Econ 603: General Equilibrium Theory
Lecture Notes

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Most of the material in these notes has been adapted from parts 2 and 3 of Microeconomic Theory *by Andreu Mas-Colell, Michael D. Whinston, and Jerry R. Green, Oxford University Press, 1995. I am grateful to Tamanna Singh Dubey, George Fenton, Nouman Khan, Paul Kindsgrab, Alexandr Moskalev, and Rosina Rodriguez Olivera for their comments on previous versions of these notes. All remaining errors are my own and not the responsibility of Mas-Colell et. al.. These notes are not to be circulated without my permission.*

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1. Pareto Efficiency

In Econ 101 we learn that for some markets economics predicts that an equilibrium outcome will be realized in which supply equals demand. More precisely, economics predicts that all trade will take place at the price at which the quantity supplied by price taking firms is equal to the quantity demanded by price taking consumers. Also, economics predicts for these markets that the quantity sold by each firm is exactly equal to their supply at the equilibrium price, and equally that the quantity bought by each consumer is exactly equal to their demand at the equilibrium price. In Econ 101 we call markets for which we make this prediction "perfectly competitive." General equilibrium theory studies the complete system of all markets in an economy, assuming that each market individually operates as a perfectly competitive market. In other words, we study an equilibrium in which supply equals demand in *all* markets.

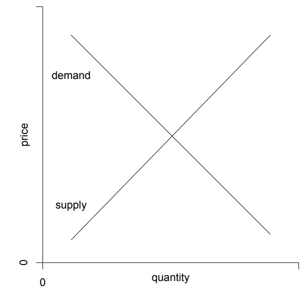


Figure 1: Perfect competition in Econ 101

One reason why we might be interested in modeling all markets together is that we can then ask whether a market system that operates as we have modeled it will allocate the resources of an economy efficiently. It is important that, as in all theory, we are analyzing an imagined world. Even if we find that markets allocate resources efficiently, the question is still open to which extent the real world resembles our imagined world. Thus, the theory of general equilibrium is just a starting point for analyzing the efficiency of the allocation of resources in real world economies.

Dictionary definition of "general:" "affecting or concerning all or most people, places, or things; widespread." General equilibrium theory is called "general" because it studies situations in which equilibrium of supply and demand is "widespread," but not because in the real world the theory could be applied to "most people, places, or things," nor because the theory strives for mathematical generality.

We begin by making precise what we mean by "efficient" use of resources. First, we model the resources available. We assume that there are L goods: $\ell = 1, 2, \dots, L$. The economy has an initial endowment with these goods denoted by $\omega = (\omega_1, \dots, \omega_L) \in \mathbb{R}^L_+$. You may think of this as the economy's endowment with natural resources, but also, for example, as the economy's endowment with time that is available for human labor. Labor, actually labor of different sorts, may be among the L goods.

In addition, the economy has some production possibilities, that is, the ability to produce from the initial endowment with some resources some other goods. We describe all the possibilities that are available to the economy by a set $Y \subseteq \mathbb{R}^L$, to which we refer as the "production set." Suppose $y = (y_1, \dots, y_L)$ is an element of Y. We refer to such a y as a feasible "production plan." If $y_\ell < 0$, the production plan uses a quantity y_ℓ of good ℓ as an input. If $y_\ell > 0$, the production plan produces a quantity y_ℓ of good ℓ as an output. Y is the complete set of all feasible production plans.

When I write: "$\omega = (\omega_1, \dots, \omega_L)$," I mean by the right hand side not a vector of vectors, but I think of it as a large vector created by concatenating smaller vectors. For example: if $x = (1, 2)$, and $y = (4, 5)$, then $(x, y) = (1, 2, 4, 5)$. I use this notation throughout these notes.

In Econ 101 we have used production functions to describe production possibilities. The concept of a production set is an elegant generalization of the concept of a production function to production with potentially multiple inputs as well as outputs. But production possibilities that are described by a production function can easily be fitted into our framework. Suppose, for example, there is a production function that assigns to every quantity

$x_1 \geq 0$ of good 1 an output $f(x_1)$ of good 2 that can be produced with input x_1. Suppose that goods 1 and 2 are the only goods in the economy. Then in our framework, the production possibilities are described by the set $Y = \left\{(-x_1, x_2) \in \mathbb{R}^2 | x_1 \leq 0 \text{ and } x_2 = f(x_1).\right\}$. In some contexts, it may be plausible to replace the equation $x_2 = f(x_1)$ by the equation $x_2 \leq f(x_1)$. We could imagine that it is always possible to waste some of x_1, or that it is always possible to throw away some of the output. We refer to this assumption as "free disposal." Obviously, in practice, not all goods can be freely disposed of.

For example, nuclear waste cannot be freely disposed of.

Which vectors of quantities can the economy achieve, using its initial endowment, and its production possibilities? The set of all such vectors is the set R which we define as follows:

$$R = \left\{x \in \mathbb{R}^L_+ | x = \omega + y \text{ for some } y \in Y\right\}.$$

A vector $\omega + y$ is feasible if $y \in Y$ because, whenever some quantities from ω are used as inputs, they are subtracted from ω, and if some quantities are produced as outputs, they are added to ω.

Now that we have described the economy's capabilities, we need to introduce consumers. Their satisfaction will be the measure of efficiency of the use of resources in the economy. Note that we don't introduce firms, and that firms' profits, or other indicators of firms' "well-being," are not directly relevant for efficiency. Firms are, in this vision of the economy, not agents in themselves. Their usefulness will depend on the extent to which they contribute to consumers' satisfaction.

Mitt Romney, the 2012 Republican candidate for the US presidency, famously claimed "corporations are people too." This is not true in general equilibrium theory.

We assume that there are I consumers: $i = 1, 2, \ldots, I$. Each consumer has a consumption set $X_i \subseteq \mathbb{R}^L_+$. Every element x_i of X_i indicates some quantities of each of the L goods that consumer i will consume. We refer to the elements of X_i as "consumption plan of consumer i." Consumer i has a rational, that is, complete and transitive preference relation $\succsim_i$ on X_i.

A "consumption plan" x is a vector of consumption plans, one for each consumer: $x = (x_1, \ldots, x_I) \in \mathbb{R}^{IL}_+$. A consumption plan is feasible if each individual consumer's consumption plan is in this consumer's consumption set, and if its total use of resources is in R. Let us introduce our first formal definition:

Definition: A consumption plan x is "feasible" if:

$$x_i \in X_i \text{ for all } i = 1, 2, \ldots, I$$
$$\text{and} \quad \sum_{i=1}^{I} x_i \in R.$$

Now we can say what we mean by "efficient" use of resources:

Definition: A feasible consumption plan x is "Pareto efficient" if there is no other feasible consumption plan $\hat{x}$ such that

$$\hat{x}_i \succsim_i x_i \text{ for all } i \in \{1, \dots, I\}$$

and

$$\hat{x}_i \succ_i x_i \text{ for at least one } i \in \{1, \dots, I\}.$$

Vilfredo Pareto, 1848-1923

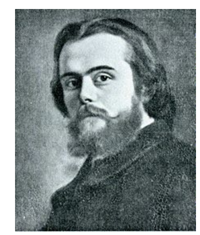
We don't ask for much from Pareto efficient consumption plans. Roughly speaking, a feasible consumption plan is Pareto efficient if there is no other feasible consumption plan that is unanimously preferred by all agents. For example, picking the production plan according to some agent i's preferences, and then giving all resources, including all output, to this agent i, is typically Pareto-efficient. Thus, there will typically be many Pareto efficient consumption plans.

Find sufficient conditions for the allocation described here to be Pareto efficient.

We might be tempted to use a more demanding definition of efficient use of resources, for example, we might require consumption plans to maximize "welfare" among all feasible consumption plans. But in Section 6 below we shall discuss some problems with the notion of "welfare."

2. Walrasian Equilibrium

Léon Walras, 1834-1910



We now formalize the concept of an outcome at which supply equals demand in all markets. We shall refer to such an outcome also as a "Walrasian equilibrium." To define Walrasian equilibrium we have to amend the model of the previous section by a description of private ownership. We assume that each consumer $i \in I$ owns some, possibly zero, of the initial resources. The "initial endowment of consumer i," i.e. what consumer i initially owns, is $\omega_i \in \mathbb{R}^L_+$. To make the model consistent with the previous section, we obviously have to assume:

$$\sum_{i=1}^{I} \omega_i = \omega.$$

We also introduce J firms: $j = 1, \dots, J$. Each firm has a certain set of production abilities $Y_j \subseteq \mathbb{R}^L$. We refer to this as firm j's production set. Elements y_j of Y_j are interpreted in the same way as elements of the production set Y in the previous section, i.e. negative entries are inputs and positive entries are outputs. If $y_j \in Y_j$, then firm j is capable of carrying out production plan y_j. To make our model consistent with the previous section, we assume:

$$Y = \left\{ y \in \mathbb{R}^L | \text{ there are } y_j \in Y_j \text{ (for all } j = 1, \ldots, J) \text{ such that } y = \sum_{j=1}^{J} y_j \right\}.$$

Short notation for this equation: $Y = \sum_{j=1}^{J} Y_j$. The right hand side of the equation in the text is thus defined to be the "sum" of the sets Y_j.

Thus, Y represents the set of all sums of production plans of all individual firms.

Firms are owned by consumers. Consumer i's share in firm j is $\theta_i^j \in [0, 1]$. We assume:

$$\sum_{i=1}^{I} \theta_i^j = 1 \text{ for all } j = 1, \ldots, J.$$

Owning share θ_i^j in firm j entitles consumer i to a share of θ_i^j in firm j's profits.

We are almost ready to define the equilibrium concept. To make that definition shorter, we introduce two auxiliary definitions.

Definition: A "price vector" is a vector $p \in \mathbb{R}^L$.

Supply of and demand for nuclear waste may equal each other only at a negative price, i.e. the seller has to pay the buyer.

Note that we don't require prices to be non-negative.

Definition: An "allocation" is a vector $(x_1, \ldots, x_I, y_1, \ldots, y_J) \in \mathbb{R}^{(I+J)L}$ such that:

$$x_i \in X_i \text{ for all } i = 1, \ldots, I$$
$$y_j \in Y_j \text{ for all } j = 1, \ldots, J$$

Note that the first I components of an allocation form a consumption plan in the sense of the previous section.

Definition: A price vector p^* and an allocation (x^*, y^*) are a "Walrasian equilibrium" if:

(i) each firm j maximizes profits, i.e. for each $j \in \{1, \ldots, J\}$:

$$p^* \cdot y_j^* \geq p^* \cdot y_j \text{ for all } y_j \in Y_j$$

$p^* \cdot y_j^*$ is the "dot product" of p^* and y_j^* which is firm j's profit. Each positive component of y_j^* is output, so multiplied by the price we obtain revenue. Each negative component of y_j^* is input, so multiplied by the price we obtain cost, which get subtracted because of the negative sign in y_j^*.

(ii) each consumer i maximizes her preferences in her budget set, i.e. for each $i \in \{1, \ldots, I\}$:

$$p^* \cdot x_i^* \leq p^* \cdot \omega_i + \sum_{j=1}^{J} \theta_i^J p^* \cdot y_j^*$$

and $x_i^* \succsim_i x_i$ for all $x_i \in X_i$ such that

$$p^* \cdot x_i \leq p^* \cdot \omega_i + \sum_{j=1}^{J} \theta_i^j p^* \cdot y_j^*$$

$p^* \cdot x_i$ is the expenditure required to buy x_i. $p^* \cdot \omega_i$ is the value of consumer i's initial endowment.

(iii) supply equals demand in all markets:

$$\sum_{i=1}^{I} x_i^* = \sum_{i=1}^{I} \omega_i + \sum_{j=1}^{J} y_j^*.$$

A few words of explanation. The inequality in condition (ii) is consumer i's budget constraint. The budget constraints are written as if consumers initially sold all their endowments to the market, collected all profits, and then spent their money on consumption. But of course, there is no such time line in our model, nor does a consumer need to sell all of her initial endowment. She might as well keep some of it. But this is the same as first selling all, and then buying some back. It is without loss of generality to write the budget constraint as we did. The condition that supply equals demand is again written imagining that all endowments, summed with the production plans, form supply, as if consumers sold all of their initial endowment to the market, and then potentially bought some back.

Walrasian equilibrium is a state where supply equals demand in all markets. But note that to define equilibrium we have not really described “demand and supply functions." In Econ 101 we would first construct such functions, and then define equilibrium to be the outcome where the value of the demand function equals the value of the supply function. Constructing the complete demand and supply functions means that we determine the quantities supplied and demanded in as well as out of equilibrium. But that is not really necessary, as the above definition shows. In the above definition x_i^* is consumer i's demand in equilibrium, and y_j^* is firm j's supply in equilibrium (including its factor demands). To check whether at p^* supply equals demand, we only need to know supply and demand for price p^*, not for other prices.

Our definition of Walrasian equilibrium is in one other way extremely parsimonious: we have not distinguished between factor markets, markets for intermediate goods, and markets for consumption goods. We didn't have to. In fact, the model allows markets to be any mixture between these types of markets. For example, we might think of a factor market as a market in which all buyers are firms, and all sellers are consumers. But the above definition allows markets in which some consumers are buyers, whereas some firms, and some consumers, are sellers.

If at this point, it is not clear to you that Walrasian equilibrium is a formalization of the idea that "supply equals demand in all markets" then I have failed in my efforts to explain this. Don't hesitate to complain to me about this.

We continue with some small observations about Walrasian equilibrium.

Observation 1: If (p^*, x^*, y^*) is a Walrasian equilibrium, and if $\lambda > 0$, then also $(\lambda p^*, x^*, y^*)$ is a Walrasian equilibrium.

Observation 1 is obvious, because multiplying all prices by a positive constant λ does not change any budget constraints, nor does it change which production plans are profit maximization. Yet, Observation 1 is also very important. It implies that the Walrasian model does not make predictions about absolute prices, it only makes predictions about price ratios, i.e. relative prices. This is because only relative prices affect agents' choices in the Walrasian model.

Modern macroeconomics uses general equilibrium models to study inflation. Inflation is about absolute prices, not relative prices. So, how do macroeconomists do this? By taking the model presented here as a starting point, and then modifying it in many ways. You need to first understand the starting point.

For the next observation recall that a preference relation $\succsim_i$ is called "locally nonsatiated" if for every $x_i \in X_i$ and every $\varepsilon > 0$ there exists an $\hat{x}_i \in X_i$ such that $\hat{x}_i \succ_i x_i$ and $||\hat{x}_i - x_i|| \leq \varepsilon$. That is, $\succsim_i$ is locally nonsatiated if for every feasible consumption plan of consumer i, and for every, arbitrarily small $\varepsilon > 0$, there is another feasible consumption plan $\hat{x}_i$ that consumer i strictly prefers to x_i, and that is only ε in Euclidean distance away from x_i.

Local nonsatiation is defined in MWG in Definition 3.B.3.

Observation 2: If every consumer i's preference $\succsim_i$ is locally nonsatiated, and if (p^*, x^*, y^*) satisfies condition (ii) in the definition of a Walrasian equilibrium, then:

$$p^* \cdot \sum_{i=1}^{I} x_i^* = p^* \cdot \sum_{i=1}^{I} \omega_i + p^* \cdot \sum_{j=1}^{J} y_j^*.$$

In words, Observation 2 says that when consumers maximize locally nonsatiated preferences, the value of demand will equal the value of supply. This, too, is almost obvious. From consumer theory we know that a consumer who has locally nonsatiated preferences, and chooses a consumption bundle that maximizes these preferences, satisfies Walras' law, i.e. fully spends her budget. Observation 2 says that this remains true when we add over all consumers. Note that Observation 2 is true regardless of whether firms maximize profits, or supply equals demand.

Walras' law for individual consumers is Proposition 3.D.2, part (ii), in MWG.

Observation 3: Suppose every consumer i's preference $\succsim_i$ is locally nonsatiated, and (p^*, x^*, y^*) satisfies

Assuming $p_L^* \neq 0$ is inelegant. One could replace this condition by sufficient conditions on preferences and technologies that imply that $p_L \neq 0$. But that would complicate the statement and proof of Observation 3 by more than the benefits could justify.

condition (ii) in the definition of a Walrasian equilibrium as well as condition (iii) for goods $\ell = 1, \dots, L-1$. Suppose also that $p_L^* \neq 0$. Then (p^*, x^*, y^*) also satisfies condition (iii) in the definition of a Walrasian equilibrium for good L.

In words, this means that if $L-1$ markets clear when consumers with locally nonsatiated preferences maximize preferences, then also the remaining market clears, provided that the price in that market is non-zero. In the Observation we identify the remaining market as the market for good L, but the choice of good L as the remaining good is arbitrary. The proof of Observation 3, unlike the proofs of Observations 1 and 2, is worth writing down.

We say that a market "clears" if supply equals demand in that market.

Proof. Re-writing the conclusion of Observation 2, we know already:

$$\sum_{\ell=1}^{L} p_\ell^* \sum_{i=1}^{I} x_{i,\ell}^* = \sum_{\ell=1}^{L} p_\ell^* \left(\sum_{i=1}^{I} \omega_{i,\ell} + \sum_{j=1}^{J} y_{j,\ell}^* \right),$$

where we write $x_{i,\ell}^*$ for the ℓ-th component of x_i^* and $y_{j,\ell}^*$ for the ℓ-th component of y_j^*. The assumption that markets $\ell = 1, \dots, L-1$ clear means that for $\ell = 1, \dots, L-1$ we have:

$$\sum_{i=1}^{I} x_{i,\ell}^* = \sum_{i=1}^{I} \omega_{i,\ell} + \sum_{j=1}^{J} y_{j,\ell}^*.$$

Multiplying both sides of this equation by p_ℓ^* we get:

$$p_\ell^* \sum_{i=1}^{I} x_{i,\ell}^* = p_\ell^* \left(\sum_{i=1}^{I} \omega_{i,\ell} + \sum_{j=1}^{J} y_{j,\ell}^* \right).$$

Summing over $\ell = 1, \dots, L-1$ we obtain:

$$\sum_{\ell=1}^{L-1} p_\ell^* \sum_{i=1}^{I} x_{i,\ell}^* = \sum_{\ell=1}^{L-1} p_\ell^* \left(\sum_{i=1}^{I} \omega_{i,\ell} + \sum_{j=1}^{J} y_{j,\ell}^* \right),$$

Subtracting this equation from our first equation, what remains is:

$$p_L^* \sum_{i=1}^{I} x_{i,L}^* = p_L^* \left(\sum_{i=1}^{I} \omega_{i,L} + \sum_{j=1}^{J} y_{j,L}^* \right).$$

Finally, because by assumption $p_L \neq 0$, we can divide both sides of this equation by p_L. Then we get:

$$\sum_{i=1}^{I} x_{i,L}^* = \sum_{i=1}^{I} \omega_{i,L} + \sum_{j=1}^{J} y_{j,L}^*,$$

which is condition (iii) in the definition of Walrasian equilibrium for good L. $\square$

3. The First Welfare Theorem

Now we can state and prove what is arguably the most important theorem of economic theory. Let us call an allocation (x^*, y^*) "feasible" if $\sum_{i=1}^{I} x_i^* = \sum_{i=1}^{I} \omega_i + \sum_{j=1}^{J} y_j^*$, and let us call a feasible allocation "Pareto efficient" if the consumption plan x^* is Pareto efficient in the sense of Section 1.

Does the First Welfare Theorem explain why the inhabitants of market economies are, on average, richer than the inhabitants of other economies?

First Welfare Theorem: Suppose all preferences $\succsim_i$ are locally nonsatiated. If (p^*, x^*, y^*) is a Walrasian equilibrium, then (x^*, y^*) is Pareto-efficient.

Note that the conclusion is weak: many consumption plans are Pareto-efficient, as we noted in Section 1. Moreover, the assumptions are hidden, but strong: price taking behavior and no externalities in consumption or production.

The modern proofs of the first and second welfare theorems, versions of which we explain in this and the next section, are due to Kenneth Arrow ("An Extension of the Basic Theorems of Classical Welfare Economics," in: Jerzy Neyman (editor), *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, 1951, pp. 507-532) and Gérard Debreu ("The Coefficient of Resource Utilization," *Econometrica* 1951.) Both acknowledged earlier work by, among others, Oskar Lange and Abba Lerner.

Proof. Suppose (x^*, y^*) were not Pareto-efficient. Then there would be another feasible consumption plan $(\hat{x}, \hat{y})$ such that $\hat{x}_i \succsim_i x_i^*$ for all i with strict preference for at least one i. We observe first that:

$$\hat{x}_i \succ_i x_i^* \Rightarrow p^* \cdot \hat{x}_i > p^* \cdot \omega_i + \sum_{j=1}^{J} \theta_i^j (p^* \cdot y_j^*).$$

This follows from the fact that x_i^* maximizes $\succsim_i$ subject to the budget constraint. Next we observe that:

$$\hat{x}_i \succsim_i x_i^* \Rightarrow p^* \cdot \hat{x}_i \geq p^* \cdot \omega_i + \sum_{j=1}^{J} \theta_i^j (p^* \cdot y_j^*).$$

We prove this indirectly. Suppose there were $\hat{x}_i$ such that $\hat{x}_i \succsim_i x_i^*$ and $p^* \cdot \hat{x}_i < p^* \cdot \omega_i + \sum_{j=1}^{J} \theta_i^j (p^* \cdot y_j^*)$. Pick a small ball around $\hat{x}_i$ such that for all x_i in the ball $p^* \cdot x_i < p^* \cdot \omega_i + \sum_{j=1}^{J} \theta_i^j (p^* \cdot y_j^*)$. By the assumption of local

This is the (only) point in the proof where we use the assumption of local nonsatiation.

nonsatiation, for one of these x_i we have: $x_i \succ_i \hat{x}_i$. By assumption $\hat{x}_i \succsim_i x_i^*$, and hence $x_i \succ_i x_i^*$. But this means that consumer i could afford x_i at prices p^* and strictly prefers x_i to x_i^*. This contradicts the assumption that (p^*, x^*, y^*) is a Walrasian equilibrium, and hence consumer i maximized his preference subject to the budget constraint when choosing x^*.

Adding up the two inequalities that we have found over all i, remembering that for at least one agent the strict inequality has to hold, we get:

$$p^* \cdot \sum_{i=1}^{I} \hat{x}_i > p^* \cdot \sum_{i=1}^{I} \omega_i + p^* \cdot \sum_{j=1}^{J} y_j^*, \tag{1}$$

where in the last term on the right hand side we have taken into account that all consumers' shares in each firm add up to 1.

Kenneth Arrow

Our final observation is that for all firms j:

$$p^* \cdot y_j^* \geq p^* \cdot \hat{y}_j.$$

This is because y_j^* is maximizes profits at prices p^*. We can now infer from (1):

$$p^* \cdot \sum_{i=1}^{I} \hat{x}_i > p^* \cdot \sum_{i=1}^{I} \omega_i + p^* \cdot \sum_{j=1}^{J} \hat{y}_j.$$

But this contradicts the assumption that $(\hat{x}, \hat{y})$ is feasible:

Gérard Debreu, 1921-2004

$$\sum_{i=1}^{I} \hat{x}_i = \sum_{i=1}^{I} \omega_i + \sum_{j=1}^{J} \hat{y}_j.$$

□

4. The Second Welfare Theorem

Although, for the reason mentioned in Section 1, we do not use the concept of welfare in these notes, we can imagine a policy maker who regards some Pareto efficient allocations as more desirable than others. For example, such a policy maker may be inequality averse. Our choice not to consider the concept of welfare in these notes only means that we do not examine in detail the logic of arguments on which a policy maker's preference for some Pareto efficient allocation rather than another may be based. One may wonder whether such a policy maker can use the market system to implement some Pareto efficient allocation rather than another. The answer is that this is possible when a very particular form of transfers can be implemented.

Definition: A system of transfers T is a vector $T = (T_1, \ldots, T_I)$ such that $\sum_{i=1}^{I} T_i = 0$.

A positive transfer $T_i > 0$ is interpreted as money that we give to agent i. A negative transfer T_i is interpreted as a tax that we charge to agent i. An important feature of the transfers that we consider them here is hidden in this definition. It is that the transfer T_i for any agent i is independent of any choice that agent i makes. For example, it does not depend on any of agent i's labor supply or consumption choices. Thus, income taxes, commodity taxes, social security payments, etc., as they exist in the real world, are all *not* transfers in the sense of the above definition. What we consider here is also called "lump sum" transfers and taxes in the literature. Note also that our definition of a system of transfers requires that transfers balance each other out, that is, there is no surplus nor any deficit.

Now we can define the concept of Walrasian equilibrium if in some way, perhaps by the government, lump sum transfers are introduced.

Definition: Let T be a system of transfers. A price vector p^* and an allocation (x^*, y^*) are a "Walrasian equilibrium with transfers T" if:

(i) each firm j maximizes profits, i.e. for each $j \in \{1, \ldots, J\}$:

$$p^* \cdot y_j^* \geq p^* \cdot y_j \text{ for all } y_j \in Y_j$$

(ii) each consumer i maximizes her preferences, i.e. for each $i \in \{1, \ldots, I\}$:

$$x_i^* \succsim_i x_i \text{ for all } x_i \in X_i \text{ such that}$$

$$p^* \cdot x_i \leq p^* \cdot \omega_i + \sum_{j=1}^{J} \theta_i^j p^* \cdot y_j^* + T_i$$

(iii) supply equals demand in all markets:

$$\sum_{i=1}^{I} x_i^* = \sum_{i=1}^{I} \omega_i + \sum_{j=1}^{J} y_j^*.$$

Will lump sum transfers disturb the efficiency of market equilibrium? The answer is “no.” You can check yourself that the proof of the first welfare theorem goes through without any substantial change even if there are lump sum transfers. Therefore, we state the following result without proof:

It will also be instructive for you to check how the proof of the first welfare theorem may fail to go through when transfers that are not “lump sum” are introduced.

First Welfare Theorem With Lump Sum Transfers: Suppose all preferences $\succsim_i$ are locally nonsatiated. Let T be a system of transfers. If (p^*, x^*, y^*) is a Walrasian equilibrium with transfers T, then (x^*, y^*) is Pareto-efficient.

Of course, transfers will influence which Pareto-efficient outcome is reached in Walrasian equilibrium. That is why we have them. The result just clarifies that, even with transfers, some Pareto-efficient outcome is reached.

The question that we raised at the beginning of this section can now be formalized as the question whether the converse of the First Welfare Theorem with Lump Sum Transfers is true. Here, we mean by the converse the claim that with appropriate lump sum taxes and transfers any Pareto-efficient allocation can be a Walrasian equilibrium outcome. The Second Welfare Theorem says that this is true under assumptions that are significantly stronger than the assumptions of the First Welfare Theorem. To make the statement of the Second Welfare Theorem as short as possible we initially state it without explicitly listing the assumptions that we shall make use of in the proof. Instead, we refer to these assumptions simply as Assumptions 1-4. Each of the four assumptions will be then be introduced in the proof, at precisely the point at which we use the assumption for the first time.

Second Welfare Theorem: Under Assumptions 1-4, if (x^*, y^*) is a Pareto efficient allocation, then there are a system of transfers T and a price vector p such that (p, x^*, y^*) is a Walrasian equilibrium with transfers T.

This result at first sight seems to make a much stronger case for markets than the first welfare theorem. But a government that wants to rely on this theorem must know which Pareto-efficient allocation it aims for, and which transfers will take us the economy to this allocation if it is to be an equilibrium. The government thus needs to know a lot. Also, as is true for the first welfare theorem, price-taking behavior and no externalities are hidden assumptions in the second welfare theorem.

Proof. The proof idea is to construct the price vector p as the vector that is perpendicular to a hyperplane that separates certain sets. We start by introducing these sets. We denote them by V and R.

$$V = \Big\{ x \in \mathbb{R}^L \,|\, x \text{ can be written as a sum } x = \sum_{i=1}^{I} x_i$$
$$\text{where the } x_i \text{ satisfy the following conditions:}$$
$$x_i \in X_i \text{ for every } i, \text{ and } x_i \succsim_i x_i^* \text{ for every } i, \text{ and}$$
$$x_i \succ_i x_i^* \text{ for at least one } i \Big\}$$

$$R = \Big\{ y \in \mathbb{R}^L \,|\, y \text{ can be written as a sum } y = \sum_{i=1}^{I} \omega_i + \sum_{j=1}^{J} y_j$$
$$\text{where } y_j \in Y_j \text{ for every } j \Big\}$$

The set R was already introduced in Section 1. We repeat its definition here, using some of the additional notation that we have introduced since Section 1.

V is the set of all resource vectors that would be enough to Pareto improve on x^* (i.e. make every consumer at least as well off as in x^*, and make at least one consumer strictly better off than in x^*), and R is the set of all resource vectors that are feasible given the economy's initial endowment and given the firms' technological capabilities.

We know:

$$V \cap R = \emptyset$$

because, if there were a feasible resource vector in R that Pareto-improved on x^*, then x^* would not be Pareto efficient. To be able to use a separating hyperplane theorem, we want to prove next:

Claim 1: V is convex, and R is convex.

To prove this we need to start making assumptions. We start with an assumption that implies that V is convex. This will be an assumption about consumers' characteristics:

Assumption 1: *For every consumer i the set X_i is convex, and the preference relation $\succsim_i$ satisfies the following two conditions:*

1. *For all $x_i, x_i', \hat{x}_i, \in X_i$, such that $x_i \succsim_i \hat{x}_i$ and $x_i' \succsim_i \hat{x}_i$ it is true that: $\lambda x_i + (1-\lambda)x_i' \succsim_i \hat{x}_i$ for all $\lambda \in (0,1)$;*
2. *For all $x_i, x_i', \hat{x}_i, \in X_i$, such that $x_i \succ_i \hat{x}_i$ and $x_i' \succsim_i \hat{x}_i$ it is true that: $\lambda x_i + (1-\lambda)x_i' \succ_i \hat{x}_i$ for all $\lambda \in (0,1)$.*

One is tempted to phrase Assumption 1 as requiring that the preference relations are "convex," however, what we assume is neither exactly identical to the convexity of preferences as defined in Definition 3.B.4 in MWG, nor to the strict convexity of preferences as defined in Definition 3.B.5 in MWG. Of course, though, our assumption is very similar to the properties defined in these two definitions.

Let us prove that this assumption implies that the set V is convex. We want to show that if $x, x' \in V$ and $\lambda \in (0,1)$, then $\lambda x + (1-\lambda)x' \in V$. Now $x \in V$ means that x can be written as a sum $x = \sum_{i=1}^{I} x_i$ where the x_i satisfy: $x_i \in X_i$ for every i, and $x_i \succsim_i x_i^*$ for every i, and $x_i \succ_i x_i^*$ for at least one i. Similarly, $x' = \sum_{i=1}^{I} x_i'$ where the x_i' have analogous properties. Now consider $\lambda x + (1-\lambda)x'$. We can write this vector as follows:

$$\lambda x + (1-\lambda)x' = \lambda \sum_{i=1}^{I} x_i + (1-\lambda) \sum_{i=1}^{I} x_i' = \sum_{i=1}^{I} (\lambda x_i + (1-\lambda)x_i')$$

which shows that we can complete our proof by showing that $\lambda x_i + (1-\lambda)x_i' \in X_i$ for every i, and $\lambda x_i + (1-\lambda)x_i' \succsim_i x_i^*$ for every i, and $\lambda x_i + (1-\lambda)x_i' \succ_i x_i^*$ for at least one i. To begin with, $\lambda x_i + (1-\lambda)x_i' \in X_i$ follows because $x_i \in X_i$ and $x_i' \in X_i$ and the set X_i is by Assumption 1 convex. Next, $\lambda x_i + (1-\lambda)x_i' \succsim_i x_i^*$ follows from $x_i \succsim_i x_i^*$ and $x_i' \succsim_i x_i^*$ and condition 1 in Assumption 1. Finally, we need to identify some i such that $\lambda x_i + (1-\lambda)x_i' \succ_i x_i^*$ for at least one i. Because $x \in V$, there is at least one i for whom $x_i \succ_i x_i^*$. Also, because $x_i' \in V$, for the same agent i it must be true that $x_i' \succsim_i x_i^*$. By condition 2 in Assumption 1 that implies $\lambda x_i + (1-\lambda)x_i' \succ_i x_i^*$. This completes the proof that V is convex.

Next, we want to make an assumption that guarantees that R is convex.

Assumption 2: *For every firm j the set Y_j is convex.*

Assumption 2 rules out increasing returns to scale. Specifically, suppose there are only two goods, and suppose that Y is derived from a production function f as explained in Section 1, that is, $Y = \{(-x_1, x_2) \in \mathbb{R}^2 | x_1 \geq 0 \text{ and } x_2 \leq f(x_1)\}$. If f is strictly convex, i.e. has increasing returns to scale, then Y is not convex. (Prove this.) Assumption 2 is thus very restrictive. As is Assumption 1, of course.

Let us prove that this assumption implies that the set R is convex. We want to show that if $r, r' \in R$ and $\lambda \in (0,1)$, then $\lambda r + (1-\lambda)r'$ in R. Now $r \in R$ means that r can be written as a sum $r = \sum_{i=1}^{I} \omega_i + \sum_{j=1}^{J} y_j$ where $y_j \in Y_j$ for every j. Similarly, $r' \in R$ means that r' can be written as a sum $r' = \sum_{i=1}^{I} \omega_i + \sum_{j=1}^{J} y_j'$ where $y_j' \in Y_j$ for every j. Now consider $\lambda r + (1-\lambda)r'$. We can write this vector as follows:

$$\begin{aligned}\lambda r + (1-\lambda)r' =& \lambda \left(\sum_{i=1}^{I} \omega_i + \sum_{j=1}^{J} y_j \right) + (1-\lambda) \left(\sum_{i=1}^{I} \omega_i + \sum_{j=1}^{J} y_j' \right) \\ =& \sum_{i=1}^{I} \omega_i + \sum_{j=1}^{J} \left(\lambda y_j + (1-\lambda) y_j' \right)\end{aligned}$$

The definition of R now means that we have proved that $\lambda r + (1-\lambda)r' \in R$ if we can show that for every j the production plan $\lambda y_j + (1-\lambda)y'_j$ is in Y_j. But this follows from Assumption 2, together with $y_j \in Y_j$ and $y'_j \in Y_j$. This completes the proof that R is convex.

So far, we know that the two sets V and R are both convex, and that they don't intersect. It will be useful for us to note one more property of the two sets: it is that, although the two sets don't intersect, they almost "touch" each other, more precisely, they get arbitrarily close to each other. This happens in the point that corresponds to the Pareto efficient allocation (x^*, y^*). Because this allocation is feasible, it must be the case that $x^* = y^* + \omega$. By the definition of R, we have that $y^* + \omega^* \in R$. The same vector is *not* in the set V. But, and this is our claim, it is on the boundary of V. This is what we mean by saying that V and R "almost touch" each other. Mathematically, the claim that x^* is on the boundary of V can be formalized as follows:

Claim 2: There is a sequence of $x_n \in V$ for all $n \in \mathbb{N}$ such that $\lim_{n\to\infty} x_n = x^*$.

To prove this claim, we make an additional assumption.

Assumption 3: *For every consumer i the preference relation $\succsim_i$ is locally nonsatiated.*

We now show how Assumption 3 implies Claim 2. To prove Claim 2, we construct the sequence of which we assert in Claim 2 that it exists. We proceed as follows: For every $n \in \mathbb{N}$ let $\varepsilon_n > 0$ be some number, and assume that $\lim_{n\to\infty} \varepsilon_n = 0$. Denote for every $i = 1, 2, \ldots, I$ and every $n \in \mathbb{N}$ by $x_i(\varepsilon_n)$ some consumption plan in X_i that is not more than ε_n away from x_i^*, i.e. $||x_i(\varepsilon_n) - x_i^*|| \leq \varepsilon_n$ where $||...||$ denotes the Euclidean distance, and that is strictly preferred to x_i^*, i.e. $x_i(\varepsilon) \succ_i x_i^*$. The local nonsatiation assumption, i.e. Assumption 3, implies that for every i and every n such a consumption plan exists. Now we set for every $n \in \mathbb{N}$:

$$x_n \equiv \sum_{i=1}^{I} x_i(\varepsilon_n).$$

We have to check that $x_n \in V$ for all $n \in \mathbb{N}$. But this is true by definition of V and of the x_n's. We also have to show that the sequence x_n converges to x^*. As each $x_i(\varepsilon_n)$ is not more than ε_n away from x_i^*, and ε_n converges to zero, we have that each of the sequences $x_i(\varepsilon_n)$ converges to x_i^*. But then also the sum of the $x_i(\varepsilon_n)$, that is, x_n, converges to the sum of the x_i^*, that is, to x^*. This completes the proof of Claim 2.

Here is a graph that shows what R and V might look like in the case that there are only two commodities:

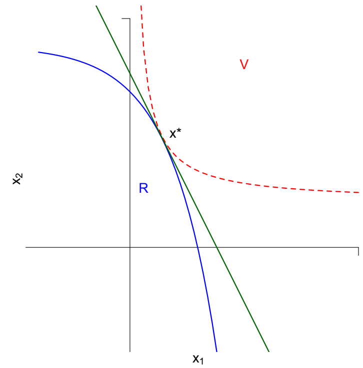


Figure 2: Sets R and V "kiss."

For the moment, ignore the straight line. The set V is above the dashed red line, and the set R is below the unbroken blue line. You can see that the two sets are drawn as convex sets. Moreover, I have drawn the red line dashed to indicate that the line does not belong to V, whereas the blue line is unbroken, to show that the blue line does belong to R. The point x^* that I have marked is on the boundary of the set V, though not in the set V, and it is on the boundary of, and thus in, the set R. In this sense, the two sets "almost touch."

Now we can apply the separating hyperplane theorem to V and R. What we need to know to apply the theorem is that $V \cap R = \emptyset$, and that V and R are convex. This is what we have already shown above. The theorem implies that there is a $p \in \mathbb{R}^L$ such that $p \neq 0$, and an $r \in \mathbb{R}$ such that $p \cdot x \geq r$ for every $x \in V$ and $p \cdot y \leq r$ for every $y \in R$.

The version of the separating hyperplane theorem that we use is the claim made in MWG on page 948 in the paragraph that follows Theorem M.G.2. Theorem M.G.2. itself is not quite what we need.

In the figure above, the straight line between V and R is the separating "hyperplane." Because the sets V and R touch in the point x^* it is now also very plausible that the point x^* itself is exactly on the separating hyperplane. This is shown in the following claim:

In two dimensions, “hyperplane” is a fancy name for a straight line.

Claim 3: $p \cdot x^* = r$.

This claim shows that x^* is on the separating hyperplane because the equation that describes all points on the separating hyperplane is $p \cdot x = r$.

To prove Claim 3, we first prove $p \cdot x^* \leq r$, and then $p \cdot x^* \geq r$. To prove the former inequality, we note that $x^* \in R$, and $p \cdot x \leq r$ for all $x \in R$, we have: $p \cdot x^* \leq r$. To prove the converse we use Claim 2. Consider the sequence x_n the existence of which is asserted in Claim 2. Because every element of this sequence is in V, we have:

$$p \cdot x_n \geq r \text{ for every } n \in \mathbb{N}.$$

Taking limits for $n \to \infty$ on the left hand side, we obtain:

$$p \cdot x^* \geq r,$$

and this completes the proof of Claim 3.

Our aim is now to complete the proof of the second welfare theorem by showing the following:

Claim 4: The vector (p, x^*, y^*) forms a Walrasian equilibrium with transfers $T_i = p \cdot x_i^* - p \cdot \omega_i - p \cdot \sum_{j=1}^{J} \theta_i^j y_j^*$ for every i.

Thus, in Claim 4 we give a specific system of transfers such that, if this system of transfers is implemented, (x^*, y^*) will be a Walrasian equilibrium allocation. Because the Pareto efficient allocation (x^*, y^*) is arbitrary, we thus have shown that for every Pareto efficient allocation we can find a system of transfers such that, if this system is implemented, (x^*, y^*) will be a Walrasian equilibrium allocation. Thus, the proof of the Second Welfare Theorem will be complete.

Let us break down Claim 4 into several sub-claims which we prove separately.

Claim 4a: For every firm j the production plan y_j^* maximizes profits: $p \cdot y_j^* \geq p \cdot y_j$ for all $y_j \in Y_j$.

We prove this indirectly. Suppose there were a firm $\hat{j}$ and a $y_{\hat{j}} \in Y_{\hat{j}}$ such that $p \cdot y_{\hat{j}} > p \cdot y_{\hat{j}}^*$. Then it would follow that:

$$p \cdot \left(\sum_{i=1}^{I} \omega_i + y_{\hat{j}} + \sum_{j \neq \hat{j}} y_j^* \right) > p \cdot \left(\sum_{i=1}^{I} \omega_i + \sum_{j=1}^{J} y_j^* \right).$$

The term in the large brackets on the left hand side is an element of R. By Claim 3 the right hand side in this inequality equals r. Therefore, our inequality contradicts $p \cdot y \leq r$ for all $y \in R$. This completes the proof of Claim 4a.

Next, we want to prove that consumers maximize their preferences subject to their budget constraint, where the budget constraint has to take into account the transfers. Before we write down the claim that we need to prove, let us calculate the right hand side of the budget constraint of an arbitrary consumer i, that is, the "income" of consumer i. It is the sum of his normal income $p \cdot \omega_i + p \cdot \sum_{j=1}^{J} \theta_i^j y_j^*$ and his transfers T_i specified in Claim 4. Adding these, we obtain:

$$p \cdot \omega_i + p \cdot \sum_{j=1}^{J} \theta_i^j y_j^* + p \cdot x_i^* - p \cdot \omega_i - p \cdot \sum_{j=1}^{J} \theta_i^j y_j^*$$

which obviously equals $p \cdot x_i^*$, which is, of course, why we have specified the transfers T_i the way we did in Claim 4, because it now follows that consumer i can afford x_i^*. Now we can state the second part of Claim 4 that we need to prove:

Claim 4b: For every consumer i the consumption plan x_i^* maximizes preferences: $x_i^* \succsim_i x_i$ for all $x_i \in X_i$ such that $p \cdot x_i \leq p \cdot x_i^*$.

Unfortunately, without additional assumptions, this claim need not be true.Therefore, we first prove something a little bit weaker: Can you find a counterexample?

Weakened Claim 4b: For every consumer i the consumption plan x_i^* maximizes preferences subject to the budget constraint written as a strict inequality: $x_i^* \succsim_i x_i$ for all $x_i \in X_i$ such that $p \cdot x_i < p \cdot x_i^*$.

The proof of the weakened Claim 4b is indirect: Suppose there were a consumer $\hat{i}$ and an $x_{\hat{i}} \in X_{\hat{i}}$ such that $x_{\hat{i}} \succ_{\hat{i}} x_{\hat{i}}^*$ and $p \cdot x_{\hat{i}} < p \cdot x_{\hat{i}}^*$. Then it would follow that:

$$p \cdot \left(x_{\hat{i}} + \sum_{i \neq \hat{i}} x_i^* \right) < p \sum_{i=1}^{I} x_i^*.$$

Now the sum in the large brackets on the left hand side is an element of V. The expression on the right hand side equals r. Thus, we have a contradiction to the fact that $p \cdot x \geq r$ for every $x \in V$. This completes the proof of the weakened Claim 4b.

How can we finish the task and prove Claim 4b itself? We use the following assumption:

Assumption 4: *For every consumer i the following is true:*

1. *$X_i = \mathbb{R}_{++}^L$ where $\mathbb{R}_{++}^L$ is the set of all vectors in $\mathbb{R}^L$ that are strictly positive in all components;*

2. *the preference relation $\succsim_i$ is continuous: For all $x_i, x_i' \in X_i$, if $x_i \succ_i x_i'$, then there is an $\varepsilon > 0$ such that $\hat{x}_i \succ_i x_i'$ for all $\hat{x}_i \in X_i$ such that $||\hat{x}_i - x_i|| \leq \varepsilon$, where $||...||$ denotes the Euclidean distance.*

Here, the first assumption is embarrassingly strong: every consumer needs strictly positive quantities of every good to survive. I, for example, as a vegetarian, can survive with quantity zero of pork. But let's use Assumption 4 to complete our proof of Claim 4b. What remains to be shown is:

Remainder of Claim 4b: For every consumer i the consumption plan x_i^* maximizes preferences subject to the budget constraint written as an equality: $x_i^* \succsim_i x_i$ for all $x_i \in X_i$ such that $p \cdot x_i = p \cdot x_i^*$.

The proof is indirect. Suppose there were some consumer i and some $x_i \in X_i$ such that $p \cdot x_i = p \cdot x_i^*$, and $x_i \succ_i x_i^*$. By reducing a little bit the quantities of those goods in x_i for which the price in the price vector p is strictly positive, and increasing a little bit the quantities of those goods in x_i for which the price in the price vector is strictly negative, we can construct another consumption bundle $\hat{x}_i$ that also satisfies $\hat{x}_i \succ_i x_i^*$ and $p \cdot \hat{x}_i < p \cdot x_i^*$. Here, we used part 1. of Assumption 4 because this assumption implies that all components of x_i are strictly positive, and therefore all of them can be increased or decreased by a little bit without leaving the set X_i. We have used part 2. of Assumption 4 because we have used that tiny changes to a consumption bundle leave strict preference unchanged. The existence of the consumption bundle $\hat{x}_i$ that we have constructed contradicts the weakened Claim 4b, and thus the proof of the remainder of Claim 4b is concluded.

To finish the proof of the second welfare theorem, we still have to show that supply equals demand:

Claim 4c: $\sum_{i=1}^{I} x_i^* = \sum_{i=1}^{I} \omega_i + \sum_{j=1}^{J} y_j^*$.

This is true because (x^*, y^*) is Pareto efficient, hence feasible. □

5. The Quasi-Linear Model

We now study a special case of the general equilibrium model: the quasi-linear model. The main point about the quasi-linear model is that in this example Walrasian equilibrium boils down to the intersection point of demand supply in the standard supply and demand diagram, and that Pareto efficiency corresponds to maximizing the sum of consumers' utilities. In much of applied modeling, in particular in the version of microeconomics taught to undergraduates, consumers' utilities are added up without much reflection. The model that we present provides underpinnings for this calculation, albeit in a very special case.

(a) Definition of the Model

There are two commodities: "money" and a "consumption good." Consumers are: $i = 1, 2, \dots, I$. Initial endowments are: $(\omega_i, 0)$. Consumption set are $X_i = \mathbb{R} \times \mathbb{R}_+$. Note that only the consumption of the second good is restricted to be non-negative. The consumption of the first good ("money") can be positive or negative. Consumer i's preferences are represented by the utility function: $u_i(m_i, x_i) = m_i + \Phi_i(x_i)$. We assume that Φ_i is differentiable with strictly positive, strictly decreasing derivative.

We follow convention and call the first commodity "money," although this is clearly a misnomer. If it really were money, it should not enter consumers' utility functions. Nobody directly derives utility from paper money notes. Rather, money facilitates decentralized transactions, and the goods acquired in these transactions provide utility. The general equilibrium model does not describe decentralized trade, at least it does not model how such trade is facilitated by money. Therefore, strictly speaking, there is no money in the general equilibrium model. We use the word "money" nonetheless to refer to the first commodity because here it stands for "all other consumption goods," and we can think that the utility of these consumption goods is summarized by the total amount of "money" that we use to purchase them.

I don't recommend eating Dollar notes.

The quasi-linear form of the utility function reflects the assumption that the marginal rates of substitution between the two commodities depends only on the quantity of the second good that is consumed, but not on the quantity of the first good that is consumed. That is, the quantity of the second good that the consumer is willing to give up for one more unit of the first good depends on the quantity of the second good that she already has, but not on the quantity of the first good that she has already has. Intuitively, if we think of the first good as "all other goods," we might motivate the quasi- form by arguing that for marginal rates of substitution it does not matter which quantity of good 1 the consumer has because satiation with this imaginary good never sets in.

The firms are $j = 1, 2, \dots, J$. Firm j's technology is: $Y_j = \left\{(-z_j, y_j) \,|\, y_j \geq 0, z_j \geq C_j(y_j)\right\}$. Here, $C_j : \mathbb{R}_+ \to \mathbb{R}_+$ is continuously differentiable with strictly positive, strictly increasing derivative. The interpretation of C_j is that $C_j(y_j)$ is the quantity of good 1 that is needed to produce y_j units of good 2 Thus, C_j is a cost function. Consumer i's share in firm j is as always denoted by θ_i^j.

An allocation is a list (m, x, z, y) where each component is in turn a vector: $m = (m_1, m_2, \dots, m_I), x = (x_1, x_2, \dots, x_I), z = (z_1, z_2, \dots, z_J), y = (y_1, y_2, \dots, y_J)$. The interpretation is that m_i is how much money agent i gets, x_i is how much of the consumption good agent i gets, z_j is how much input firm j uses, and y_j is how much output firm j produces. An allocation (m, x, z, y) is feasible if:

$$
\begin{aligned}
(m_i, x_i) &\in X_i \quad \forall i \\
(z_j, y_j) &\in Y_j \quad \forall j \\
\sum_{i=1}^{I} m_i &= \sum_{i=1}^{I} \omega_i - \sum_{j=1}^{J} z_j \\
\sum_{i=1}^{I} x_i &= \sum_{j=1}^{J} y_j
\end{aligned}
$$

The second to last equation says that supply and demand for money must be the same, and the last equation says that supply and demand of the consumption good must be the same.

(b) Walrasian Equilibrium Allocations

It is easy to describe Walrasian equilibrium in the quasi-linear model. As you recall, we may normalize one price. Here, we shall set the price of money equal to 1. We denote the price of the consumption good by p. We focus on the market for the consumption good. From our earlier analysis, we know that we may omit one market when determining equilibrium, and the market that we shall omit is the money market. Thus, we find a Walrasian equilibrium simply by finding a price p at which demand equals supply in the market for the consumption good.

We first determine demand. Consumer i chooses $m_i \in \mathbb{R}$ and $x_i \in \mathbb{R}_+$ to maximize:

How would the following analysis change if we required $m_i \geq 0$?

$$m_i + \Phi_i(x_i)$$

subject to the budget constraint:

$$m_i + px_i \leq \omega_i + \sum_{j=1}^{J} \left(\theta_i^j (py_j - C_j(y_j) \right).$$

We can re-write this problem by solving the budget constraint for m_i and substituting the result into the utility function. We obtain as our simplified maximization problem: choose $x_i \in \mathbb{R}_+$ to maximize:

$$\omega_i + \sum_{j=1}^{J} \left(\theta_i^j (py_j - C_j(y_j) \right) - px_i + \Phi_i(x_i).$$

Note that the first term and second term are independent of consumer i's choice. We can omit them from the utility function without changing the solution. Thus, it remains that consumer i maximizes:

$$\Phi_i(x_i) - px_i.$$

It is immediate from first and second order conditions that the solution is (implicitly) given by:

$$\Phi_i'(x_i) = p \text{ if } p \leq \Phi_i'(0) \text{ and } x_i = 0 \text{ otherwise.}$$

In words, this says that the consumer continues increasing consumption until the marginal utility from consumption equals the price, but consumes nothing, if the price is larger than the marginal utility of even the first unit of consumption, in which case demand is zero. We thus obtain demand following the Econ 101 rule for utility maximization: marginal utility equals price. For each consumer we obtain a demand function, and we obtain market demand by adding up all consumers' demand functions.

Firm j chooses y_j to maximize:

$$py_j - C_j(y_j)$$

which leads to the solution:

$$C_i'(y_j) = p \text{ if } p \geq C_i'(0) \text{ and } y_j = 0 \text{ otherwise.}$$

Thus, firm j uses the Econ 101 rule: marginal cost equals price, to determine its supply, except in case that the price is lower than the marginal cost of producing even the first unit, in which case supply is zero. We obtain for each firm a supply function, and market supply is the sum of all firms' supply functions.

To find an equilibrium price we now set supply equal to demand in the market for the consumption good. Graphically, we draw market demand and supply functions, and their intersection point corresponds to the Walrasian equilibrium. This graph is just the Econ 101 supply and demand diagram. We have re-interpreted this graph as representing equilibrium in a general equilibrium model.

(c) Pareto Efficient Allocations

A feasible allocation is Pareto efficient if there is no other feasible allocation that everyone prefers. We have given the formal definition of Pareto efficiency in Section 1. That definition applies here. In this section we characterize Pareto efficient allocations in the quasi-linear model.

Proposition: In the quasi-linear model, an allocation (m, x, z, y) is Pareto efficient if and only if the vectors x and y maximize

$$\sum_{i=1}^{I} \Phi_i(x_i) - \sum_{j=1}^{J} C_j(y_j)$$

subject to:

$$x_i \geq 0 \quad \forall i, \qquad y_j \geq 0 \quad \forall j, \qquad \text{and} \qquad \sum_{i=1}^{I} x_i = \sum_{j=1}^{J} y_j,$$

and the vectors m and z satisfy:

$$\sum_{i=1}^{I} m_i = \sum_{j=1}^{I} \omega_i - \sum_{j=1}^{J} z_j \quad \text{and} \quad z_j = C_j(y_j) \quad \forall j.$$

The expression that is maximized by a Pareto efficient allocation according to this proposition is the difference between the sum of all consumers' utility from consuming the consumption good and the firms' cost of producing the consumption good. This difference is also called the "surplus" in Econ 101. At first sight, one might object that this difference really has no meaning. We cannot add up utilities if these are non-comparable among individuals, and in any case only determined up to increasing transformations. Which meaning, moreover, would it have to subtract the cost of production from utility? Aren't these numbers with very different units attached to them? In the quasi-linear model it turns out that the special structure of the model overcomes these objections, and makes the apparently very naive operation of summing utilities, and then subtracting cost, meaningful.

It is also interesting that the commodity to which we referred as "money" does not show up at all in the optimization problem. In fact, once the production and consumption vectors for the consumption commodity are optimally chosen, it is clear how much "money" will be used by the firms as input. The remaining money can be allocated to consumers in any arbitrary way. There is no inefficient allocation of money, as long as all remaining money goes to someone. This is because all consumers have the same marginal utility of money. Note also that there is also no feasibility constraint for money. This is because by assumption agents can hold "negative" amounts of money.

Proof. Step 1: We first show that a feasible allocation is Pareto efficient if and only if it maximizes among all feasible allocations the sum of consumers' utilities:

$$\sum_{i=1}^{I} \left(m_i + \Phi_i(x_i) \right).$$

Prove that what I claim to be clearly true is really true.

If a feasible allocation maximizes the sum of consumers' utilities, it clearly must be Pareto-efficient. We only prove the other direction. The proof is indirect. Suppose (m, x, z, y) were a Pareto efficient feasible allocation, and suppose it did not maximize the sum of consumers' utilities. Let us introduce some notation: We denote the utility of consumer i in the postulated efficient allocation by u_i. We denote the sum of all consumers utilities in this postulated allocation by W. We denote an allocation that does maximize the sum of all consumers' utilities by (m^*, x^*, z^*, y^*). Also, we denote by u_i^* consumer i's utility in this allocation, and denote by W^* the sum of consumers' utilities in this allocation. By construction: $W^* - W > 0$.

We complete the indirect proof by showing that there is a feasible allocation $(\hat{m}, \hat{x}, \hat{z}, \hat{y})$ which Pareto dominates (m, x, z, y). This contradicts the postulate that (m, x, z, y) is Pareto efficient. This new allocation is defined as follows:

$$\hat{m}_i = m_i^* + u_i - u_i^* + \frac{W^* - W}{I}$$

What is agent i's utility in this new allocation? It is:

$$\hat{u}_i = m_i^* + u_i - u_i^* + \frac{W^* - W}{I} + \Phi(x_i^*) = u_i + \frac{W^* - W}{I} > u_i$$

where the second equality follows from the fact that $m_i^* - u_i^* + \Phi(x_i^*) = 0$, and the inequality follows from $W^* - W > 0$. Because this calculation applies to any consumer i, the new allocation makes everyone strictly better off.

It remains to verify that $(\hat{m}, \hat{x}, \hat{z}, \hat{y})$ is feasible. This is true for the consumption good because the quantities produced and consumed are the same as in the allocation (m^*, x^*, z^*, y^*). Thus, we only have to verify that the quantity of money consumed is the quantity of money left over after production, that is:

$$\begin{aligned}
\sum_{i=1}^{I}\left(m_i^* + u_i - u_i^* + \frac{W^* - W}{I}\right) &= \sum_{i=1}^{I}\omega_i - \sum_{j=1}^{J} z_j^* \Leftrightarrow \\
\sum_{i=1}^{I} m_i^* + \sum_{i=1}^{I} u_i - \sum_{i=1}^{I} u_i^* + \sum_{i=1}^{I}\frac{W^* - W}{I} &= \sum_{i=1}^{I}\omega_i - \sum_{j=1}^{J} z_j^* \Leftrightarrow \\
\sum_{i=1}^{I} m_i^* + W - W^* + W^* - W &= \sum_{i=1}^{I}\omega_i - \sum_{j=1}^{J} z_j^* \Leftrightarrow \\
\sum_{i=1}^{I} m_i^* &= \sum_{i=1}^{I}\omega_i - \sum_{j=1}^{J} z_j^*
\end{aligned}$$

which is true because (m^*, x^*, z^*, y^*) is feasible.

Step 2: Now, we calculate the sum of agents' utility in a feasible allocation. It is:

$$\sum_{i=1}^{I} (m_i + \Phi_i(x_i)) = \sum_{i=1}^{I} \omega_i + \sum_{i=1}^{I} \Phi_i(x_i) - \sum_{j=1}^{J} C(y_j).$$

The equality follows from the feasibility condition. Next, we notice that when we maximize the term on the right hand side, we can ignore the sum of the initial endowments, because the value of this sum is a constant, that is, independent of the allocation. Thus, to maximize the sum of agents' utilities, we choose among all feasible allocations one that maximizes:

$$\sum_{i=1}^{I} \Phi_i(x_i) - \sum_{j=1}^{J} C(y_j).$$

Observe that the money allocation does not appear at all in this expression. Thus, we obtain a Pareto efficient allocation if and only if we choose x_is and y_js so that the above expression is maximized, and so that the feasibility condition for the consumption commodity is satisfied:

$$\sum_{i=1}^{I} x_i = \sum_{j=1}^{J} y_j,$$

and if we combine this with any allocation of money that respects the feasibility constraints for money. This is what the Proposition asserts. $\square$

We now investigate this optimization problem further. Denote the total quantity produced and consumed of the consumption good by Q. We can solve the optimization problem of the previous proposition in two steps. First, for any given $Q \geq 0$, we determine how best to allocate it among consumers to maximize the sum of their utilities, and how best to allocate the production duties to firms to minimize the sum of their production cost. The second step is then the problem of choosing Q optimally.

Elementary optimization theory implies that we maximize the first term in (2) subject to the constraint that $\sum_{i=1}^{I} x_i = Q$ by choosing x_i such that for all $i, \hat{i}$:

$$\Phi_i'(x_i) = \Phi_{\hat{i}}'(x_{\hat{i}}) \text{ if } x_i > 0 \text{ and } x_{\hat{i}} > 0, \text{ and } \Phi_i'(x_i) \geq \Phi_{\hat{i}}'(x_{\hat{i}}) \text{ if } x_i > 0 \text{ and } x_{\hat{i}} = 0.$$

Similarly, elementary optimization theory implies that we maximize the second term in (2) subject to the constraint that $\sum_{j=1}^{J} y_j = Q$ by choosing the y_j such that for all $j, \hat{j}$:

$$C_j'(y_j) = C_{\hat{j}}'(y_{\hat{j}}) \text{ if } y_j > 0 \text{ and } y_{\hat{j}} > 0, \text{ and } C_j'(y_i) \leq C_{\hat{j}}'(y_{\hat{j}}) \text{ if } y_j > 0 \text{ and } y_{\hat{j}} = 0.$$

Thus, for given Q, the optimal allocation will involve a common value for the derivative $\Phi_i'(x_i)$ for all consumers i who consume strictly positive quantities, and a common value of the derivative $C_j'(y_j)$ for all firms j that produce strictly positive quantities. Finally, turning to the second step of our solution to the optimization problem, again from elementary optimization theory, we see that in the optimum: $\Phi_i'(x_i) = C_j'(y_j)$ whenever i consumes a strictly positive quantity, and j produces a strictly positive quantity.

We have thus again obtained simple Econ 101 rules: the marginal utilities of all consumers must be the same, except for consumers whose marginal utility is so low that efficiency requires them to consume nothing. Similarly, efficiency requires that the marginal costs of all firms are the same, except for firms whose marginal cost are so high that efficiency requires them to produce nothing. The optimal quantity is determined by the rule that marginal utility must equal marginal cost.

These considerations give rise to a simple procedure for finding Pareto-efficient allocations. For every value $c \geq 0$, for every firm j we determine the quantity $q_j(c)$ at which firm j's marginal cost equal c. If firm j's marginal cost are larger than c for all quantities, then we set $q_j(c) = 0$, and if firm j's marginal cost are smaller than c for all quantities, then we set $q_j(c) = \infty$. For every c we denote by $S(c)$ the sum $\sum_j q_j(c)$. Similarly, we construct for each value u for every consumer i the quantity $q_i(u)$ at which consumer i's marginal utility $\Phi_i'(q_i(u))$ equals u. If consumer i's marginal utility is smaller than u for all quantities, then we set $q_i(u) = 0$, and if it is larger than u for all quantities, then we set $q_i(u) = \infty$. We denote by $D(u)$ the sum $\sum_i q_i(u)$. We represent both functions in the same diagram, putting c and u on the vertical axis, and the quantities on the horizontal axis. The total quantity of production in a Pareto efficient allocation corresponds to a point in which $c = u$, and $S(c) = D(u)$.

But notice that the diagram that we have just described in words is exactly the same diagram as the supply and demand diagram that we constructed in subsection (b) to determine the Walrasian equilibrium. Thus, we have shown that a feasible allocation is Pareto efficient if and only if the production and consumption of the consumption good is described by the intersection point of supply and demand functions in the market for the consumption good. Any feasible distribution of money is then compatible with Pareto efficiency. The Walrasian equilibrium is given by the same supply equals demand condition, and results in some particular distribution of money that depends on agents' initial endowment. Thus, we have found a graphical interpretation of the first welfare theorem.

By the fundamental theorem of calculus, the "surplus" is the area between supply and demand function - another Econ 101 rule.

Can you also show the second welfare theorem using the results derived in this section?

6. Welfare

Suppose a policy maker can implement lump-sum transfers, and has to choose among different Pareto-efficient allocations. Does economics have anything to say about which allocation to choose? Or suppose a policy maker has to choose among policies none of which achieves full Pareto-efficiency. How should such policies be evaluated? That is not difficult if one policy leads to an allocation that makes everyone in society better off than the allocation arising from the other policy. But that will rarely be the case.

The examples of the previous paragraph show that it would be useful to have a criterion for comparing allocations, and a criterion that goes beyond saying that one allocation is better than another if everyone prefers the former. Note that in a sense any such criterion has to weigh different individuals' preferences against each other.

One might argue that nothing "scientific" can be said about which criterion for comparing allocations one might choose because no agreement on a single such criterion can ever be achieved. This is a little too defeatist. For example, there is perhaps currently no consensus on the causes of business cycles, yet the scientific investigation of these causes seems a worthwhile enterprise.

One might argue that the discussion of any such criterion cannot be scientific because, because no "ought" statement can be derived from an "is" statement, and that science only deals with "is" statements. That is not convincing because "theory" as we pursue it in this book considers analytical connections between statements, but not their empirical validity. It is possible to study analytical connections between different "ought" statements.

In the first paragraphs of this Section, I am pretending to be a philosopher. It is good for you to keep in mind that I am not.

One might argue that the discussion of criteria for ordering different allocations belongs into philosophy's subfield of ethics, but not into economics. But it is just a fact that these discussions are simultaneously conducted in economics and in philosophy.

Economists often speak of "optimal policies." For example, you will find that there is a large literature on "optimal taxation." The reference to "optimal policies" suggests that economics has come up with some scientifically ground conclusion about how we ought to order allocations resulting from different policies. My view is that this by far exaggerates what has been achieved in this area.

If we think of our objective here as developing a complete order of allocations, then we might try to find a numerical representation of such an order. Such a representation is called a "welfare function." Often, in the literature you will find statements that treat the numerical values of welfare functions as informative, and thus you find claims that say that some policy will "increase welfare by 10%." It is worth in such cases digging deep to see how the author defines "welfare" and what the motivation for this definition is.

The analytical result on orders of allocations that we provide in this section is a negative one. It shows that there is no way of ordering allocations that satisfies a certain set of very plausible requirements ("axioms"). The result is due to Kenneth Arrow, and is called "Arrow's impossibility theorem." This theorem is one of the major accomplishments of economic theory. The theorem is typically presented in a setting that is more general than the setting of general equilibrium theory. It concerns any decision that has consequences for the preferences of a set of individuals, not just allocation decisions in the sense of general equilibrium theory. We shall present the theorem first in this general setting, and then discuss its implications for general equilibrium theory.

Arrow's impossibility theorem illustrates two claims that I made earlier in this Section: (i) There can be interesting analytical insights into "ought" statements. (ii) Unambiguous statements about "optimal" policies exaggerate what has been achieved in this field.

To consider the general setting, let us "reset" notation. That is, in this Section we don't use the notation used in the other parts of these notes, but the notation used in this section is only used here. We start with a finite set of individuals, say the set $N = \{1, 2, \ldots, n\}$, and a finite set of alternatives, denoted by A. Let $\mathcal{R}$ be the set of all rational preferences over A, ruling out indifferences. A preference of agent i is an element R_i of $\mathcal{R}$. A preference profile is a vector of preferences, one for each agent: $R = (R_1, R_2, \ldots, R_n)$. The set of all preference profiles is thus: $\mathcal{R}^n$. We want to investigate how, starting from a preference profile R, one can come up with society's preference. We denote society's preference by: $f(R) \in \mathcal{R}$. We want to consider all possible preference profiles R, and therefore, f will be a function of the form:

$$f : \mathcal{R}^n \to \mathcal{R}.$$

The assumption that f has domain $\mathcal{R}^n$ is often referred to as the assumption of a "universal domain." The interpretation of that assumption is this: Even if we know the actual profile of individuals' preferences, we want to come up with a systematic method for determining society's preference for any arbitrary profile of individuals' preferences. This method is formalized by the function f. It describes society's preferences for all conceivable, hypothetical preference profiles. Thus, it describes how we answer arbitrary questions of the form: "But how would you define society's preferences if the profile of individuals' profile were not R, but some other profile?"

Arrow asked whether f could satisfy a certain set of actions. We now list these axioms. The first is:

Pareto Axiom: Suppose $a, b \in A$, and suppose the preference profile R is such that aR_ib for all $i \in N$. Then $af(R)b$.

This axiom requires that if individuals unanimously prefer a over b, then society prefers a over b. That seems eminently reasonable.

No-Dictatorship Axiom: There is no $i \in I$ such that $f(R) = R_i$ for all $R \in \mathcal{R}^n$.

This axiom requires that no single individual alone determines society's preference regardless of what everyone else's preferences are. This, too, seems reasonable. The third axiom is perhaps more controversial.

I try to use as few acronyms as possible. But IIA is really useful!

Independence of Irrelevant Alternatives Axiom (IIA): Suppose $a, b \in A$, and suppose R is a preference profile such that $af(R)b$. Let $\hat{R}$ be some other preference profile, but suppose that preferences over a and b are the same in R and in $\hat{R}$, that is: $a\hat{R}_i b$ if and only if $aR_i b$. Then $af(\hat{R})b$.

Decision theory also has an axiom called "independence of irrelevant alternatives." It is related to Arrow's IIA, but not quite the same.

In words, this says that society's preferences over a and b only depend on individuals' preferences over these two alternatives, and not on individuals' preferences over any other alternatives. For example, if c is an alternative other than a and b, it does not matter for society's ranking of a and b whether individual 1 ranks c between a and b, or below both a and b. This is a little controversial because in the former case, perhaps, the individual's preference for a over b could be seen as "stronger" than in the latter case.

Arrow's "impossibility theorem" is this result:

Proposition: Suppose there are at least three alternatives: $\#A \geq 3$. Then no $f : \mathcal{R}^n \to \mathcal{R}$ satisfies the Pareto axiom, the No-Dictatorship axiom, and IIA.

This first step is included for pedagogical reasons only. We could omit it, and the proof could start with Step 2. What we are displaying in Step 1 is called the "Condorcet cycle," named after the Marquis de Condorcet (1743-1794) whose life and work are worth studying.

Proof: *Step 1:* .We show that majority voting does not satisfy the axioms. We focus on the case that there are only three alternatives: $A = \{a, b, c\}$ and three individuals: $I = \{1, 2, 3\}$. We mean by "majority voting" the system by which $af(R)b$ if more than one half of all individuals in N prefer a over b. Suppose that the three individuals have the following preferences: aR_1bR_1c, cR_2aR_2b, and bR_3cR_3a. Then a majority prefers a over b, a majority prefers b over c, and a majority prefers c over a. Thus majority voting does not produce a rational preference over $A = \{a, b, c\}$.

Step 2: We show that generalized majority voting satisfies the axioms only if it is dictatorial. Here, we mean by "generalized majority voting" a system where there is a set $\mathcal{G}$ of winning coalitions such that $af(R)b$ if and only if the set of all individuals who prefer a over b is in $\mathcal{G}$: $af(R)b \Leftrightarrow \{i \in N | aR_ib\} \in \mathcal{G}$. Moreover, for the system to be called "generalized majority voting," we will require $\mathcal{G}$ to satisfy the following two assumptions:

1. For every non-empty set $G \subseteq N$ either $G \in \mathcal{G}$ or $N \setminus G \in \mathcal{G}$, but not both.

2. $G \in \mathcal{G}$ and $G' \supseteq G$ imply $G' \in \mathcal{G}$.

If we did not make the first of these two assumptions, our system would not be well-defined. It would then be possible that all individuals in some winning coalition G prefer x to y, but all individuals in the coalition $N \setminus G$

prefer y to x, and this coalition is also winning. Then, society would have to prefer x to y and y to x, a contradiction The second assumption is a natural generalization of a property of "majorities," namely, if they get larger, they remain majorities.

Now let us define k to be the size of the smallest coalition in $\mathcal{G}$, that is:

$$k = \min_{G \in \mathcal{G}} \#G.$$

We distinguish two cases. The first case is that $k \geq 2$. We shall show that in this case an argument like the argument given in Step 1 shows that society's preferences may become cyclical. Let's pick one "smallest" G, i.e. a G with k elements. Let's partition G into two non-empty subsets: G_1 and G_2. Hence $\emptyset \neq G_1 \subseteq G$, $\emptyset \neq G_2 \subseteq G$, $G_1 \cap G_2 = \emptyset$ and $G_1 \cup G_2 = G$. Let's pick three alternatives x, y, z from A, and let us assume that xR_iyR_iz for all $i \in G_1$, zR_ixR_iy for all $i \in G_2$, and yR_izR_ix for all $i \notin G$. Because xR_iy for all $i \in G$, we have $xf(R)y$. Because yR_iz for all $i \notin G_2$, we have $yf(R)z$ (because G_2 is not a winning coalition, $N \setminus G_2$ is a winning coalition). Finally, because zR_ix for all $i \notin G_1$, we have $zf(R)x$ (because G_1 is not a winning coalition, $N \setminus G_1$ is a winning coalition). Thus, we have concluded, as in Step 1, that society's preference is cyclical. This is not a system that satisfies Arrow's conditions.

We are generalizing the Condorcet cycle in Step 2.

We are left with the case $k = 1$, which means that there must be some i such that $\{i\} \in \mathcal{G}$. Then every $G \subseteq N$ such that $i \in G$ is an element of $\mathcal{G}$ (because supersets of winning coalitions are winning, property 2 above), and every $G \subseteq N$ such that $i \notin G$ is not an element of $\mathcal{G}$ (because complements of winning coalitions are not winning, property 1 above). Thus, the system is the dictatorship of agent i.

Step 3: We show that the axioms imply that the system is generalized majority voting. Pick any two alternatives $x, y \in A$, and denote by $\mathcal{G}_{x,y}$ the set of all $G \subseteq N$ such that $xf(R)y$ for the preference profiles for which xR_iy if $i \in G$, and yR_ix if $i \notin G$. Notice that $\mathcal{G}_{x,y}$ is well defined because of the IIA axiom.

It is Step 3 that makes the proof of Arrow's theorem a little involved. We shall simplify it by taking a shortcut (which I shall point out).

Consider any two pairs of alternatives, x, y and a, b. We begin by proving that the axioms imply that $\mathcal{G}_{a,b} = \mathcal{G}_{x,y}$. To prove this, we show that $G \in \mathcal{G}_{x,y}$ implies $G \in \mathcal{G}_{a,b}$. This is sufficient because it implies $\mathcal{G}_{x,y} \subseteq \mathcal{G}_{a,b}$, and swapping the roles of a, b, x, y we can then also conclude: $\mathcal{G}_{a,b} \subseteq \mathcal{G}_{x,y}$, and therefore $\mathcal{G}_{a,b} = \mathcal{G}_{x,y}$.

We give the argument only for the case that x, y, a, b are four distinct alternatives. This a shortcut that we take to simplify the proof. Suppose $G \in \mathcal{G}_{x,y}$. We want to show that $G \in \mathcal{G}_{a,b}$. Consider all preference profiles $R \in \mathcal{R}^n$ such that aR_ib for all $i \in G$ and bR_ia for all $i \in N \setminus G$. Let us denote the set of all such preference profiles by $\mathcal{R}^n(a, b)$. We have to show that for all such preference profiles $af(R)b$. We consider separately two types of such preference profiles.

Type 1: Preference profiles $R \in \mathcal{R}^n(a,b)$ such that $aR_ixR_iyR_ib$ for all $i \in G$ and $yR_ibR_iaR_ix$ for all $i \in N \setminus G$. For such profiles Pareto implies $af(R)x$ and $yf(R)b$. Moreover, because $G \in \mathcal{G}_{x,y}$ we have: $xf(R)y$. Transitivity of $f(R)$ implies then what we wanted to show: $af(R)b$.

Type 2: Preference profiles $R \in \mathcal{R}^n(a,b)$ that are not of Type 1. For such preference profiles IIA implies $af(R)b$.

Now it remains to show that $\mathcal{G}_{x,y}$ (which we now know not to depend on the identity of the alternatives x and y) has the properties 1. and 2. that we earlier made part of our definition of "generalized majority voting." Property 1. is true by construction. We have to prove property 2. We want to prove that if $G \in \mathcal{G}_{x,y}$ and if $G' \supseteq G$, then $G' \in \mathcal{G}_{x,y}$. Pick any alternative z other than x and y. We are going to show that $G' \in \mathcal{G}_{x,z}$. This is sufficient because, by our earlier argument, we know that $\mathcal{G}_{x,z}$ and $\mathcal{G}_{x,y}$ are the same.

Consider all preference profiles $R \in \mathcal{R}^n$ such that xR_iz for all $i \in G'$ and zR_ix for all $i \in N \setminus G'$. Let us denote the set of all such preference profiles by $\mathcal{R}^n(x,z)$. We have to show that for all such preference profiles $xf(R)z$. We consider separately two types of such preference profiles.

Type 1: Preference profiles $R \in \mathcal{R}^n(x,z)$ such that xR_iyR_iz for all $i \in G$, yR_ixR_iz for all $i \in G' \setminus G$, and yR_izR_ix for all $i \in N \setminus G'$. Because $G \in \mathcal{G}_{x,y}$, we have for every such profile $xf(R)y$. By Pareto, we have for every such profile: $yf(R)z$. Because $f(R)$ is transitive, this implies $xf(R)z$, as we wanted to show.

Type 2: Preference profiles $R \in \mathcal{R}^n(x,z)$ that are not of Type 1. For such preference profiles IIA implies $xf(R)z$.

This concludes the proof of Arrow's Theorem. $\square$

I promised earlier to discuss the implications of Arrow's Theorem for general equilibrium theory. You might argue that the setting of Arrow's Theorem is too general for the purposes of general equilibrium theory. There are two ways in which this is indeed the case: (i) the set of society's options is more specific in general equilibrium theory than just some set consisting of $x, y, z, \ldots$. (The set is also infinite in general equilibrium theory, whereas it is finite in Arrow's theorem. This is a technical point of minor importance.) Moreover, preferences are not "arbitrary," but, for example, it is perhaps natural that consumers have monotone preferences. There exist versions of Arrow's theorem for general equilibrium theory that address these concerns. The theorem remains true if these concerns are addressed. I presented the more abstract theorem here only because it is of importance in economics far beyond general equilibrium theory. It seemed desirable for you to know the general version.

Of course, economists' conversation about "welfare" has not stopped with Arrow's theorem. Perhaps the second most famous theorem in this field, for example, is by John Harsanyi, and provides a foundation for evaluating outcomes by adding up different individuals' Bernoulli utilities (the utility functions referred to in expected utility theory). A precise examination of this theorem would go beyond the scope of this course, as would any further discussion of analytical results in welfare economics. The negative message of Arrow's theorem forms the starting point, and it is most important for the beginner to fully understand this message.

7. Aggregating Supply Correspondences

Now we shall change our focus a little bit. We shall ask whether supply and demand in general equilibrium are as if there was just one firm and just one consumer. This assumption is popular in some parts of applied general equilibrium theory. We now focus not just on supply and demand in equilibrium, but on supply and demand functions. Studying these functions will later allow us to consider uniqueness and comparative statics of Walrasian equilibria. The aggregation theory that we consider here and in the next section however also of interest in itself.

For every firm j and every $p \in \mathbb{R}^L_+$ define the set of profit maximizing production plans of firm j as:

$$S_j(p) = \left\{ y_j \in Y_j | p \cdot y_j \geq p \cdot \hat{y}_j \text{ for all } \hat{y}_j \in Y_j \right\}.$$

If we don't make assumptions about the production set Y_j, for some $p \in \mathbb{R}^L_+$ the set $S_j(p)$ may be empty. It may also have more than one element. This is why we refer to S as a "correspondence," rather than a function, where a correspondence is really just a set-valued function. For every p, $S(p)$ is an element of the set of all subsets of $\mathbb{R}^L$. We can think of the set of all subsets of $\mathbb{R}^L$ as the co-domain of S. The correspondence S_j is called the "supply correspondence of firm j."

The question that we shall ask is: if there are many firms in the economy, does the aggregate supply correspondence, that is, the sum of all firms' supply correspondences, look like the supply correspondence of just one firm? If the answer is "yes," we shall call that firm a "representative firm." It "represents" the production sector of the economy. But first, to explain the precise meaning of our question, we have to define what we mean by the "sum of supply correspondences." We'll define this below, but just to understand the question that we are posing, you may think of the special case in which the supply correspondence of every firm is a function with values in $\mathbb{R}^L$, that is, for all prices profit maximizing supply is unique. Then it is obvious what we mean by the sum of the firms' supply correspondences, namely the sum of all firms' optimal production plans. We also have to say what we mean by the phrase "looks like the supply correspondence of just one firm." By this we mean: Is there a production set Y such that, if there was just one firm that had this production set Y, then this firm's supply correspondence would equal the sum of the supply correspondences of all firms. If this is true, then equilibrium prices and quantities are the same in the economy with J firms, and in the economy with just one firm that has

If there was just one firm, maybe it would act as a monopolist? Here, we go along with the fiction that this one firm acted as a price taker.

production set Y. The bottom line of this section will be that typically the answer to our question is "yes." Thus, typically, one may assume the existence of just one "representative firm" without loss of generality.

We now define what we mean by the sum of the supply correspondences. For this, we first define what we mean by the "(Minkowski) sum" of two sets. Let M_1 and M_2 be subsets of $\mathbb{R}^n$ for some $n \in \mathbb{N}$. Then we define:

$$M_1 + M_2 = \{x \in \mathbb{R}^n | \exists x_1 \in M_1, x_2 \in M_2 \text{ such that } x = x_1 + x_2\}$$

For example, if $n = 1$, $M_1 = \{1, 2\}$ and $M_2 = \{0, 4\}$, then $M_1 + M_2 = \{1, 2, 5, 6\}$. Note that we can also definite the sum of not just two but any finite number of sets, and that the standard laws of algebra hold for set addition. Now we define the sum $S = S_1 + S_2$ of two correspondences S_1 and S_2 that have identical domains by setting $S(p) = S_1(p) + S_2(p)$ for every p in the domain of $S_1(p)$ and $S_2(p)$.

We can now state the main result of this section: there is always a representative firm. Its production set is the sum of all firms' production sets.

Proposition: Define:

$$Y = Y_1 + Y_2 + \ldots + Y_J.$$

Let S be the supply correspondence of a firm with technology Y. Then

$$S = S_1 + S_2 + \ldots + S_J.$$

Proof: We have to show for every $p \in \mathbb{R}^L_+$ that:

$$y \in S(p) \Leftrightarrow \text{ There exist } y_j \in S_j(p) \text{ for all } j \text{ such that } y = \sum_{j=1}^{J} y_j.$$

Step 1: We prove $\Rightarrow$. Suppose $y \in S(p)$. Then there must be some $y_j \in Y_j$ for all j such that $y = \sum_{j=1}^{J} y_j$. Suppose now there were $\hat{j}$ such that $y_{\hat{j}} \notin S_{\hat{j}}(p)$. Then there must be some $\hat{y}_{\hat{j}} \in Y_{\hat{j}}$ such that

$$p \cdot \hat{y}_{\hat{j}} > p \cdot y_{\hat{j}}.$$

Define $\hat{y} = \sum_{j \neq \hat{j}} y_j + \hat{y}_{\hat{j}}$. Then $\hat{y} \in Y$ and:

$$p \cdot \hat{y} - p \cdot y = \sum_{j \neq \hat{j}} p \cdot y_j + p \cdot \hat{y}_{\hat{j}} - \sum_{j=1}^{J} p \cdot y_j = p \cdot \hat{y}_{\hat{j}} - p \cdot y_{\hat{j}} > 0$$

which contradicts $y \in S(p)$.

Step 2: We prove $\Leftarrow$. Assume there exist $y_j \in S_j(p)$ for all j such that $y = \sum_{j=1}^{J} y_j$ but $y \notin S(p)$. Then there must be $\hat{y} \in Y$ such that:

$$p \cdot \hat{y} > p \cdot y.$$

We can re-write the right hand side as follows:

$$p \cdot \hat{y} > \sum_{j=1}^{J} p \cdot y_j.$$

We can make a similar replacement on the left hand side. Because $\hat{y} \in Y$ we can write $\hat{y} = \sum_{j=1}^{J} \hat{y}_j$ where $\hat{y}_j \in Y_j$ for every j. Then we obtain:

$$\sum_{j=1}^{J} p \cdot \hat{y}_j > \sum_{j=1}^{J} p \cdot y_j.$$

But this can only be true if there is at least one j such that:

$$p \cdot \hat{y}_j > p \cdot y_j$$

which contradicts $y_j \in S_j(p)$.

Q.E.D.

8. Aggregating Demand Functions

Our interest in this section is in the properties of the sum of all consumers' demand functions. In particular, after seeing the main result of the previous section on the existence of a representative firm, we are interested in whether the sum of consumers' demand functions looks like a demand function resulting from a single agent's preference maximization. This agent could then be regarded as a "representative consumer." The representative consumer would be a fiction. But her existence would be useful, because it would allow us a simple shortcut for modeling aggregate demand.

Note that we are only discussing in these notes the existence of what the literature calls a "positive" representative consumer. The representative consumer, if she exists, helps us to simplify the analysis of the structure of aggregate demand. There is no presumption at all that the representative consumer's preference will represent society's "welfare," whatever that is. Indeed, nowhere in these notes appears a "welfare function." Thus, our positive representative consumer is not what the literature calls a "normative representative consumer."

In your future studies, watch out for papers that equate "welfare" and the utility of the representative consumers. Ask yourself how the authors of those papers justify this.

The bottom line of this section is that a positive representative consumer typically does not exist. We also discuss the even stronger result that aggregate demand need not have properties that are necessary, but not sufficient, for the existence of a positive representative consumer.

(a) Income Aggregation

We begin by asking a simple question: When can aggregate demand for every good be expressed as a function of the sum of all consumers' incomes, and thus does not depend on income distribution among the consumers? This is necessary, but of course not sufficient, for the existence of a representative consumer. A moment of reflection indicates that even this requirement, that demand for a good depends only on the sum of incomes, is very restrictive. The demand for Jaguar cars, perhaps, depends not only on the sum of incomes, but also on the proportion of consumers with very high income. Thus, it will not be surprising that the conditions that guarantee that demand for all goods depends only on the sum of incomes are extremely restrictive. In other words, we obtain the result only for very small sets of preference profiles.

Suppose for every agent i there is a function: $D_i : \mathbb{R}^L_{++} \times \mathbb{R}_+ \to \mathbb{R}^L_+$ where for every $(p, y_i) \in \mathbb{R}^L_{++} \times \mathbb{R}_+$ we interpret $D_i(p, y_i)$ as the vector of quantities demanded by consumer i if consumer i's income is y_i. Note that for the moment we make no assumption that demand results from utility maximization. We do make, though, the implicit assumption that demand is uniquely defined for every p and y_i. That is, there is neither a problem of existence nor a problem of multiplicity. We also assume without further mentioning that for each i the function D_i is continuous.

Our question, put into mathematics, is: Under which condition does a function $D : \mathbb{R}^L_{++} \times \mathbb{R}_+ \to \mathbb{R}^L_+$ exist such that:

$$\sum_{i=1}^{I} D_i(p, y_i) = D\left(p, \sum_{i=1}^{I} y_i\right) \text{ for all } (p, y_1, y_2, \ldots, y_I) \in \mathbb{R}^L_{++} \times \mathbb{R}^I_+.$$

Let us say that in this case *the functions $D_1, D_2, \ldots, D_I$ allow income aggregation*. Intuitively, this means that aggregate demand only depends on aggregate income, and not on how this income is distributed among consumers. Note that this is really a purely mathematical question. No assumptions have been made that would distinguish the functions that we are considering as "demand functions" from other functions.

The following is known as *Antonelli's Theorem:*

Proposition: Suppose that $D_i(p, 0) = 0$ for all $i = 1, 2, \ldots, I$ and $p \in \mathbb{R}^L_{++}$. Then the demand functions $D_1, D_2, \ldots, D_I$ allow income aggregation if and only if

(i) $D_i = D_j$ for all $i, j \in \{1, 2, \ldots, I\}$;

(ii) There is a function $\alpha : \mathbb{R}^L_{++} \to \mathbb{R}^L_+$ such that

$$D_i(p, y_i) = y_i \alpha(p) \text{ for all } i \in \{1, 2, \ldots, I\} \text{ and all } (p, y_i) \in \mathbb{R}^L_{++} \times \mathbb{R}_+.$$

We omit the proof of this result. In words, the necessary and sufficient condition for income aggregation are thus (i) that all consumers have the same demand function, and (ii) that this demand function has linear Engel curves. Linear Engel curves means, of course, that the proportion of income spent on any particular good does not depend on the size of income. If demand functions are then also identical, it is then immediate that aggregate demand does not depend on the income distribution. The less trivial part of the proposition is the converse. Finally, as an example, we mention that the conditions of this proposition hold if all consumers have preferences that can be represented by the same Cobb-Douglas utility function.

Another, more limited question is as follows. Let $0 < a < b < \infty$. Under which conditions does a function $D : \mathbb{R}^L_{++} \times [Ia, Ib] \to \mathbb{R}^L_+$ exist such that:

$$\sum_{i=1}^{I} D_i(p, y_i) = D\left(p, \sum_{i=1}^{I} y_i\right) \text{ for all } (p, y_1, y_2, \ldots, y_I) \in \mathbb{R}^L_{++} \times [a, b]^I.$$

Let us say that in this case *income aggregation on* $[a, b]$ is possible.

Proposition: Income aggregation on $[a, b]$ is possible if and only if there are a function $\alpha : \mathbb{R}^L_{++} \to \mathbb{R}^L_+$ and, for every $i = 1, 2, \ldots, I$, a function $d_i : \mathbb{R}^L_{++} \to \mathbb{R}^L$ such that for all :

$$D_i(p, y_i) = d_i(p) + \alpha(p) y_i \text{ for all } i \in \{1, 2, \ldots, I\} \text{ and all } (p, y_i) \in \mathbb{R}^L_{++} \times [a, b].$$

Again, we omit the proof. In words, the necessary and sufficient condition for income aggregation on $[a, b]$ is thus that all consumers have for all goods affine Engel curves, where the Engel curves for the same good have for all consumers the same slope. The conditions of this proposition hold whenever those of the previous proposition hold. Moreover, some additional cases are admitted, for example the case that all consumers have preferences that can be represented by the same quasi-linear utility function.

Can you come up with an example in which consumers' don't have identical demand functions?

(b) Aggregate Demand and The Weak Axiom of Revealed Preferences

Let us abandon the project of income aggregation. We shall make this possible by fixing, in a sense, the income distribution. Specifically, we assume that whenever income is $Y \geq 0$, then individual i's income is $\beta_i Y$ where $\beta_i > 0$ for every i and $\sum_{i=1}^{I} \beta_i = 1$. Starting with individual demand functions D_i we can then without further complications construct aggregate demand as a function of the sum of all incomes:

$$D(p, Y) = \sum_{i=1}^{I} D_i(p, \beta_i Y).$$

Now we can ask: under which conditions does D look as if a representative agent maximized some preference relation subject to the budget constraint defined by p and Y.

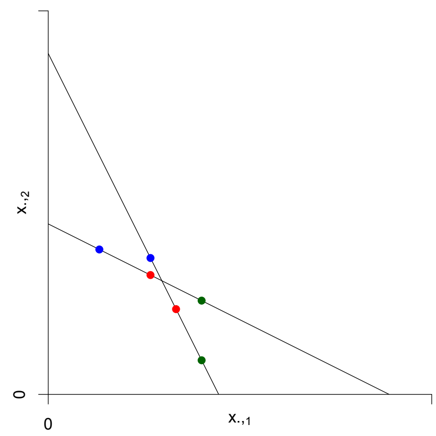
The weak axiom of revealed preferences (WARP) is discussed in Section 2.F of MWG.

If aggregate demand is the result of the preference maximization of a representative agent, it must satisfy the weak axiom of revealed preference.Note that the weak axiom of revealed preference is only necessary, but not sufficient for demand being the result of rational choice by a single agent. We now show that aggregate demand does not even have to satisfy the weak axiom of revealed preference.

The strong axiom of revealed preferences (SARP) is necessary and sufficient for demand being the result of preference maximization. See Section 3.J of MWG.

Suppose two consumers have identical endowments, and identical shares in all firms. Then, at all prices, they will have identical incomes. Thus, in the notation introduced above, we have $\beta_1 = \beta_2 = 0.5$. Suppose that there are only two goods, and that both consumers' consumption sets are $\mathbb{R}^2_+$. We represent both consumers' demand choices in the same two-dimensional diagram, drawing just one budget set for both consumers. In Figure 3 we show both consumers' budget sets for two different price vectors. For the steeper budget line the relative price of good 1 is larger than for the flat budget set. We have denoted consumer 1's choices from the two budget sets by blue dots, and consumer two's choices from the two budget sets by green dots. Clearly, neither consumer's choices violates the weak axiom of revealed preferences, and therefore both consumers' choices are compatible with the maximization of some preference relation.

Figure 3: Aggregate Demand Violates the Weak Axiom



Now we want to investigate whether aggregate demand satisfies the weak axiom of revealed preferences. To obtain aggregate demand we sum up the two consumers' incomes, and then sum up their demands, for each of the two price vectors represented in Figure 3. To continue to use Figure 3 we then divide aggregate income, and also aggregate consumption, by 2. Thus, we look at average consumption as a function of average income. Average income is the same as each individual's income, because they have identical incomes. Therefore, the budget set with average income is the same as the budget set of each of the two individuals. We can therefore use again Figure 3. The economy's average consumption, moreover, is just the mid point between each of the two consumers' consumption bundles. We have indicated the two midpoints in Figure 3 in red. It is obvious then that the average demand violates the weak axiom of revealed preferences.

(c) The Law of Demand and Aggregation

The weak axiom is known to be equivalent under mild conditions to the *law of compensated demand* for price changes. This is the following condition: for all $p, p' \in \mathbb{R}^L_{++}$ and $Y > 0$:

$$(p' - p) \cdot [D(p', p' \cdot D(p, Y)) - D(p, Y)] \leq 0$$

with strict inequality if $D(p', p' \cdot D(p, Y)) \neq D(p, Y)$.We have already seen in Section (b) that the weak axiom does not aggregate: Even if all individual demand functions satisfy the weak axiom, aggregate demand function need not satisfy the weak axiom. Equivalently, we can now say that the law of compensated demand does not aggregate.

A proof that this condition is equivalent to the weak axiom of revealed preference as we know it is in MWG (proof of Proposition 2.F.1 on pages 31 and 32).

MWG show in the second half of the proof of Proposition 4.C.1 that the law of demand implies the law of compensated demand.

We shall now consider a more restrictive monotonicity condition. We say that the *law of demand* holds if for all $p, p' \in \mathbb{R}^L_{++}$ and $Y > 0$:

$$(p' - p) \cdot [D(p', Y) - D(p, Y)] \leq 0$$

with strict inequality if $D(p', Y) \neq D(p, Y)$. Why is this called the law of demand? Suppose the price of only one good changed. Then the above inequality says that the quantity demanded has to move into the opposite direction, that is, In undergraduate microeconomics the "law of demand" says that demand falls as price increases. An important lesson of undergraduate microeconomics is that the law of demand is not true, more precisely, that it is not an implication of preference maximization. Goods may be Giffen goods if they are inferior goods and the income effects of price changes are very strong.

Now suppose that the law of demand holds for all consumers. Does it also hold for aggregate demand? The answer is yes. This is so immediate that it is not quite clear what to write down as a "proof." Nonetheless, maybe the following argument is useful. The law of demand can be written as:

$$(p' - p) \cdot D(p', Y) \leq (p' - p) \cdot D(p, Y).$$

If the law of demand holds for each individual, then we have for each i:

$$(p' - p) \cdot D_i(p', \beta_i Y) \leq (p' - p) \cdot D_i(p, \beta_i Y).$$

If we write this inequality for every $i \in I$, and then add up these inequalities, then we obtain the law of demand for aggregate demand.

(d) Properties of Excess Demand Functions

For the purposes of general equilibrium theory we want to ask yet another question. Let us consider an exchange economy. Assume also for simplicity that $X_i = \mathbb{R}^L_+$ for all $i \in \{1, 2, \ldots, I\}$. Define the demand correspondence by:

$$D_i(p) = \{x \in X_i | p \cdot x \leq p \cdot \omega_i, x \succsim_i x' \text{ for all } x' \in X_i \text{ such that } p \cdot x' \leq p \cdot \omega_i\}$$

Note that because we are considering an exchange economy we now work with the endogenous wealth $p \cdot \omega_i$. We call D_i a "correspondence" because it is not necessarily singleton-valued. But suppose that $\succsim_i$ satisfies local nonsatiation, and is continuous and strictly convex. Then D_i is singleton-valued, and we can treat it with a small abuse of notation as a function. Moreover, D_i is homogenous of degree zero, continuous, and satisfies Walras' law: $p \cdot D_i(p) = p \cdot \omega_i$.

Now define consumer i's excess demand function as:

"ζ" is the Greek letter: "zeta."

$$\zeta_i(p) = D_i(p) - \{\omega_i\}.$$

Again, if $\succsim_i$ satisfies local nonsatiation, and is continuous and strictly convex, then ζ_i is homogeneous of degree zero, continuous, and satisfies Walras' law.

Next, define aggregate excess demand as:

$$\zeta(p) = \sum_{i=1}^{I} \zeta_i(p)$$

It is easy to see that if each $\zeta_i(p)$ is homogeneous of degree zero, continuous, and satisfies Walras' law, then so does ζ. But what else can we infer about ζ? This is important because a Walrasian equilibrium is simply a price vector p at which $\zeta(p)$ is zero, and therefore anything that can be said about the existence, uniqueness, and comparative statics of Walrasian equilibria must follow from what we know about the function ζ. Unfortunately, nothing beyond what we have just said can be deduced about ζ. This is formalized in the following result that is named after the people who discovered it the *Sonnenschein-Mantel-Debreu Theorem*.

Proposition: Suppose $\zeta : \mathbb{R}^L_{++} \rightarrow \mathbb{R}^L$ is homogenous of degree zero, continuous, and satisfies Walras' law. Then for every $\varepsilon > 0$ there exist L consumers with continuous, strictly convex, and nondecreasing preferences, and endowments for these consumers, such that ζ is the aggregate excess-demand function for these L consumers for all p such that $p_\ell / ||p|| \geq \varepsilon$ for all $\ell = 1, 2, \ldots L$.

A theorem that says that nothing can be said.

We omit the proof of this result. This result is of central importance to economics. Without more detailed knowledge about individuals' demand behavior, we can infer nothing about aggregate excess demand beyond some very simple properties. An implication is that any substantial empirical predictions that economics makes about market equilibria, for example that prices go up when some demand goes down, must be based on more than just consumer rationality.

9. Existence of General Equilibrium

Our purpose now will be to find sufficient conditions for the existence of at least one Walrasian equilibrium. Note the two important words: "sufficient conditions" - finding necessary and sufficient conditions seems to be very difficult - and "at least one" - there might be more than one.

What are sufficient conditions for the existence of at least one Walrasian equilibrium is not obvious. This is in contrast with the partial equilibrium model, where existence of Walrasian equilibrium can easily be ensured through appropriate continuity assumptions and assumptions about the monotonicity of demand and supply functions, or about their behavior close to the boundary of the non-negative orthant. In the general equilibrium model, one has to worry that any adjustment that creates equilibrium of supply and demand in one market, has ripple effects in other markets where it may create disequilibrium.

The seminal paper on the existence of a Walrasian equilibrium is by Kenneth Arrow and Gérard Debreu ("Existence of an Equilibrium for a Competitive Economy," *Econometrica*, 1954).

If we denote by $\zeta(p)$ the L-dimensional vector of excess demand, i.e. demand minus supply, at prices p, then seeking sufficient conditions for Walrasian equilibrium is equivalent to seeking sufficient conditions for the existence of at least one p such that $\zeta(p) = 0$. We shall begin by finding sufficient conditions that are directly formulated for the function ζ. Then we ask which conditions for the basic exogenous components of an economy, i.e. preferences, endowments, technologies, etc., ensure that the conditions for ζ that guarantee the existence of at least one equilibrium hold.

We shall restrict our attention to price vectors, one for each commodity, such that all prices are non-negative, and such that the prices add up to 1. Let us call the set of such price vectors Δ. Thus:

$$\Delta = \left\{ (p_1, p_2, \ldots, p_L) | p_\ell \geq 0 \text{ for all } \ell \text{ and } \sum_{\ell=1}^{L} p_\ell = 1 \right\}$$

This is essentially just a normalization of prices, as long as we assume that no price is negative, and at least one price is strictly positive. If that is true for equilibrium prices, then we can find at least one equilibrium price in Δ, because the equilibrium conditions only pin down relative prices, but not absolute prices. We do not lose anything in this case by restricting attention to price vectors in Δ.

We shall now assume that Δ is the domain of the excess demand function:

$$\zeta : \Delta \rightarrow \mathbb{R}^L$$

There is an implicit assumption here, which is that the excess demand for any commodity is finite even if the price of that commodity is zero. One might instead think that this demand should be allowed to be infinite.

We assume that excess demand is finite even at price zero because this makes the proof below much simpler. We will justify this assumption later by assuming that the amount of any commodity that any consumer can consume is bounded from above. That seems a reasonable assumption.

See condition (iv) in the second proposition in this section.

In the proposition that follows, we use the following notation: If ℓ is some commodity, then we write p_ℓ for the price of commodity ℓ, and $\zeta_\ell(p)$ for the excess demand for commodity ℓ at price vector p.

Proposition: Suppose that

(i) ζ is continuous;

(ii) ζ satisfies Walras' Law: $p \cdot \zeta(p) = 0$ for all $p \in \Delta$;

(iii) for every $p \in \Delta$ if for some good ℓ we have $p_\ell = 0$, then $\zeta_\ell(p) > 0$;

then there is at least one $p \in \Delta$ such that $\zeta(p) = 0$.

We have seen conditions (i) and (ii) before. Condition (iii) says: if the price of a good is zero, then excess demand for that good will be positive. This seems very natural if preferences for all goods are monotone.

Proof: The proof strategy is to define a function $f : \Delta \to \Delta$ that mimics a dynamic price adjustment process where prices are raised if there is excess demand and lowered if there is excess supply. A fixed point of f, where $f(p) = p$ and hence no price is changed, will then be a price vector at which excess demand is zero, that is, a Walrasian equilibrium price vector. We then show that f satisfies the assumptions of Brouwer's fixed point theorem and conclude that there is at least one fixed point of f, i.e. at least one Walrasian equilibrium.

Brouwer's fixed point theorem: If X is a non-empty, convex, compact subset of some finite dimensional Euclidean space, and $f : X \to X$ is continuous, then at least one $x \in X$ is a "fixed point" of f, that is, $f(x) = x$.

A first, naïve definition of the price adjustment function might be:

$$f_\ell(p) \stackrel{?}{=} p_\ell + \zeta_\ell(p),$$

where "f_ℓ" stands for the ℓ-th component of f. That, however, does not work, because $f(p)$, if defined in this way, need not be in Δ. There are two reasons for this. Firstly, there is no guarantee that the adjusted price of commodity ℓ is not negative. Secondly, after adjustment the prices need no longer add up to 1. We shall remedy this as follows. First, we shall add to prices only if excess demand is positive, not if it is negative. Thus, we shall add $\max\{\zeta_\ell(p), 0\}$. To make sure that prices remain in Δ we shall then re-normalize the adjusted prices, i.e. divide by the sum of all adjusted prices. This will also lower the relative price of commodities for which there is excess supply rather than excess demand, as we intended to. This motivates that we define f as follows:

$$f_\ell(p) = \frac{p_\ell + \max\{\zeta_\ell(p), 0\}}{\sum_{k=1}^{L} p_k + \max\{\zeta_k(p), 0\}} \text{ for all } p \in \Delta \text{ and all } \ell = 1, 2, \ldots, L$$

Now let us check that fixed points of this function f are indeed Walrasian equilibria, that is, we want to prove that $f(p) = p$ implies $\zeta(p) = 0$. We first show the somewhat weaker claim that $f(p) = p$ implies $\zeta_\ell(p) \leq 0$ for all ℓ. First note that here can't be a $p \in \Delta$ such that $\zeta_\ell(p) > 0$ for all ℓ, because this would imply $p \cdot \zeta(p) > 0$, and thus contradict Walras' Law. Now suppose $\zeta_\ell(p) > 0$ for some ℓ, and $\zeta_\ell(p) \leq 0$ for some other ℓ.Then $f_\ell(p) < p$ for all ℓ such that $\zeta_\ell(p) \leq 0$, because in the formula for $f_\ell(p)$ the numerator for such an ℓ is p_ℓ, but the denominator is greater than 1, because it consists of the sum of all prices, which equals 1, and some other strictly positive term. But if $f_\ell(p) < p$ for some ℓ, then we can't have a fixed point. Thus, we conclude that $f(p) = p$ implies $\zeta_\ell(p) \leq 0$ for all ℓ.

Note that the reverse is obviously true, that is, if $\zeta(p) = 0$, then $f(p) = p$. Thus, the set of Walrasian equilibrium prices equals the set of all fixed points of f.

As an intermediate step, we now note that $\zeta_\ell(p) \leq 0$ for all ℓ implies $p_\ell > 0$ for all ℓ, which means, using the result of the previous paragraph, that all prices are strictly positive in any fixed point of f. This follows directly from condition (iii) in the proposition. If $p_\ell = 0$, then we would have $\zeta_\ell(p) > 0$.

Now we can finally prove that $\zeta(p) = 0$ in any fixed point of f. What we have to show is that it cannot be that $\zeta_\ell(p) < 0$ for some ℓ if p is a fixed point of f. But as all prices are positive, and all components of $\zeta(p)$ are non-positive, this would imply $p \cdot \zeta(p) < 0$, which contradicts Walras' Law, condition (ii) of the proposition. Thus, we can conclude that if $f(p) = p$ then $\zeta(p) = 0$.

We conclude the proof by showing that f has at least one fixed point. We use Brouwer's fixed point theorem for this. The existence of at least one fixed point of f follows if f satisfies the conditions of Brouwer's theorem: the domain of f must be compact and convex, and f must be continuous. The domain of f is Δ which is obviously compact and convex. The function f is continuous because we have assumed that the function ζ is continuous.

Q.E.D.

It remains to find assumptions about the basic features of the underlying economy that imply that ζ has the properties listed in the previous proposition. This is done in the next proposition.

This proposition makes very restrictive assumptions. This allows us to present a comparatively simple proof.

Proposition: Suppose for every consumer $i \in \{1, 2, \ldots, I\}$:

(i) $X_i = [0, m]^L$ where $Im > \max\limits_{\ell \in \{1,2,\ldots,L\},\ x \in \left\{\sum_{i=1}^{I} \omega_i\right\} + \sum_{j=1}^{J} Y_j} x_\ell$,

(ii) $\omega_i \gg 0$,

Condition (i) is requires each consumer's consumption set to be bounded from above by some number m, where m must be sufficiently large. Specifically, if all I consumers demand m of any commodity ℓ, then the total demand, Im, is larger than the quantity of ℓ that could be produced under any feasible production plan. In this sense, although consumption sets are bounded from above, the upper bound must not "bind."

(iii) $\succsim_i$ is continuous, strictly increasing, and satisfies the following convexity condition:

$$x_i \neq x_i', x_i \sim_i x_i' \text{ and } \lambda \in (0,1) \Rightarrow \lambda x_i + (1-\lambda)x_i' \succ_i x_i;$$

and for every firm $j \in \{1, 2, \ldots, J\}$:

(iv) $0 \in Y_j$, Y_j is compact, and Y_j satisfies the following convexity condition:

$$y_j, y_j' \in Y_j, y_j \neq y_j', \text{ and } \lambda \in (0,1) \Rightarrow \hat{y}_j \gg \lambda y_j + (1-\lambda)y_j' \text{ for some } \hat{y}_j \in Y_j,$$

then the economy's excess demand function $\zeta : \Delta \to \mathbb{R}^L$ is well-defined, and satisfies assumptions (i)-(iii) of the previous proposition.

Proof: The existence of optimal consumption and production plans for any p is implied by Weierstrass' theorem because the consumers' and firms' maximization problems have non-empty, compact choice sets and continuous objectives. We don't spell out further details. Note that demand exists even if some price is zero.Even then the choice set is compact because we have assumed the existence of an upper bound m for every agent's consumption of every good.

Weierstrass' Theorem: Every continuous function $f : X \to \mathbb{R}$ assumes a maximum in X if X is a non-empty subset of a finite-dimensional Euclidean space, and compact.

. The uniqueness of optimal consumption and production plans for any p follows from our convexity assumptions. Consider consumer's decisions first. Suppose for some consumer i, at some prices p, there were two optimal consumption plans. Then by condition (iii) any convex combination of these two consumption plans is strictly preferred. Moreover, because consumption sets are convex, such a convex combination will be in the consumption set, and because the budget constraint is linear in quantities, any such convex combination will also satisfy the budget constraint. Thus, the consumer can afford the convex combination, and it is strictly preferred to the other two plans. This contradicts the assumption that the two original consumption plans are optimal.

One reason why we have assumed condition (iv) in the proposition is that it renders the last sentence of the first paragraph true.

Now consider firms. Suppose for some firm j, at some prices p, there were two optimal production plans. Then, because profits are a linear function of quantities, any convex combination of these two production plans is also optimal. By condition (iv) of the proposition, there is also a production plan that is in every component strictly larger than the convex combination. Such a production plan yields higher profits, because all prices are non-negative and some prices are strictly positive. This contradicts the assumption that both original production plans are optimal.

Walras' Law follows from the assumption that every agent's preferences are strictly increasing. Condition (iii) of the previous proposition, i.e. $\zeta_\ell(p) > 0$ if $p_\ell = 0$, follows from the monotonicity assumption, which implies that every agent will demand m of good ℓ if good ℓ has price zero, together with the assumption that $Lm > \sum_{i=1}^{I} \omega_{i,\ell}$.

It remains to show that ζ is continuous. This follows from the theorem of the maximum if we can show that for all firms and consumers the correspondence that maps prices into admissible choices is a continuous correspondence. This is obvious for firms, as the correspondence is constant. Thus, supply functions are continuous. It remains to prove for consumers that the budget correspondence:

$$\mathcal{B}_i(p) = \left\{ x_i \in X_i | p \cdot x_i \leq p \cdot \omega_i + \sum_{j=1}^{J} \theta_i^j \left(p \cdot y_j(p) \right) \right\}$$

is continuous on Δ. That $\mathcal{B}_i$ is upper hemi-continuous is obvious, given that we have already shown that $y_j(p)$ is continuous in p for every j. Thus, it remains to prove that $\mathcal{B}_i$ is lower hemi-continuous.

The proof that $\mathcal{B}_i$ is lower hemi-continuous is the only somewhat tricky part of the proof, and it is a somewhat tedious proof. It is less important that you study this part of the proof.

Let $p \in \Delta$ and $x_i \in \mathcal{B}_i(p)$. Suppose $p_n \in \Delta$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} p_n = p$. We have to prove that for every $n \in \mathbb{N}$ there is some $x_{i,n} \in \mathcal{B}_i(p_n)$ such that $\lim_{n\to\infty} x_{i,n} = x_i$.

Before we start the proof, we note two facts. The first is that the consumer's income in the limit is strictly positive:

$$p \cdot \omega_i + \sum_{j=1}^{J} \theta_i^j \left(p \cdot y_j(p) \right) > 0.$$

This is true because at least one commodity has a positive price, ω_i is strictly positive in every component, and profits cannot be less than zero because $0 \in Y_j$ for all $j \in \{1, \ldots J\}$. The second fact we note is that the income is continuous in the limit, and therefore the limit of the incomes is strictly positive:

$$\lim_{n\to\infty} \left(p_n \cdot \omega_i + \sum_{j=1}^{J} \theta_i^j \left(p_n \cdot y_j(p_n) \right) \right) = p \cdot \omega_i + \sum_{j=1}^{J} \theta_i^j \left(p \cdot y_j(p) \right) > 0.$$

This claim is obvious given that, as we have already noted, y_j is continuous in p for every j.

We now distinguish two cases: The first case is that $p \cdot x_i = 0$. In this case $\lim_{n\to\infty} p_n \cdot x_i = 0$. Because on the other hand the limit income is strictly positive, for large enough n the consumption bundle x_i will be affordable at prices p_n. Thus, for large enough n we can set $x_{i,n} = x_i$. For small n we can choose $x_{i,n}$ arbitrarily. Obviously, then $\lim_{n\to\infty} x_{i,n} = x_i$.

The second case is that $p \cdot x_i > 0$. In this case, if x_i is affordable at p_n, that is, if $p_n \cdot x_i \leq p_n \cdot \omega_i + \sum_{j=1}^{J} \theta_i^j (p_n \cdot y_j(p_n))$, then we set $x_{i,n} = x_i$. But we cannot be sure that this is always possible. If x_i is not affordable at p_n, then we define $x_{i,n}$ to be an appropriately re-scaled version of x_i:

$$x_{i,n} = \frac{p_n \cdot \omega_i + \sum_{j=1}^{J} \theta_i^j (p_n \cdot y_j(p_n))}{p_n \cdot x_i} x_i.$$

We first note that $x_{i,n}$ is in the consumption set X_i. This is the case because $x_{i,n}$ is a convex combination of x_i, which is in X_i by assumption, and of the zero vector, and because the consumption set X_i is convex and includes the zero vector. Next, we verify that $x_{i,n}$ is affordable at prices p_n:

The definition of $x_{i,n}$ is the only trick in this proof.

$$p_n \cdot x_{i,n} = \frac{p_n \cdot \omega_i + \sum_{j=1}^{J} \theta_i^j (p_n \cdot y_j(p_n))}{p_n \cdot x_i} p_n \cdot x_i = p_n \cdot \omega_i + \sum_{j=1}^{J} \theta_i^j (p_n \cdot y_j(p_n)).$$

The right hand side is income at prices p_n. Thus, x_n satisfies the budget constraint at prices p_n as an equality.

Finally, we have to show:

$$\lim_{n \to \infty} x_{i,n} = x_i.$$

To prove this, it is sufficient to focus on the case in which $x_{i,n} \neq x_i$. But in that case the formula that defines $x_{i,n}$ shows that $x_{i,n}$ is equal to x_i times a re-scaling factor that converges to 1 as $n \to \infty$. Thus the proof is complete.

Q.E.D.

We conclude that under the assumptions of the second Proposition there is at least one Walrasian equilibrium. We have proven this under assumptions that are much stronger than needed. This has allowed us to present a proof that exposes the core arguments of the typical proof of existence of Walrasian equilibrium without being distracted by other issues. You can find more general versions of the existence result in advanced microeconomics textbooks.

The most important set of assumptions that cannot really be relaxed are the convexity assumptions. In our proof, the convexity assumptions were used to ensure the uniqueness of optimal choices by consumers and firms. A proof can be constructed in which these choices are not unique. We would then have to work with excess demand correspondences rather than functions, and could use Kakutani's fixed point theorem for correspondences to prove existence of Walrasian equilibrium. Kakutani's theorem applies, however, only to convex valued correspondences. To ensure that excess demand is a convex-valued correspondence one typically ensures that individual demand and supply correspondences are convex valued, and to ensure this one assumes weak convexity of preferences and production sets. Thus, our strict convexity assumptions can be replaced by weak convexity assumptions, and the result still holds.

10. Uniqueness and Comparative Statics of General Equilibrium

We now want to study whether the theory of general equilibrium allows us to make predictions. We shall focus about predictions about equilibrium prices. Of course, we could also focus on predictions about equilibrium quantities. But for simplicity we will maintain a narrow focus in this section. Suppose, for example, that it becomes easier to produce laptops. Our question is: how will equilibrium prices respond? Note that we use the plural, "prices," because we are not only interested in the response of the price of laptops, but we are also interested in the price of other personal electronic devices, and perhaps also in the price of other consumption goods.

Recall that equilibrium prices are prices p^* where excess demand is zero: $\zeta(p^*) = 0$. At a very abstract level, we are asking how the solution of the equation $\zeta(p^*) = 0$ will change if ζ changes. To get a good handle on this question we shall assume that there are some parameters that enter ζ in addition to prices. Let us denote these parameters by a. Assume a is a finite dimensional vector of real numbers. For example, a could be an element of $\mathbb{R}$ that indicates how easy it is to produce laptops, in other words it shifts the production set of laptop producers. We can include these parameters among the arguments of ζ and write $\zeta(p, a)$, and then our question is: How does the solution of $\zeta(p, a) = 0$ with respect to p change if the parameter a changes. In particular, we will be interested in the derivative of the solution, that is, the equilibrium prices, with respect to a.

Notice that the formulation of our question that we have used so far implicitly assumes that there is a unique vector of equilibrium prices. Obviously, this can only be true up to normalization. But is it true that there is a unique, up to normalization, vector of equilibrium prices? It turns out that this is the case only under very special assumptions. But, most of the time, if for some excess demand function ζ the price vector p^* is an equilibrium price vector, then there is no other solution of $\zeta(p, a) = 0$ that is very close to p^*. More precisely, we can draw a circle, perhaps small, around p^* and there are no equilibrium prices other than p^* in that circle. In this case we say that the equilibrium prices are "locally unique," and it turns out that this is enough to give meaning to our question about the effect of changes in a on equilibrium prices. Specifically, we can ask: how do equilibrium prices change as we change a, assuming that even after we have changed a we only consider equilibrium prices in a small neighborhood of p^*.

The main mathematical tool for studying the question as we have now formulated it will be the Implicit Function Theorem. Let us begin by recalling the precise statement of this theorem: In the following formulation of this theorem, disregard the meaning that the symbols that I use have had in other parts of these lecture notes. Just imagine we made a new beginning, and interpret all notation in the precise sense defined in the statement of the theorem. In the theorem a variable called "x" will later play the role of equilibrium prices, and a variable called "y" will later play the role of the arguments a that enter the excess demand function. It may be helpful to keep this in mind when reading the result.

Theorem 9.28 in W. Rudin, *Principles of Mathematical Analysis*, 3rd edition, 1976.

The Implicit Function Theorem: *Let $f : E \to \mathbb{R}^n$, where $E \subseteq \mathbb{R}^{n+m}$ is an open set. Suppose f is continuously differentiable. Suppose $f(x^*, y^*) = 0$ for some $(x^*, y^*) \in E$. Let A be the matrix of partial derivatives of f in (x^*, y^*):*

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} & \frac{\partial f_1}{\partial y_1} & \dots & \frac{\partial f_1}{\partial y_m} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{\partial f_n}{\partial x_1} & \dots & \frac{\partial f_n}{\partial x_n} & \frac{\partial f_n}{\partial y_1} & \dots & \frac{\partial f_n}{\partial y_m} \end{pmatrix}$$

Let A_x be the matrix consisting of the first n columns of A, and let A_y be the matrix consisting of the last m columns of A. Assume that A_x is invertible.

Then there are open sets $U \subseteq \mathbb{R}^{n+m}$ and $W \subseteq \mathbb{R}^m$ with $(x^*, y^*) \in U$ and $y^* \in W$ such that for every $y \in W$ there is a unique x such that $(x, y) \in U$ and $f(x, y) = 0$. Denote this x by $g(y)$. Thus, $g : W \to \mathbb{R}^n$. The function g is continuously differentiable, and if B denotes the matrix of g's partial derivatives in y^*, then:

$$B = -(A_x)^{-1} A_y.$$

This theorem concerns the solutions of the equation $f(x, y) = 0$ with respect to x, where y is a vector of parameters that enters f. The theorem asserts that under certain conditions three things are true: (i) the solution to $f(x, y) = 0$ with respect to x is locally unique, that is, there is a neighborhood of the solution such that in that neighborhood there is no other solution; (ii) if we change y only a little bit, then there will be unique solutions to the equation, provided that we only consider solutions that stay close to the original solution; (iii) derivative of the solutions x to the equation $f(x, y)$ with respect to y is given by a simple formula. In our application, we thus know that under certain conditions equilibrium price vectors are locally unique, and that they remain locally unique if we change parameters in the excess demand function a little bit. Moreover, we have a formula for the derivative of the equilibrium price vectors with respect to the parameters of excess demand function.

Note that we are only considering small changes of the exogenous parameters because we focus on the derivative with respect to these parameters. Also, we are assuming that equilibrium prices change only a little, and don't jump, as we change those parameters a little. Notice that we are assuming this. We don't argue why this might be so.

Before we apply the implicit function theorem to our context, we have to take some preliminary steps. First, we note that the absolute value of equilibrium prices cannot be uniquely determined, not even locally, because only relative prices determine supply and demand. We therefore have to choose some normalization of prices. It is useful to set the price of one commodity, say commodity L, equal to 1. Thus, essentially, we express the price of all commodities ℓ relative to good L. This normalization has the advantage that there is a natural candidate for the domain of ζ, namely the set $\mathbb{R}^{L-1}_{++}$. This is an open set, as it should to be for an application of the implicit function theorem. With other normalizations of prices, it would have been slightly more awkward to choose the domain of ζ so that the implicit function theorem can be used.

Now it appears that the domain of ζ is $L-1$-dimensional whereas the codomain is the set of all L-dimensional excess demand vectors. The application of the implicit function theorem, however, requires that domain and co-domain have the same dimension. However, if ζ satisfies Walras' law, that is, if $p\zeta(p) = 0$ for all p (where, for a moment, we take p again to be in $\mathbb{R}^{L}_{++}$) then we can ignore excess demand for any one of the goods, because if $L-1$ markets are in equilibrium, that is, excess demand equals zero, it will also equal zero in the remaining market. We shall drop the excess demand for good L from the excess demand function. Let us call the new excess demand function η. Thus, $\eta : \mathbb{R}^{L-1}_{++} \to \mathbb{R}^{L-1}$ maps every vector of relative prices of goods $\ell = 1, 2, \dots, L-1$ (relative to the price of good L) into the vector of excess demand for goods $\ell = 1, 2, \dots, L-1$.

The following result on local uniqueness is an immediate consequence of the Implicit Function Theorem.

Proposition: Suppose that η is continuously differentiable, that $\eta(p^*) = 0$ for some $p^* \in \mathbb{R}^{L-1}_{++}$, and suppose the matrix of partial derivatives of η in p^* is invertible. Then p^* is a locally unique equilibrium price vector, that is, there is $\varepsilon > 0$ such that $||p - p^*|| < \varepsilon$ and $p \neq p^*$ implies $\eta(p) \neq 0$.

Definition: Suppose that η is continuously differentiable, that $\eta(p^*) = 0$ for some $p^* \in \mathbb{R}^{L-1}_{++}$, and suppose the matrix of partial derivatives of η in p^* is invertible. Then p^* is called a *regular* equilibrium.

Intuitively, the condition that the matrix of partial derivatives of η is invertible means the following: Using the local linear approximation of η, if we are given any vector of changes in the quantities demanded for all $L-1$

goods, we can infer unambiguously which changes in the prices of the $L-1$ goods have caused this change in the quantities demanded. We might wonder how restrictive this assumption is. This is answered by the following proposition that we only state informally.

Informal Proposition: "Almost all" economies have an odd number of equilibria all of which are regular.

Now let us study comparative statics. Suppose that there is an m-dimensional vector q of real numbers that enters excess demand. This vector could parametrize, for example, agents' preferences, their initial endowments, or the firms' production technologies. Suppose for some parameter value Q^* we have a locally unique regular equilibrium p^*. Our interest is in how the equilibrium prices change as we change q. The implicit function theorem makes clear that this depends on what we know about the matrix of partial derivatives of η with respect to prices. One case in which we can draw some conclusions is the case in which we know that this matrix is negative semi-definite. We shall comment later on the economic meaning of this condition.

Proposition: *Suppose excess demand is:* $\eta : \mathbb{R}^{L-1}_{++} \times W \to \mathbb{R}^{L-1}$ *where* $W \subseteq \mathbb{R}^m$ *is an open set. Assume* η *is continuously differentiable, and suppose that for some* q^* *we have* $\eta(p^*, q^*) = 0$*. Moreover, denoting by* A_p *the matrix of partial derivatives of* η *with respect to p, assume that* A_p *is invertible and negative semi-definite. Denote by* B_q *the derivative of equilibrium prices with respect to q. Then for every vector* $dq \in \mathbb{R}^m$*:*

$$(A_q dq)^T \cdot B_q dq \geq 0.$$

Let us consider what this theorem says in words. Consider any vector $dq \in \mathbb{R}^m$ of changes in the parameter q. Then $A_q dq$ is the $L-1$ dimensional vector of the demand changes that result, according to the local linear approximation of demand, at equilibrium prices p^* if the parameters q change by dq. Similarly, $B_q dq$ is the $L-1$ dimensional vector of price changes that result, according to the local linear approximation of demand, if the parameters q change by dq. The conclusion of the proposition is that the vector product of these two vectors is non-negative. This is a weaker statement than saying: if $A_q dq$ is positive in some component ℓ, that is, if the change in q causes demand for good ℓ to go up, then price goes up. The proposition says that this has to be true if we sum over all commodities. For example, if the demand for good 1 goes up by 2 units, yet its price goes down by 2 units, and if the demand for good 2 goes down by 1 unit, then it must be that the price of good 2 goes down by at least 4 units. Only then is the vector product referred to in the Proposition non-negative. In fact, if the price of good 2 goes down by exactly 4 units, then the vector product is: $2 \cdot (-2) + (-1) \cdot (-4) = 0$.

Proof: We have to show:

For simplicity we omit the transpose sign "T" in the proof.

$$A_q dq \cdot B_q dq \geq 0$$

We use the formula that the implicit function theorem gives us to substitute for B_q:

$$\begin{aligned} A_q dq \cdot \left(- (A_p)^{-1} A_q dq \right) &\geq 0 \\ (A_q dq) \cdot (A_p)^{-1} (A_q dq) &\leq 0 \end{aligned}$$

This is true because the assumption that A_p is negative semi-definite implies that also $(A_p)^{-1}$ is negative semi-definite.

Q.E.D.

It remains to examine the economic meaning of the condition that the matrix A_p is negative semi-definite. In fact, it is a weak monotonicity assumption on demand. The vector product of demand changes caused by a price change, and the price change itself, must be non-positive. If it were true for each good, then an increase in price must cause a decrease in quantity, and vice versa. Because we sum over all goods, in a sense this monotonicity has to hold only on average. Note that the condition has to hold only locally, at the equilibrium price vector.

There is another assumption that allows us to obtain results on equilibrium comparative statics. It is that all goods are gross substitutes, i.e. if the price of one good increases, the demand for all other goods goes up. Clearly, that is an extremely strong assumption.

Definition: *All goods are gross substitutes at some price vector p if at that price vector all derivatives $\partial\zeta_\ell(p)/\partial p_\ell$ are negative, and all derivatives $\partial\zeta_\ell(p)/\partial p_{\ell'}$ where $\ell' \neq \ell$ are positive.*

In this definition, we mean by ζ the excess demand function as defined in Section 8 (d), that is, $\zeta(p)$ is defined for all price vectors, and no normalization is assumed.

One can show that this assumption is stronger than the assumption that A_p is negative semi-definite. It also allows a stronger conclusion:

Proposition: *Suppose excess demand is: $\eta : \mathbb{R}^{L-1}_{++} \times W \to \mathbb{R}^{L-1}$ where $W \subseteq \mathbb{R}^m$ is an open set. Assume η is continuously differentiable, and suppose that for some q^* we have $\eta(p^*, q^*) = 0$. Moreover, suppose that at (p^*, q^*) all goods are gross substitutes. Consider a vector $dq \in \mathbb{R}^m$ which satisfies: $A_q dq \ll 0$. Then:*

$$B_q dq \ll 0.$$

This means in words the following: Suppose at some equilibrium prices all goods are gross substitutes, and suppose a change in the exogenous parameters q has, using a local linear approximation, the effect that demand for all $L-1$ goods decreases. Then the relative price of all $L-1$ goods has to decrease.

Proof: We have to show:

$$B_q dq \ll 0$$

We use the formula that the implicit function theorem gives us to substitute for B_q:

$$\begin{aligned} -\left(A_p\right)^{-1} A_q dq &\ll 0 \\ \left(A_p\right)^{-1} A_q dq &\gg 0 \end{aligned}$$

A linear algebra result in MWG shows that the gross substitutes assumption implies that all entries of $\left(A_p\right)^{-1}$ are negative. But then our assertion follows immediately.

Proposition 17.G.3 in MWG.

Q.E.D.

Now we have seen some conditions on the matrix A_p that imply that we can do meaningful comparative statics. But what can one say in general about the matrix A_p? It turns out that one can say nothing. This should not be a complete surprise, as we have already seen in the Debreu-Sonnenschein-Mantel theorem that there are almost no general properties of aggregate excess demand that hold in general.

Proposition: For every invertible $(L-1) \times (L-1)$-matrix $\mathcal{M}$ there is an exchange economy with excess demand function η and a price vector $p^* \in \mathbb{R}^{L-1}_{++}$ such that $\eta(p^*) = 0$ and the matrix of partial derivatives of η at p^* is $\mathcal{M}$.

We omit the proof of this result.

So far, we have only considered local uniqueness of equilibrium. When considering comparative statics, we have assumed that small changes in exogenous parameters do not cause large changes in equilibrium prices. We have no justification for this assumption. Our comparative statics results would be much stronger if we knew that overall there is just a single equilibrium. Let us say that in this case equilibrium is “globally unique.” We now present two sufficient, or almost sufficient, conditions for global uniqueness of equilibrium.

The first result assumes that aggregate demand satisfies the weak axiom of revealed preferences. This assumption is formulated ignoring that consumers might have profit income. This is not problematic because the result also assumes that all technologies have constant returns to scale, and hence all firms will have zero profits in equilibrium.

Proposition: *Consider an economy such that for every consumer i and every price vector $p \in \Delta$ there is a unique $x_i(p) \in X_i$ that maximizes $\succsim_i$ subject to the constraint $x_i' \in X_i$ and $p \cdot x_i' \leq p \cdot \omega_i$. Assume that Walras Law holds for every consumer i, i.e. $p \cdot x_i(p) = p \cdot \omega_i$ for all $p \in \Delta$. Define $z(p) = \sum_{i=1}^{I} (x_i(p) - \omega_i)$. Suppose that z satisfies the weak axiom of revealed preferences: $p \cdot z(p') \leq 0$ and $p' \cdot z(p) \leq 0$ implies $z(p) = z(p')$. Suppose that for every firm j the production set Y_j has constant returns to scale, i.e. if $y_j \in Y_j$ and $\lambda \geq 0$, then $\lambda y_j \in Y_j$. Then the set of Walrasian equilibrium prices is convex.*

The conclusion of this Proposition is not what I had promised: the proposition does not say that there is a unique Walrasian equilibrium, but that the set of Walrasian equilibrium prices is convex. But this is actually al;most the same as saying the equilibrium prices are unique. This is because a convex set has either no, one, or infinitely many elements. But, as we saw before, for almost all economies, the set of equilibria is finite. Therefore a convex set of equilibrium prices must be either empty or have a unique element.

Here, we are sloppy. We don't know whether the statement that for almost all economies the set of equilibria is finite is still true if we restrict attention to only those economies that satisfy the conditions of the Propositions.

Proof: We begin by observing that p is a Walrasian equilibrium price vector if and only if

(i) $p \cdot y_j \leq 0$ for every j and for every $y_j \in Y_j$;

(ii) there is a $y \in \sum_{j=1}^{J} Y_j$ such that $z(p) = y$.

Condition (i) is necessary because $p \cdot y_j > 0$ would imply that firm j could choose as its production plan λy_j for arbitrarily large $\lambda \geq 0$, and make unbounded profits $\lambda p \cdot y_j$. In other words, a profit maximizing production plan would not exist. The production plan λy_j is feasible because Y_j has constant returns to scale. Condition (ii) is necessary because there must be equilibrium production plans y_j at which supply equals demand. We can take $y = \sum_{j=1}^{J} y_j$. This production plan satisfies: $p \cdot y = 0$. To see this observe that condition (i) implies that $p \cdot y \leq 0$. But if $p \cdot y_j < 0$ for some j then firm j would not maximize profits, because it could could choose production plan 0. We know $0 \in Y_j$ because of constant returns to scale, setting $\lambda = 0$. Now $p \cdot y = 0$ means that profit income is zero for all consumers. Therefore, consumer i's income is given by $p \cdot \omega_i$, and therefore, the budget constraint subject to which $x_i(p)$ maximizes consumer i's preferences is indeed the budget constraint that consumer i faces in equilibrium. For supply to equal demand we have to have: $z(p) = y$.

It remains to show that if $p, p' \in \Delta$ satisfy (i) and (ii), and if $\alpha \in (0,1)$ then also $p'' = \alpha p + (1-\alpha)p'$ satisfies (i) and (ii). First we show that $p'' y_j \leq 0$ for every j and every $y_j \in Y_j$. Note that:

$$p'' \cdot y_j = \alpha p \cdot y_j + (1-\alpha) p' \cdot y_j.$$

Because each of the two terms in the sum of the right hand side is non-negative, also the sum is non-negative.

The crucial step is to prove (ii). This is where we shall use the weak axiom of revealed preferences. First, we note that (i), and the fact that $z(p)$ and $z(p')$ both satisfy (ii) together implies $p'' z(p) \leq 0$ and $p'' z(p') \leq 0$. This means that at prices p'' both $z(p)$ and $z(p')$ can be afforded. Now we show the reverse as well, that is, that $z(p'')$ must be affordable either at prices p or at prices p' or at both. If we can show that, then the weak axiom of revealed preferences implies immediately that either $z(p'') = z(p)$ or $z(p'') = z(p')$ or both, and therefore, because $z(p)$ and $z(p')$ satisfy (ii), so does $z(p'')$.

Walras Law implies that: $p'' \cdot z(p'') = 0$, hence $\alpha p \cdot z(p'') + (1-\alpha) p' \cdot z(p'') = 0$, and hence at least one of the two terms on the left hand side must be non-positive. This means that either $p \cdot z(p'') \leq 0$, in which case $z(p'')$ is affordable at prices p or $p' \cdot z(p'') \leq 0$ in which case $z(p'')$ is affordable at prices p'.

Q.E.D.

Our second result uses a very strong assumption, and then concludes that there is a unique equilibrium price vector, provided that all prices are strictly positive.

Proposition: *Suppose ζ is homogeneous of degree zero, continuously differentiable, and suppose that at all price vectors all goods are gross-substitutes. Then there is at most one $p^* \in \Delta$ such that $p^*_\ell \neq 0$ for all ℓ and $\zeta(p^*) = 0$.*

Proof: Suppose $p', p'' \in \Delta$ were equilibrium price vectors, and $p \neq p''$. Define

$$\lambda = \max_{\ell=1,\ldots,L} \frac{p'_\ell}{p''_\ell}.$$

Then $\lambda p''_\ell \geq p'_\ell$ for all ℓ, and $\lambda p''_{\hat{\ell}} = p'_{\hat{\ell}}$ for at least one $\hat{\ell}$. Moreover, because $p'_\ell \neq p''_\ell$ we have $\lambda p''_\ell > p'_\ell$ for at least one ℓ. Now at price vector p' we have $\zeta(p') = 0$. Now suppose we change the price vector p' by successively raising the price of each commodity to $\lambda p''_\ell$, starting with commodity 1, and working our work successively to commodity L, leaving out commodity $\hat{\ell}$, because that commodity in any case has the same price in p' and in $\lambda p''$. At each step, if we raise the price of some commodity other than $\hat{\ell}$, the excess demand for good $\hat{\ell}$ will either stay the same or increase. Because we raise at least one price strictly, at the end the excess demand for good $\hat{\ell}$ will have increased. This implies that $\zeta(\lambda p''_{\hat{\ell}}) > 0$, and hence, because ζ is homogeneous of degree zero, $\zeta(p'') \neq 0$, which contradicts the assumption that p'' is an equilibrium price vector.

Q.E.D.

11. Time and Uncertainty in General Equilibrium

To include time and uncertainty in general equilibrium theory we can simply re-interpret the static Arrow-Debreu model. What we have so far called a "good," having in mind some particular physical good, such as apples, we now define to be a particular physical good, say apples, at a particular point in time, say October 30, 2015, and in a particular state of the world, say the state defined by the condition that next year's apple harvest in Michigan exceeds some threshold. Let us call a contract that delivers a particular physical good at a particular time in a particular state of the world an "Arrow-Debreu commodity." The theory that we have studied so far does not need to be changed if we interpret goods as Arrow-Debreu commodities.

Arrow-Debreu commodities can be viewed as contracts. Such contracts exist in practice, but there are not as many such contracts as there could be. For example, there does not seem to be a market for the apple contract that I gave as an example in the first paragraph. This poses a difficulty because we have assumed so far that agents' preferences only depend on the commodities that are actually traded. But my preference might be over hypothetical contracts that don't exist as well as over the actually existing contracts. If that is the case, then the first and second welfare theorems need not be true any longer. Our primary purpose in this section is to investigate how incomplete the system of tradable contracts may be if the welfare theorems are to remain true.

We shall work with a very simple model. There are only two time periods: $t = 0, 1$. There is a finite set of possible states of the world, $s \in \{1, 2, \cdots, S\}$. The state is realized in period 1. In period 0 it is unknown which state will be realized, but in period 1 the state will be observed by everyone. For simplicity, we consider an exchange economy. In period 0, future contracts for period 1 are traded, but no consumption takes place, and indeed no endowments exist. Endowments are realized in period 1, and their size may depend on the state of the world. Preferences are defined over consumption bundles that specify for each period 1 state of the world the consumption of each physical good in that state.

We now introduce the notation and formal assumptions. The consumers are $i \in \{1, 2, \ldots, I\}$. The physical goods are $\ell \in \{1, 2, \ldots, L\}$. Consumer i's consumption set is $X_i = \mathbb{R}_+^{LS}$. An element x_i of X_i is an LS-dimensional vector where entry $x_{i\ell s}$ specifies consumer i's consumption of physical good ℓ in state s. Consumer

i's endowment is $\omega_i \in \mathbb{R}^{LS}_{+}$. Consumer i has rational preferences $\succsim_i$ over X_i. Note that we do not assume that agents have probabilities over S and then maximize the expected value of a von Neumann Morgenstern utility function. This assumption would restrict the preferences $\succsim_i$ that are allowed, and we don't need this assumption in what follows. We do assume, however, that for every i the preferences $\succsim_i$ are strictly monotonically increasing in every component.

Now we explain which markets are assumed to exist. Note that we take as exogenous which markets exist. We are leaving out the big issue of how to explain which markets exist and why. In period 0 agents can sell K assets. Each asset is a promise by the market to provide in each possible state of the world a certain quantity of good 1 that may depend on the state. This quantity may be negative in which case the asset is really a claim to provide in that state a certain quantity of physical good 1. We use here good 1 as "numéraire," which means nothing else but that all assets are written as contracts regarding physical good 1. This is a simplification. It does not really matter which exact physical good is promised to be provided or delivered, as agents will be able to swap good 1 for other goods in period 1.

As mentioned, which markets exist is exogenous. Specifically, this now means that it is exogenous which assets can be traded in period 0. There are assumed to be K assets. These assets are described by a returns matrix R:

$$R = \begin{pmatrix} r_{11} & \dots & r_{1K} \\ \dots & \dots & \dots \\ r_{S1} & \dots & r_{SK} \end{pmatrix}$$

This matrix is of dimension $S \times K$, and entry r_{sk} describes the quantity of physical good 1 promised by asset k in state s. The prices of these assets in period 0 will be denoted by $q = (q_1, \dots, q_K) \in \mathbb{R}^K$, and the quantities of each asset that consumer i buys in period 0, consumer i's "portfolio," are written as $z_i = (z_{i1}, \dots, z_{iK})$. If consumer i purchases asset k, then $z_{ik} \geq 0$, and if consumer i sells asset k, then $z_{ik} \leq 0$. Consumer i's budget constraint in period 0 is $q \cdot z_i \leq 0$. To understand this inequality recall that there are no initial endowments in period 0. Thus, all purchases of assets have to be financed from sales of other assets. This motivates the budget constraint.

To simplify some arguments that follow below, we now introduce an assumption that ensures in a sense that the assets available are "attractive." The assumption is that there is at least one portfolio $z \in \mathbb{R}^K$ such that $R \cdot z \geq 0$ and $R \cdot z \neq 0$. This implies that, if there were no budget constraint in period 0, the consumer could find portfolios that purchase arbitrarily much of good 1 in at least one state of the world, and that would not require her to sell good 1 in any other state of the world. Interesting asset structures will satisfy this assumption.

Do you agree with the last sentence of this paragraph? Find an example of an asset structure that violates the assumption introduced in this paragraph.

In period 1 the state and the (state contingent) endowments are realized. There are then L spot markets for each of the physical goods. We then have to add to consumer i's endowment the quantity of good 1 to which the asset that he purchased in period 0 entitles him, and we have to subtract the quantities of physical good 1 that consumer i promised to deliver, according to his asset purchases in period 0. This gives him a sort of "net" physical endowment, which he then sells. He can use the return to purchase his actual physical consumption in period 1. If we denote by p_s the vector of prices of the L physical goods in state s, by ω_{is} consumer i's endowment in state s, and by x_{is} consumer i's consumption in state s, then consumer i's budget constraint in state s is:

$$p_s \cdot x_{is} \leq p_s \cdot \omega_{is} + p_{1s} \sum_{k=1}^{K} z_{ik} r_{sk}.$$

We can now formally define a Walrasian equilibrium for our model. In honor of NYU professor Roy Radner, such an equilibrium is called a "Radner equilibrium."

Definition: *A* Radner equilibrium *is a list of prices* $q^* \in \mathbb{R}^K, (p_1^*, \ldots, p_S^*) \in \mathbb{R}^{SL}$ *and, for each agent* i*, quantities* $z_i^* \in \mathbb{R}^K, (x_{i1}^*, \ldots, x_{iS}^*) \in \mathbb{R}_+^{SL}$ *such that the following three conditions are satisfied. To formulate the conditions we define agent* i*'s budget set as:*

$$\mathcal{B}_i = \Bigg\{ (z_i, x_{i1}, \ldots, x_{iS}) \in \mathbb{R}^K \times \mathbb{R}_+^{LS} \,|\, q^* \cdot z_i \leq 0 \text{ and, for all states } s \in S,$$

$$p_s^* \cdot x_{is} \leq p_s^* \cdot \omega_{is} + p_{1s}^* \sum_{k=1}^{K} z_{ik} r_{sk} \Bigg\}$$

(i) for all $i : (z_i^*, x_{i1}^*, \ldots, x_{iS}^*) \in \mathcal{B}_i$*;*

(ii) for all $i : (x_{i1}^*, \ldots, x_{iS}^*) \succsim_i (x_{i1}, \ldots, x_{iS})$ *for all* $(x_{i1}, \ldots, x_{iS})$ *for which there is some* $z_i \in \mathbb{R}^K$ *such that* $(z_i, x_{i1}, \ldots, x_{iS}) \in \mathcal{B}_i$*;*

(iii) $\sum_{i=1}^{I} z_i^* = 0$ *and* $\sum_{i=1}^{I} x_{is}^* = \sum_{i=1}^{I} \omega_{is}$ *for all* $s \in S$*.*

The first condition says that consumer i's plan must be in her budget set. The second condition says that the plan must maximize her preference among all consumption plans that are possible. The third condition says that in all asset markets in period 0 and in all spot markets in all states in period 1 supply must equal demand. Note that in the market clearing condition for period 1 the physical good purchases or claims that result from asset purchases in period 0 don't appear. This is because for all assets in period 0 the sum of purchases and sales of that asset must equal zero. Thus, in period 1 in any state the claims to commodity 1 that result from transactions in some particular asset in period 0 exactly equal the obligations to deliver physical good 1 that result from transactions in that particular asset in period 0.

To understand Radner equilibria, we shall compare them to Arrow-Debreu equilibria, that is, to equilibria of the model in which in period 0 every physical good can be traded contingent on every period 1 state s. Before we can get to our main result, though, we prove some intermediate results that are of interest in their own right.

Proposition: *If for every agent i preferences $\succsim_i$ are strictly monotone, then in a Radner equilibrium the spot market prices p_{1s} are strictly positive for every s. Moreover, the price vector q^* is* arbitrage free, *that is, there is no $z \in \mathbb{R}^K$ such that:*

$$q^* \cdot z \leq 0, R \cdot z \geq 0, \text{ and at least one of these inequalities does not hold as an equality.}$$

The proof below will clarify why the condition listed in the proposition is referred to as "arbitrage freeness."

Proof: Suppose any of the spot market prices p_{1s} were not strictly positive. Then in that state every consumer i would have infinite demand for good 1. This cannot be an equilibrium.

Now suppose in a Radner equilibrium a vector z as described in the proposition exists. Suppose first the second inequality did not hold as an equality. Then every agent i can modify their period 0 portfolio and add the vector z to it. $R \cdot z \geq 0$ and $R \cdot z \neq 0$ means that there is at least one state $s \in S$ in which this strictly increases agent i's state s budget, and therefore agent i can increase the right hand side of his budget constraint. But, because strictly increasing preferences imply that the budget constraints for all states s hold in equilibrium as equalities, this means that she can increase her consumption of good 1. Thus, her original portfolio and consumption plans cannot have been optimal.

Suppose the first inequality did not hold as an equality. Then every agent i can modify their period 0 portfolio and add the vector z to it. There is no state in the second period in which this would reduce agent i's budget. However, $q^* \cdot z < 0$ means that in the first period the budget constraint is now satisfied as a strict inequality. So, further asset purchases are possible. But are they strictly advantageous? This is assured by our earlier assumption that the asset structure is "attractive," i.e. there is at least one portfolio that yields in at least one state a positive quantity, and in no state a negative quantity. Agent i can buy a small proportion of this portfolio. This is advantageous by the same argument as in the previous paragraph, and it is affordable, because the first period budget constraint, after the first modification of the portfolio, is satisfied as a strict inequality. Again, the original portfolio and consumption plans cannot have been optimal.

Q.E.D.

To prove the result use Stiemke's alternative.

The condition in the Proposition can be re-written in a straightforward way using one of the many versions of Farkas' Lemma. We omit the proof for the next result.

Proposition: A price vector q^* is arbitrage free if and only if there is a vector $\mu \in \mathbb{R}^S_{++}$ such that:

$$q^* = \mu \cdot R.$$

The vector μ has a simple interpretation. μ assigns to good 1 for every state s a price that an agent has to pay in period 0 for acquiring one unit of good 1 to be delivered in state s of period 1. The asset prices are then constructed by adding up for each asset the period 1 prices of the quantities of good 1 delivered or promised for period 1.

We now turn to the comparison between Radner and Arrow-Debreu equilibria. We first formally define Arrow-Debreu equilibria. These are equilibria of a market system in which in period 0 there are contingent markets for every good for every state. Trading only takes place in period 0. The following definition is the standard definition, adapted to this market system.

Definition: *An* Arrow-Debreu equilibrium *is a list of prices* $(p_1^*, \ldots, p_S^*) \in \mathbb{R}^{SL}$ *and, for each agent* i*, quantities* $(x_{i1}^*, \ldots, x_{iS}^*) \in \mathbb{R}^{SL}$ *such that the following three conditions are satisfied. To formulate the conditions we define agent* i*'s budget set as:*

$$\mathcal{B}_i^{AD} = \left\{ (x_{i1}, \ldots, x_{iS}) \in \mathbb{R}^{LS}_+ | \sum_{s=1}^{S} p_s^* \cdot x_{is} \leq \sum_{s=1}^{S} p_s^* \cdot \omega_{is} \right\}$$

(i) for all $i : (x_{i1}^*, \ldots, x_{iS}^*) \in \mathcal{B}_i$*;*

(ii) for all $i : (x_{i1}^*, \ldots, x_{iS}^*) \succsim_i (x_{i1}, \ldots, x_{iS})$ *for all* $(x_{i1}, \ldots, x_{iS}) \in \mathcal{B}_i^{AD}$*;*

(iii) $\sum_{i=1}^{I} x_{is}^* = \sum_{i=1}^{I} \omega_{is}$ *for all* $s \in S$*.*

When comparing Radner and Arrow-Debreu equilibria what really matters are only the state contingent consumption quantities for every agent. Those quantities are what consumers care about. Therefore, we focus on these quantities. We call a vector of quantities $(x_1^*, \ldots, x_I^*)$ where for every i: $x_i^* = (x_{i1}^*, \ldots, x_{iS}^*) \in \mathbb{R}^{LS}$

a "Radner equilibrium allocation" if there is a Radner equilibrium of which it is one component. Denote the set of all Radner equilibrium allocations by $\mathcal{R}^*$. We call a vector of quantities $(x_1^*, \ldots, x_I^*)$ where for every i: $x_i^* = (x_{i1}^*, \ldots, x_{iS}^*) \in \mathbb{R}^{LS}$ an "Arrow-Debreu equilibrium allocation" if there is an Arrow-Debreu equilibrium of which it is one component. Denote the set of all Arrow-Debreu equilibrium allocations by $\mathcal{AD}^*$. Our focus in this section is on the comparison between the sets $\mathcal{R}^*$ and $\mathcal{AD}^*$. The main result is:

Proposition: Suppose all agents' preferences $\succsim_i$ are strictly monotonically increasing. Also, suppose that the rank of the matrix R equals S, the number of states. Then

$$\mathcal{R}^* = \mathcal{AD}^*.$$

If the rank of the matrix R is equal to S, then the lack of tradable contracts is thus innocuous. We therefore refer to this case still as a case of "complete" markets. When markets are complete, the theory that we have developed in the previous sections can be applied without change to Radner equilibria. If the rank of the matrix R is smaller than S, however, the situation is very different, and we arrive at a model that cannot be covered in this course.

There is a small gap in our argument here. To extend the second welfare theorem to this setting, we would need an equivalence result for Walrasian equilibria with transfers in the Arrow-Debreu and the Radner model. Such a result, postulating budget balanced transfers in period 0, can be obtained by adapting the arguments of the proof.

Proof: *Step 1:* We begin by showing that every Radner equilibrium allocation is also an Arrow-Debreu equilibrium allocation. Consider a given Radner equilibrium allocation. Let q^* and $(p_1^*, \ldots, p_S^*)$ be the corresponding equilibrium prices. Note that the prices q^* have to be arbitrage-free because we have assumed that consumers' preferences are strictly monotonically increasing. Let μ^* be the vector of period 0 prices of good 1 the existence of which is implied by our characterization of arbitrage-free equilibrium asset prices. To show that the given Radner equilibrium allocation is also an Arrow-Debreu equilibrium allocation. We define the period 0 vector of Arrow-Debreu prices of all good to be delivered in period 1 in state s to be:

$$\frac{\mu_s^*}{p_{s1}^*} p_s^*.$$

We define agent i's demand for good ℓ in state s to be $x_{is\ell}^*$, i.e., the same as it is in the Radner equilibrium allocation. By the second part of condition (iii) in the definition of Radner equilibrium, all period 0 markets then clear.

Next we need to show that agent i's consumption plan $(x^*_{i1},\dots,x^*_{iS})$ is in agent i's Arrow-Debreu budget set. Because $(x^*_{i1},\dots,x^*_{iS})$ is part of a Radner equilibrium, there is a z^*_i such that

$$R\cdot z^*_i=\left(\frac{p^*_1\cdot(x^*_{i1}-\omega_{i1})}{p^*_{11}},\dots,\frac{p^*_S\cdot(x^*_{iS}-\omega_{iS})}{p^*_{S1}}\right)$$

and such that

$$q^*_i\cdot z^*_i\leq 0.$$

Using the implicit prices that the consumer pays in period 0 for goods to be delivered in state s in in period 1, we can re-write the last inequality as:

$$\mu^*\cdot R\cdot z^*_i\leq 0,$$

which is equivalent to:

$$\mu^*\cdot\left(\frac{p^*_1\cdot(x^*_{i1}-\omega_{i1})}{p^*_{11}},\dots,\frac{p^*_S\cdot(x^*_{iS}-\omega_{iS})}{p^*_{S1}}\right)\leq 0\Leftrightarrow$$
$$\sum_{s=1}^{S}\mu^*_s\frac{p^*_s\cdot(x^*_{is}-\omega_{is})}{p^*_{s1}}\leq 0,$$

which is the Arrow-Debreu budget constraint, given our definition of the period 0 Arrow-Debreu prices.

What remains for us to check is that every agent i maximizes her preference in her budget set. For this it is sufficient to show that for every $(x_{i1},\dots,x_{iS})$ in the Arrow-Debreu budget set there is a z_i such that $(z_i,x_{i1},\dots,x_{iS})$ is in the Radner budget set. This is sufficient because we know that $(x_{i1},\dots,x_{iS})$ maximizes $\succsim_i$ in the Radner equilibrium budget set.

A consumption plan is in the Radner equilibrium budget set if and only there is a z_i such that

$$R\cdot z_i=\left(\frac{p^*_1\cdot(x_{i1}-\omega_{i1})}{p^*_{11}},\dots,\frac{p^*_S\cdot(x_{iS}-\omega_{iS})}{p^*_{S1}}\right)$$

and such that

$$q^*\cdot z_i\leq 0.$$

Using the implicit prices that the consumer pays in period 0 for good 1 to be delivered in state s in in period 1, we can re-write the last inequality as:

$$\mu^*\cdot R\cdot z_i\leq 0.$$

We thus have to show that every consumption plan in the Arrow-Debreu budget set satisfies the above conditions. The required z_i that covers the period 1 requirements always exists because matrix R has rank S. Therefore, it remains to be shown:

$$\mu^* \cdot \left(\frac{p_1^* \cdot (x_{i1} - \omega_{i1})}{p_{11}^*}, \ldots, \frac{p_S^* \cdot (x_{iS} - \omega_{iS})}{p_{S1}^*} \right) \leq 0.$$

Writing out the vector product, we obtain:

$$\sum_{s=1}^{S} \mu_s^* \frac{p_s^* \cdot (x_{is} - \omega_{is})}{p_{s1}^*} \leq 0.$$

which is true because we are considering a consumption plan that is in the Arrow-Debreu budget set. Because, by choosing $(x_{i1}^*, \ldots, x_{iS}^*)$, consumer i maximizes $\succsim_i$ in the Radner budget set, she does so also in the Arrow-Debreu budget set. This completes the first half of our proof.

Step 2: Now suppose an Arrow-Debreu equilibrium allocation were given. We want to show that it is also a Radner equilibrium allocation. We thus construct a Radner equilibrium. We define the period 1 spot prices of all goods to be the prices of the corresponding forward contracts in the Arrow-Debreu equilibrium in period 0. Asset prices in period 0 are:

$$q^* = p_1^* \cdot R.$$

Any agent i's demand for goods in period 1 is the same as in the Arrow-Debreu equilibrium.To define agent i's demand for assets in period 0, we select from the matrix R a subset of S columns such that these S columns of R form an invertible square matrix. This is always possible because R has rank S. We let $\tilde{R}$ denote this sub matrix of R, and we let $\tilde{R}^{-1}$ denote its inverse. Agent i's demand for the S assets that correspond to the columns of $\tilde{R}$ is:

$$z_i^* = \tilde{R}^{-1} \cdot \left(\frac{p_1^* \cdot (x_{i1}^* - \omega_{i1})}{p_{11}^*}, \ldots, \frac{p_S^* \cdot (x_{iS}^* - \omega_{iS})}{p_{S1}^*} \right)$$

For all other assets, agent i's demand is zero.

To show that agent i satisfies the budget constraints of the Radner model, we note first that the right hand side vector in large brackets in the previous equation indicates the quantities of good 1 that consumer i needs in each period 1 state to balance her budget in that state. Therefore, z_i^* is an asset purchase that allows agent i to obtain those quantities. It remains to be verified that z_i^* is affordable in period 0. This means:

$$
\begin{aligned}
q^* \cdot z_i^* &\le 0 \Leftrightarrow \\
p_1^* \cdot \tilde{R} \cdot \tilde{R}^{-1} \cdot \left(\frac{p_1^* \cdot (x_{i1}^* - \omega_{i1})}{p_{11}^*}, \dots, \frac{p_S^* \cdot (x_{iS}^* - \omega_{iS})}{p_{S1}^*} \right) &\le 0 \Leftrightarrow \\
p_1^* \cdot \left(\frac{p_1^* \cdot (x_{i1}^* - \omega_{i1})}{p_{11}^*}, \dots, \frac{p_S^* \cdot (x_{iS}^* - \omega_{iS})}{p_{S1}^*} \right) &\le 0 \Leftrightarrow \\
\sum_{s=1}^{S} p_{s1}^* \frac{p_s^* \cdot (x_{is}^* - \omega_{is})}{p_{s1}^*} &\le 0 \Leftrightarrow \\
\sum_{s=1}^{S} p_s^* \cdot (x_{is}^* - \omega_{is}) &\le 0
\end{aligned}
$$

This is true because the consumption plan satisfies the budget inequality in the Arrow-Debreu model.

To show that agent i's asset purchases and consumption plans maximize $\succsim_i$ in the Radner budget set it is sufficient to show that for every $(z_i, x_{i1}, \dots, x_{iS})$ that is in the Radner budget set, $(x_{i1}, \dots, x_{iS})$ is in the Arrow-Debreu budget set. This is sufficient because we know that $(x_{i1}, \dots, x_{iS})$ maximizes $\succsim_i$ in the Arrow-Debreu budget set. Suppose $(z_i, x_{i1}, \dots, x_{iS})$ is in the Radner budget set. It is without loss of generality to assume that the asset purchases are restricted to the S assets that correspond to the columns of $\tilde{R}$. All purchases have to cover the same quantities of good 1 in the period 1 states, and by the definition of asset prices incur the same expenses. Thus, we can set:

$$
z_i = \tilde{R}^{-1} \cdot \left(\frac{p_1^* \cdot (x_{i1} - \omega_{i1})}{p_{11}^*}, \dots, \frac{p_S^* \cdot (x_{iS} - \omega_{iS})}{p_{S1}^*} \right)
$$

Because $(z_i, x_{i1}, \dots, x_{iS})$ is in the Radner budget set, the plan z_i satisfies the budget inequality:

$$
q^* z_i \le 0.
$$

By the argument that we used in the previous paragraph, this is equivalent to:

$$
\sum_{s=1}^{S} p_s^* \cdot (x_{is} - \omega_{is}) \le 0,
$$

which means that the consumption plan is contained in the Arrow-Debreu budget set, which is what we wanted to show.

It remains to verify that for every asset demand equals supply in period 0. There is only demand or supply for those assets that correspond to columns in $\tilde{R}$. We therefore proceed as if these were the only assets. Recall that agent i's asset demand is:

$$z_i^* = \tilde{R}^{-1} \cdot \left(\frac{p_1^* \cdot (x_{i1}^* - \omega_{i1})}{p_{11}^*}, \ldots, \frac{p_S^* \cdot (x_{iS}^* - \omega_{iS})}{p_{S1}^*} \right)$$

Therefore,

$$\begin{aligned} \tilde{R} \cdot z_i^* &= \tilde{R} \cdot \tilde{R}^{-1} \cdot \left(\frac{p_1^* \cdot (x_{i1}^* - \omega_{i1})}{p_{11}^*}, \ldots, \frac{p_S^* \cdot (x_{iS}^* - \omega_{iS})}{p_{S1}^*} \right) \\ &= \left(\frac{p_1^* \cdot (x_{i1} - \omega_{i1})}{p_{11}^*}, \ldots, \frac{p_S^* \cdot (x_{iS}^* - \omega_{iS})}{p_{S1}^*} \right) \end{aligned}$$

Taking sums over i, we get:

$$\begin{aligned} \sum_{i=1}^{I} \tilde{R} \cdot z_i^* &= \sum_{i=1}^{I} \left(\frac{p_1^* \cdot (x_{i1}^* - \omega_{i1})}{p_{11}^*}, \ldots, \frac{p_S^* \cdot (x_{iS}^* - \omega_{iS})}{p_{S1}^*} \right) \\ &= \left(\frac{p_1^* \cdot \sum_{i=1}^{I} (x_{i1}^* - \omega_{i1})}{p_{11}^*}, \ldots, \frac{p_S^* \cdot \sum_{i=1}^{I} (x_{iS}^* - \omega_{iS})}{p_{S1}^*} \right) \end{aligned}$$

By the definition of Radner equilibrium we have:

$$\sum_{i=1}^{I} (x_{is}^* - \omega_{is}) = 0$$

for every state s. Therefore, the right hand side is the zero vector, and we obtain:

$$\tilde{R} \cdot \sum_{i=1}^{I} z_i^* = (0, \ldots, 0)$$

But because we chose $\tilde{R}$ to be invertible, there is only one vector that we can multiply R with and obtain the zero vector, namely the zero vector. Therefore, we must have:

$$\sum_{i=1}^{l} z_i^* = 0$$

which is what we wanted to prove.

Q.E.D.