpha_b+\beta_b) \\ = \;& (1,1) - (\beta_g-\alpha_g, \alpha_b-\beta_b) \\ = \;& \lambda_1[(1,1)-(1-\alpha_g,\alpha_b)] + \lambda_2[(1,1)-(\beta_g,1-\beta_b)] + (1-\lambda_1-\lambda_2)[(1,1)-(0,0)] \\ = \;& \lambda_1(\alpha_g, 1-\alpha_b) + \lambda_2(1-\beta_g,\beta_b) + (1-\lambda_1-\lambda_2)(1,1)\end{aligned}$$

where we use in the third line our result for $(\beta_g-\alpha_g, \alpha_b-\beta_b)$. The last line shows that $(1-\beta_g+\alpha_g, 1-\alpha_b+\beta_b) \in co(Z^A \cup Z^B)$ because $(\alpha_g, 1-\alpha_b) \in Z^A$, $(1-\beta_g, \beta_b) \in Z^B$, and $(1,1) \in Z^A$ (of course also $(1,1) \in Z^B$).

Step 2: To prove that (9) holds any joint signal distribution that is different from the one in Table 4, we note first that any such joint distribution corresponds to disagreement probabilities $d_g = \Delta_g$ and $d_b = \alpha_b - \beta_b + \Delta_b$ where both Δ_g and Δ_b are non-negative and at least one of them is strictly positive. The corresponding Markov matrix is:

$$P^{\Delta} = \begin{array}{c} \\ \omega_g \\ \omega_b \end{array} \begin{array}{c} \begin{array}{cccc} s_g^A s_g^B & s_g^A s_b^B & s_b^A s_g^B & s_b^A s_b^B \end{array} \\ \begin{pmatrix} \alpha_g - \Delta_g & \Delta_g & \beta_g - \alpha_g + \Delta_g & 1 - \beta_g - \Delta_g \\ 1 - \alpha_b - \Delta_b & \Delta_b & \alpha_b - \beta_b + \Delta_b & \beta_b - \Delta_b \end{pmatrix} \end{array}.$$

If we multiply P^{Δ} by the strategy $\tilde{\sigma} = (1, 0, 0, 0)$ we obtain the vector:

$$\tilde{\rho} = (\alpha_g - \Delta_g, 1 - \alpha_b - \Delta_b)\,.$$

If we multiply P^{Δ} by the strategy $\hat{\sigma} = (0, 0, 0, 1)$, we obtain:

$$\hat{\rho} = (1 - \beta_g - \Delta_g, \beta_b - \Delta_b)\,.$$

Our proof strategy is to show that $\tilde{\rho} \in co(Z^A \cup Z^B)$ and $\hat{\rho} \in co(Z^A \cup Z^B)$ leads to a contradiction.

By the first part of this proof $co(Z^A \cup Z^B) = Z^*$. Thus, our hypothesis can also be written as: $\tilde{\rho} \in Z^*$ and $\hat{\rho} \in Z^*$. We begin by showing that for all $\rho \in Z^*$:

(10) $$\rho \cdot ((1 - \alpha_b), -\alpha_g) \leq 0 \text{ and } \rho \cdot (-\beta_b, (1 - \beta_g)) \leq 0.$$

Intuitively: if signals follow the joint distribution given by P^*, then no observation provides strong enough evidence for the good state to make a bet that pays $(1 - \alpha_b)$ in the good state and that pays $-\alpha_g$ in the bad state attractive, and no observation provides strong enough evidence of the bad state to make a bet that pays $-\beta_b$ in the good state and $1 - \beta_g$ in the bad state attractive relative to not betting. Mathematically, the proof of (10) is based on the observation that for every $\rho \in Z^*$ there is a strategy $\sigma = (\sigma_1, \sigma_2, \sigma_3) \in [0, 1]^3$ such that

$$\rho = (\rho_g, \rho_b) = \sigma_1 (\alpha_g, 1 - \alpha_b) + \sigma_2 (\beta_g - \alpha_g, \alpha_b - \beta_b) + \sigma_3 (1 - \beta_g, \beta_b)\,.$$

Therefore:

$$\begin{aligned} & \rho \cdot ((1 - \alpha_b), -\alpha_g) \\ = \; & (\sigma_1 (\alpha_g, 1 - \alpha_b) + \sigma_2 (\beta_g - \alpha_g, \alpha_b - \beta_b) + \sigma_3 (1 - \beta_g, \beta_b)) \cdot ((1 - \alpha_b), -\alpha_g) \\ = \; & \sigma_2 ((1 - \alpha_b)\beta_g - \alpha_g(1 - \beta_b)) + \sigma_3 ((1 - \alpha_b)(1 - \beta_g) - \alpha_g \beta_b)\,. \end{aligned}$$

Inequality (7) implies that the first term on the right hand side is not positive, and inequality (1) implies that the second term on the right hand side is not positive.

Likewise:

$$\begin{aligned} & \rho \cdot (-\beta_b, (1-\beta_g)) \\ = \; & (\sigma_1(\alpha_g, 1-\alpha_b) + \sigma_2(\beta_g - \alpha_g, \alpha_b - \beta_b) + \sigma_3(1-\beta_g, \beta_b)) \cdot (-\beta_b, (1-\beta_g)) \\ = \; & \sigma_1\left((1-\alpha_b)(1-\beta_g) - \alpha_g\beta_b)\right) + \sigma_2\left(\alpha_b(1-\beta_g) - \beta_b(1-\alpha_g)\right). \end{aligned}$$

Inequality (1) implies that the first term on the right hand side is not positive, and inequality (7) implies that the second term on the right hand side is not positive.

Now, using the first inequality in (10), we obtain that $\tilde{\rho} \in Z^*$ implies:

$$\begin{aligned} \tilde{\rho} \cdot \left((1-\alpha_b), -\alpha_g\right) & \leq 0 \Leftrightarrow \\ (\alpha_g - \Delta_g, 1-\alpha_b - \Delta_b) \cdot \left((1-\alpha_b), -\alpha_g\right) & \leq 0 \Leftrightarrow \\ \alpha_g\Delta_b - (1-\alpha_b)\Delta_g & \leq 0 \Leftrightarrow \\ \Delta_g & \geq \frac{\alpha_g}{1-\alpha_b}\Delta_b. \end{aligned} \tag{11}$$

Using the second inequality in (10), we obtain that $\hat{\rho} \in Z^*$ implies:

$$\begin{aligned} \hat{\rho} \cdot (-\beta_b, (1-\beta_g)) & \leq 0 \Leftrightarrow \\ (1-\beta_g - \Delta_g, \beta_b - \Delta_b) \cdot (-\beta_b, (1-\beta_g)) & \leq 0 \Leftrightarrow \\ \beta_b\Delta_g - (1-\beta_g)\Delta_b & \leq 0 \Leftrightarrow \\ \Delta_g & \leq \frac{1-\beta_g}{\beta_b}\Delta_b. \end{aligned} \tag{12}$$

If $\Delta_b = 0$, then (12) implies $\Delta_g = 0$, which contradicts our assumption that at least one of Δ_g and Δ_b is strictly positive. But If $\Delta_b > 0$, then by (2):

$$\frac{\alpha_g}{1-\alpha_b}\Delta_b > \frac{1-\beta_g}{\beta_b}\Delta_b,$$

and (11) and (12) cannot hold simultaneously.

Appendix B. Proofs for Section 5

B.1. **Proof of Proposition 6.**

When

$$(\alpha_g, \alpha_b, \beta_g, \beta_b) \in [0.5-\varepsilon, 0.5+\varepsilon]^4$$

then the lowest posterior probabilities of ω_g that the decision maker may have after observing the realizations of conditionally independent signals is:

$$\frac{(0.5-\varepsilon)^2}{(0.5-\varepsilon)^2+(0.5+\varepsilon)^2}.$$

A simple lower bound for this probability can be found as follows:

$$\frac{(0.5-\varepsilon)^2}{(0.5-\varepsilon)^2+(0.5+\varepsilon)^2}=\frac{0.25+\varepsilon^2-\varepsilon}{0.5+2\varepsilon^2}=0.5-\frac{\varepsilon}{0.5+2\varepsilon^2}\geq 0.5-2\varepsilon$$

A symmetric calculation applies to the largest posterior probabilities of ω_g that the decision maker may have after observing the realizations of conditionally independent signals, and thus we may conclude that the posterior provabilities of ω_g that have positive probabilities when signals are conditionally independent are contained in the interval:

$$[0.5-2\varepsilon, 0.5+2\varepsilon].$$

Proposition 5 in Börgers (2024) implies that (d_g, d_b) Blackwell dominates conditionally independent signals if the following four inequalities hold:

$$\frac{\alpha_g-d_g}{\alpha_g-d_g+(1-\beta_b)-d_b}<0.5-2\varepsilon, \qquad \frac{(1-\beta_g)-d_g}{(1-\beta_g)-d_g+\alpha_b-d_b}<0.5-2\varepsilon,$$

$$\frac{d_g}{d_g+(\beta_b-\alpha_b)+d_b}>0.5+2\varepsilon, \qquad \frac{(\beta_g-\alpha_g)+d_g}{(\beta_g-\alpha_g)+d_g+d_b}>0.5+2\varepsilon.$$

The left hand sides of these four inequalities are the four posterior probabilities that can arise under any joint distribution of s^A and s^B.

Consider the first two inequalities. We begin by deriving an upper bound for $(\alpha_g - d_g)$ and $(1-\beta_g-d_g)$. For $\alpha_g - d_g$ such an upper bound can be obtained as follows:

$$\alpha_g-d_g\leq\alpha_g-(\min\{\alpha_g,1-\beta_g\}-\delta)=\max\{0,1-\alpha_g-\beta_g\}+\delta\leq 2\varepsilon+\delta.$$

A symmetric argument obtains an identical upper bound for $(1-\beta_g-d_g)$. We conclude that our two inequalities are implied by:

$$\frac{2\varepsilon+\delta}{2\varepsilon+\delta+(1-\beta_b)-d_b}<0.5-2\varepsilon \quad \text{and} \quad \frac{2\varepsilon+\delta}{2\varepsilon+\delta+\alpha_b-d_b}<0.5-2\varepsilon.$$

Next, we derive a lower bound for $(\alpha_b - d_b)$ and $(1-\beta_b-d_b)$. For $(\alpha_b - d_b)$ such a lower bound can be obtained as follows:

$$\alpha_b-d_b\geq\alpha_b-(\max\{0,\alpha_b-\beta_b)\}+\delta)=\min\{\alpha_b,\beta_b\}-\delta\geq 0.5-\varepsilon-\delta.$$

A symmetric argument obtains an identical upper bound for $(1-\beta_b-d_b)$. We conclude that our two inequalities are implied by:

$$\begin{aligned}\frac{2\varepsilon+\delta}{2\varepsilon+\delta+0.5-\varepsilon-\delta} &< 0.5-2\varepsilon \Leftrightarrow \\ \frac{2\varepsilon+\delta}{0.5+\varepsilon} &< 0.5-2\varepsilon \Leftrightarrow \\ \delta &< 0.25-2.5\varepsilon-2\varepsilon^2\end{aligned}$$

We next turn to the third and fourth inequalities that imply Blackwell dominance. We begin by deriving lower bounds for d_g and $(\alpha_g-\beta_g+d_g)$. For d_g such a lower bound can be obtained as follows:

$$d_g \geq \min\{\alpha_b, 1-\beta_b\}-\delta \geq 0.5-\varepsilon-\delta.$$

A symmetric calculation shows that the same lower bound also holds for $(\alpha_g-\beta_g+d_g)$. We can therefore conclude that our two inequalities are implied by:

$$\frac{0.5-\varepsilon-\delta}{0.5-\varepsilon-\delta+(\beta_b-\alpha_b)+d_b} > 0.5+2\varepsilon \quad \text{and} \quad \frac{0.5-\varepsilon-\delta}{0.5-\varepsilon-\delta+d_b} > 0.5+2\varepsilon.$$

Next, we obtain upper bounds for d_b and $(\beta_b-\alpha_b+d_b)$. For d_b such an upper bound can be obtained as follows:

$$d_b \leq \max\{0, \alpha_b-\beta_b\}+\delta \leq 2\varepsilon+\delta.$$

A symmetric calculation shows that the same lower bound also holds for $(\beta_b-\alpha_b+d_b)$. We conclude that our two inequalities are implied by:

$$\begin{aligned}\frac{0.5-\varepsilon-\delta}{0.5-\varepsilon-\delta+2\varepsilon+\delta} &> 0.5+2\varepsilon \Leftrightarrow \\ \frac{0.5-\varepsilon-\delta}{0.5+\varepsilon} &> 0.5+2\varepsilon \Leftrightarrow \\ \delta &< 0.25-2.5\varepsilon-2\varepsilon^2, \qquad (13)\end{aligned}$$

which is the same inequality as we obtained before.

We can now choose $\bar{\varepsilon}$ so that $\varepsilon < \bar{\varepsilon}$ implies that the right hand side of (13) is strictly positive, and then, for given $\varepsilon < \bar{\varepsilon}$, choose $\bar{\delta}$ to be the minimum of the right hand side of (13), $\bar{d}_g - \underline{d}_g$ and $\bar{d}_b - \underline{d}_b$.

Appendix C. Proofs for Section 6

C.1. **Proof of Proposition 8.**

Table 11 shows that under the conditions of the proposition agreement reveals the state, and that the good state is revealed with probability $(\alpha_g + \beta_g - 1)/2$ and the bad state is revealed with probability $(\alpha_b + \beta_b - 1)/2$. If $(\alpha_g, \alpha_b, \beta_g, \beta_b) \in (2/3, 1)^4$, then a lower bound for both probabilities is $1/6$.

By Corollary 1, if there is a distribution with interior parameters (d_g, d_b) that strictly Blackwell dominates the distribution with parameters $(\bar{d}_g, \bar{d}_b)$ then it is also strictly dominated by a distribution with boundary parameters (d_g, d_b). We prove that no distribution on the boundary that is different from $(\bar{d}_g, \bar{d}_b)$ strictly dominates $(\bar{d}_g, \bar{d}_b)$.

Because any strictly Blackwell dominating distribution must imply a posterior distribution that is a mean preserving spread of the posterior distribution implied by $(\bar{d}_g, \bar{d}_b)$, any strictly Blackwell dominating distribution must imply probabilities of at least $1/6$ for the posteriors 0 and 1 each.

If one of the two disagreement parameters is neither equal to its maximal nor to its minimal value, then all joint signal realizations have strictly positive probability in the corresponding state. But then only this state can receive posterior probability 1 with positive probability. It is impossible for the other state. Therefore, for posteriors 0 and 1 to be attached strictly positive probability, it is necessary that both disagreement probabilities equal either their maximum or their minimum values.

It remains to consider the joint distributions corresponding to $(\underline{d}_g, \underline{d}_b)$, $(\bar{d}_g, \underline{d}_b)$, and $(\underline{d}_g, \bar{d}_b)$. The joint distribution corresponding to $(\underline{d}_g, \underline{d}_b)$ is shown in Table 12 which is the general version of Tables 3 and 4. Table 12 shows that agreement will not imply a posterior of 0 or 1 in this case. Therefore, for posterior 0 and posterior 1 to have positive probabilities, one of the two types of disagreement must only occur in state ω_g and reveal ω_g and the other type of disagreement must only occur in state ω_b and reveal ω_b. This is the case if and only if either $\alpha_g > \beta_g$ and $\alpha_b > \beta_b$ or $\alpha_g < \beta_g$ and $\alpha_b < \beta_b$. In the first case, the probabilities of the two types of disagreement are $(\alpha_g - \beta_g)/2$ and $(\alpha_b - \beta_b)/2$, which, under the assumption of the proposition, are both strictly less than $1/6$. The second case is similar. Thus, the posteriors 0 and 1 can only occur with a probability that is smaller than their value if disagreement probabilities are maximal, and the joint distribution with minimal disagreement probabilities does not strictly Blackwell dominate the joint probability distribution with maximum disagreement probabilities.

	s_g^B	s_b^B
s_g^A	$\min\{\alpha_g, \beta_g\}$	$\max\{0, \alpha_g - \beta_g\}$
s_b^A	$\max\{\beta_g - \alpha_g, 0\}$	$\min\{1-\beta_g, 1-\alpha_g\}$

ω_g

	s_g^B	s_b^B
s_g^A	$\min\{1-\beta_b, 1-\alpha_b\}$	$\max\{\beta_b - \alpha_b, 0\}$
s_b^A	$\max\{0, \alpha_b - \beta_b\}$	$\min\{\alpha_b, \beta_b\}$

ω_b

TABLE 12. The joint distribution corresponding to $d_g = \underline{d}_g$ and $d_b = \underline{d}_b$.

Finally consider the joint distributions corresponding to $(\bar{d}_g, \underline{d}_b)$. The case of the joint distribution corresponding to $(\underline{d}_g, \bar{d}_b)$ is analogous. When the probability of disagreement is maximal in state ω_g but minimal in state ω_b, then the joint distribution of signals in state ω_g is the one shown in Table 11 on the left, and the joint distribution of the two signals in state ω_b is the one shown in the bottom part of Table 12. These tables show that the good state is revealed only by disagreement, and the probability of revealing disagreement is at $(1-\alpha_g)/2$ or $(1-\beta_g)/2$. But if α_g and β_g are contained in $(2/3, 1)$ this expression is strictly less than $1/6$. Therefore, the state ω_g is revealed less frequently than it is when both disagreement probabilities are maximized, and the joint distribution with maximal disagreement probabilities is not strictly Blackwell dominated by the distribution with parameters $(\bar{d}_g, \underline{d}_b)$.

C.2. **Proof of Proposition 9.**

(i) We prove that, if the joint signal distribution corresponding to $d_g = \bar{d}_g$ and $d_b = \underline{d}_b$ is strictly Blackwell dominated, then it is strictly Blackwell dominated by the joint signal distribution corresponding to $d_g = \underline{d}_g$ and $d_b = \bar{d}_b$. The converse to this statement also holds but we omit the proof because it is symmetric to the proof that we present. The proposition then follows because the two joint distributions cannot strictly Blackwell dominate each other, and hence at least one of them is strictly Blackwell undominated.

By Corollary 1, if there is a distribution with interior parameters (d_g, d_b) that strictly Blackwell dominates the distribution with parameters $(\bar{d}_g, \underline{d}_b)$ then it is also strictly dominated by a distribution with boundary parameters (d_g, d_b). We prove that no distribution on the boundary that is different from $(\underline{d}_g, \bar{d}_b)$ strictly dominates $(\bar{d}_g, \underline{d}_b)$.

ω_g:

	s_g^B	s_b^B
s_g^A	$\max\{0, \alpha_g + \beta_g - 1\}$	$\min\{\alpha_g, 1 - \beta_g\}$
s_b^A	$\min\{\beta_g, 1 - \alpha_g\}$	$\max\{1 - \alpha_g - \beta_g, 0\}$

ω_b:

	s_g^B	s_b^B
s_g^A	$\min\{1 - \beta_b, 1 - \alpha_b\}$	$\max\{\beta_b - \alpha_b, 0\}$
s_b^A	$\max\{0, \alpha_b - \beta_b\}$	$\min\{\alpha_b, \beta_b\}$

TABLE 13. The joint distribution corresponding to $d_g = \bar{d}_g$ and $d_b = \underline{d}_b$.

Table 13 indicates that the joint distribution with $(\bar{d}_g, \underline{d}_b)$ leads with positive probability to posteriors 0 and 1 because at least one type of agreement occurs with probability 0 in state ω_g, and at least one type of disagreement occurs with probability 0 in state ω_b. Because any strictly Blackwell dominating distribution must result from a mean preserving spread of the posterior distribution, also any strictly Blackwell dominating distribution must imply a positive probability that the posterior is 0 and a positive probability that the posterior is 1.

If one of the two disagreement parameters is neither equal to its maximal nor to its minimal value, then all joint signal realizations have strictly positive probability in the corresponding state. But then this state only can receive posterior probability 1 with positive probability. It is impossible for the other state. Therefore, for posteriors 0 and 1 to have strictly positive probability, it is necessary that both disagreement probabilities equal either their maximum or their minimum.

It remains to consider the joint distributions corresponding to $(\underline{d}_g, \underline{d}_b)$ and $(\bar{d}_g, \bar{d}_b)$. The joint distribution corresponding to $(\underline{d}_g, \underline{d}_b)$ was shown in Table 12. It is easily seen in this table that agreement cannot reveal the state because agreement occurs with positive probability in both states. Therefore, for posteriors 0 and 1 to have positive probabilities, one of the two types of disagreement must only occur in state ω_g and reveal ω_g and the other type of disagreement must only occur in state ω_b and reveal ω_b. The disagreement that reveals ω_g occurs with probability zero in state ω_b, and therefore it also occurs with probability zero in state ω_b under the signal distribution $(\bar{d}_g, \underline{d}_b)$, but under this signal distribution the disagreement occurs more frequently in state ω_g, because in that state disagreement probabilities are maximal rather than minimal. Therefore, the posterior belief 1 is induced with higher probability by $(\bar{d}_g, \underline{d}_b)$, and therefore $(\underline{d}_g, \underline{d}_b)$ does not strictly Blackwell dominate $(\bar{d}_g, \underline{d}_b)$.

Finally, consider the joint distribution corresponding to $(\bar{d}_g, \bar{d}_b)$. In this case disagreement cannot reveal the state, because both types of disagreement occur with positive probability in both states, as can be seen from Table 1 and inequality (3). Therefore, for posteriors 0 and 1 to have positive probabilities, it must be that one of the two types of agreement only occurs in state ω_g and reveals ω_g and the other type of agreement only occurs in state ω_b and reveals ω_b. The type of agreement that reveals ω_b occurs with probability 0 in state ω_g, but with positive probability in state ω_b. It will also occur with zero probability in state ω_g under the distribution $(\bar{d}_g, \underline{d}_b)$. But it will occur with higher probability under the distribution $(\bar{d}_g, \underline{d}_b)$ in state ω_b than under the distribution $(\bar{d}_g, \bar{d}_b)$ because the former distribution maximizes the probability of agreement in state ω_b. Therefore, the posterior belief 0 is induced with higher probability by $(\bar{d}_g, \underline{d}_b)$, and therefore $(\bar{d}_g, \bar{d}_b)$ does not strictly Blackwell dominate $(\bar{d}_g, \underline{d}_b)$.

(ii) Note that under the assumptions of the proposition:

$$\alpha_g + \beta_g > 1 \text{ and } \alpha_b + \beta_b > 1.$$

Recall also that the proposition assumes:

$$(\alpha_g - \beta_g)(\alpha_b - \beta_b) \neq 0.$$

We now consider two cases: (ii.a) $\alpha_g > \beta_g$ and $\alpha_b < \beta_b$; (ii.b) $\alpha_g > \beta_g$ and $\alpha_b > \beta_b$. Other cases are symmetric.

First consider case (ii.a). We display the two joint signal distributions referred to in the proposition in Tables 14 and 15.

ω_g:

	s_g^B	s_b^B
s_g^A	β_g	$\alpha_g - \beta_g$
s_b^A	0	$1 - \alpha_g$

ω_b:

	s_g^B	s_b^B
s_g^A	0	$1 - \alpha_b$
s_b^A	$1 - \beta_b$	$\alpha_b + \beta_b - 1$

TABLE 14. The joint distribution corresponding to $d_g = \underline{d}_g$ and $d_b = \bar{d}_b$ when $\alpha_g + \beta_g > 1$, $\alpha_b + \beta_b > 1$, $\alpha_g > \beta_g$, and $\beta_b > \alpha_b$.

ω_g:

	s_g^B	s_b^B
s_g^A	$\alpha_g + \beta_g - 1$	$1 - \beta_g$
s_b^A	$1 - \alpha_g$	0

ω_b:

	s_g^B	s_b^B
s_g^A	$1 - \beta_b$	$\beta_b - \alpha_b$
s_b^A	0	α_b

TABLE 15. The joint distribution corresponding to $d_g = \bar{d}_g$ and $d_b = \underline{d}_b$ when $\alpha_g + \beta_g > 1$, $\alpha_b + \beta_b > 1$, $\alpha_g > \beta_g$, and $\beta_b > \alpha_b$.

In Table 14 the posterior probability of of ω_g equals 1 with probability $\beta_g/2$ whereas in Table 15 it equals 1 with the lower probability $(1 - \alpha_g)/2$. On the other hand, in Table 14 the posterior probability of ω_g equals 0 with probability $(1 - \beta_b)/2$ whereas in Table 15 it equals 0 with the higher probability $\alpha_b/2$. It follows that neither distribution can be a mean-preserving spread of the other, and therefore neither of the two joint distributions strictly Blackwell dominates the other. By the result in part (i) of the proposition, the claim follows.

Next consider case (ii.b). The joint distribution corresponding to $d_g = \underline{d}_g$ and $d_b = \bar{d}_b$ is the same as case (ii.a). The joint distribution corresponding to $d_g = \bar{d}_g$ and $d_b = \underline{d}_b$ is shown in Table 16.

ω_g:

	s_g^B	s_b^B
s_g^A	$\alpha_g + \beta_g - 1$	$1 - \beta_g$
s_b^A	$1 - \alpha_g$	0

ω_b:

	s_g^B	s_b^B
s_g^A	$1 - \alpha_b$	0
s_b^A	$\alpha_b - \beta_b$	β_b

TABLE 16. The joint distribution corresponding to $d_g = \bar{d}_g$ and $d_b = \underline{d}_b$ when $\alpha_g + \beta_g > 1$, $\alpha_b + \beta_b > 1$, $\alpha_g > \beta_g$, and $\alpha_b > \beta_b$.

In Table 14 the posterior probability of of ω_g equals 1 with probability $\beta_g/2$ whereas in Table 16 it equals 1 with the lower probability $(1-\beta_g)/2$. On the other hand, in Table 14 the posterior probability of ω_g equals 0 with probability $(1-\beta_b)/2$ whereas in Table 16 it equals 0 with the higher probability $\beta_b/2$. It follows that neither distribution can be a mean-preserving spread of the other, and therefore neither of the two joint distributions strictly Blackwell dominates the other. By the result in part (i) of the Proposition, the claim follows.

C.3. **Proof of Proposition 10.**

The joint signal distribution corresponding to $\left(\underline{d}_g, \underline{d}_b\right)$ is shown in Table 3 and the joint signal distribution corresponding to $\left(\bar{d}_g, \bar{d}_b\right)$ is shown in Table 11. We are going to use Proposition 2 to prove that the joint distribution in Table 11 strictly Blackwell dominates the joint distribution in Table 3. We take the partition of signal realizations to which Proposition 2 refers to consist of the two sets: $S_1 = \left\{(s_g^A, s_g^B), (s_g^A, s_b^B)\right\}$ and $S_2 = \left\{(s_b^A, s_g^B), (s_b^A, s_b^B)\right\}$. We prove that with this definition the two sufficient conditions for strict Blackwell dominance in Proposition 2 hold. First we note that the probability that the signal realizations are in S_1 is, under both joint distributions, equal to α_g in state ω_g and equal to $1-\alpha_b$ in state ω_b. Similarly, the probability that the signal realizations are in S_2 is, under both joint distributions, equal to $1-\alpha_g$ in state ω_g and equal to α_b in state ω_b. These probabilities are both strictly positive. Therefore the assertion follows from Proposition 2 if we can prove the strict Blackwell dominance of both conditional signals.

Both conditional signals have just two signal realizations, and therefore generate posterior distributions with two elements in their support. This allows us to use Theorem 2 in Wu (2023) to establish strict Blackwell dominance. According to this theorem, applied to the case that there are only two states and that the supports of the posterior distributions have only two elements, one signal Blackwell dominates another signal if and only if the lower posterior in the support is weakly lower for the dominating signal than it is for the dominated signal, and the higher posterior in the support is weakly higher for the dominating signal than it is for the dominated signal. The Blackwell dominance is strict if and only if at least one of these inequalities is strict.

Conditional on S_1, if the joint signal distribution is parametrized by $\left(\underline{d}_g, \underline{d}_b\right)$, the posterior probability of ω_g corresponding to signal realization (s_g^A, s_g^B) is

$$\frac{\beta_g}{\beta_g + \alpha_g}$$

and the posterior probability of ω_g corresponding to signal realization $\left(s_g^A, s_b^B\right)$ is 1. If the joint signal distribution is parameterized by $\left(\bar{d}_g, \bar{d}_b\right)$, the corresponding posterior probability for (s_g^A, s_b^B) is:

$$\frac{1-\beta_g}{1-\beta_g+\alpha_g}.$$

and, for (s_g^A, s_g^B) it is 1. Now $\beta_g > 0.5$ implies:

$$\frac{1-\beta_g}{1-\beta_g+\alpha_g} < \frac{\beta_g}{\beta_g+\alpha_g}.$$

Therefore, we can infer from Theorem 2 in Wu (2023) that conditional on S_1 the joint signal distribution corresponding to $\left(\bar{d}_g, \bar{d}_b\right)$ strictly Blackwell dominates the joint signal distribution corresponding to $\left(\underline{d}_g, \underline{d}_b\right)$.

Conditional on S_2, if the joint signal distribution is parametrized by $\left(\underline{d}_g, \underline{d}_b\right)$, the posterior probability of ω_g corresponding to signal realization $\left(s_b^A, s_g^B\right)$ is 0 and the posterior probability corresponding to signal realization (s_b^A, s_b^B) is:

$$\frac{\alpha_b}{\alpha_b+\beta_b}.$$

If the joint signal distribution is parameterized by $\left(\bar{d}_g, \bar{d}_b\right)$, the posterior probability for (s_b^A, s_b^B) is 0 and for (s_b^A, s_g^B) it is:

$$\frac{\alpha_b}{\alpha_b+1-\beta_b}.$$

Now $\beta_b > 0.5$ implies:

$$\frac{\alpha_b}{\alpha_b+1-\beta_b} > \frac{\alpha_b}{\alpha_b+\beta_b}.$$

Therefore, we can infer from Theorem 2 in Wu (2023) that conditional on S_2 the joint signal distribution corresponding to $\left(\bar{d}_g, \bar{d}_b\right)$ strictly Blackwell dominates the joint signal distribution corresponding to $\left(\underline{d}_g, \underline{d}_b\right)$. Putting together what we have shown and applying Proposition 2 we obtain the assertion.