DEHN FILLING IN SEMISIMPLE LIE GROUPS

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Abstract. We generalize one part of Thurston's hyperbolic Dehn filling theorem to abitrary-rank semisimple Lie groups by showing that certain deformations of extended geometrically finite subgroups of a semisimple Lie group are still extended geometrically finite. As a special case, our theorem gives a criterion which guarantees that a deformation of a relatively Anosov subgroup is (non-relatively) Anosov, and also ensures that limit sets vary continuously. Our result also applies to several higher-rank examples in convex projective geometry which are outside of the relatively Anosov setting.

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1. Introduction

Thurston's celebrated hyperbolic Dehn surgery theorem is a foundational result in the theory of hyperbolic manifolds. Given a single noncompact finite-volume hyperbolic 3-manifold M_∞, the theorem provides a construction for a countable family M_n of pairwise distinct isometry classes of hyperbolic 3-manifolds, each of which is a topological Dehn filling of M_∞; moreover the hyperbolic structures on M_n converge to the hyperbolic structure on M_∞ in a geometric sense.

Thurston's theorem is striking both because it provides an abundance of closed hyperbolic 3-manifolds and because it closely connects the topological study of hyperbolic manifolds to the analysis of character varieties of their fundamental groups. It is natural to ask if similar phenomena can occur for other types of geometric manifolds, *not* locally modeled on $\mathbb{H}^3$, and in fact examples of "geometric Dehn filling" have since been observed in several different contexts. See e.g. [Sch89; Sch07; Aco16; Aco19; MR18; CLM20; BDLM].

In this article, our goal is to provide a general framework for studying and constructing analogs and generalizations of hyperbolic Dehn filling in the context of "geometrically finite" discrete subgroups of a semisimple Lie group G with arbitrary rank. This framework gives a means to study geometric convergence of manifolds and orbifolds locally modeled on homogeneous G-spaces; it describes several instances of "geometric Dehn filling" previously studied by other authors, as well as some new examples to be explored in forthcoming work.

Our main result gives general conditions which guarantee that, if Γ is a relatively hyperbolic group, G is a semisimple Lie group, and $\rho : \Gamma \to G$ is a representation which is "geometrically finite" in a precise sense, then a sequence of representations ρ_n converging to ρ also converges *strongly* to ρ, and gives a countable family of representations which are also "geometrically finite." To make all of this concrete, we work in the context of (relativized) *Anosov representations.*

1.1. **Anosov representations and extended geometrical finiteness.** Originally introduced in a seminal paper of Labourie [Lab06], and subsequently studied by many other authors (see e.g. [GW12; GGKW17; KLP17; BPS19]), Anosov representations provide a natural arbitrary-rank generalization of convex cocompact subgroups of rank-one Lie groups. There are many equivalent definitions of an Anosov representation; the most relevant one for this paper is in terms of a *boundary embedding*, i.e. an equivariant embedding from the Gromov boundary of a hyperbolic group into a *flag manifold* associated to a semisimple Lie group.

Several notions of *relative* Anosov representation have recently been introduced to study generalizations of *geometrical finiteness* in higher rank [KL23; Zhu21; CZZ22; ZZ22]. In this paper, we mainly work with the most general definition available: that of an *extended geometrically finite* (or EGF) representation, previously introduced by the author in [Wei22]. There are two main advantages to doing so. First, several of the previously-studied examples of discrete groups undergoing "Dehn filling" phenomena are simply not covered by alternative definitions. Second, also unlike other definitions, EGF representations are well-suited to an understanding of the transition betwen two relatively hyperbolic subgroups Γ, Γ' of some semisimple Lie group in the situation where Γ and Γ' have qualitatively different peripheral behavior. This is precisely what is needed to describe both Dehn filling in $\mathrm{PSL}(2, \mathbb{C})$ and many of its analogs in other Lie groups.

Although our results apply in the broad context of EGF representations, our theorems are new even when we consider special cases corresponding to more specific notions of relative Anosov representation (see Theorem 1.8). In fact, we obtain useful results even in the case of geometrically finite groups in rank one. See Theorem 1.10 and Theorem 1.11 below.

1.2. **The general Dehn filling theorem.** To get an idea of our main theorem, first note that the proof of Thurston's hyperbolic Dehn filling theorem can essentially be broken down into two parts:

1. Prove that if $\Gamma < \mathrm{PSL}(2, \mathbb{C})$ is the holonomy group of a finite-volume hyperbolic 3-manifold M with n cusps, then the character variety $X(\Gamma, \mathrm{PSL}(2, \mathbb{C}))$ is locally an n-dimensional complex manifold in a neighborhood of the inclusion $\Gamma \hookrightarrow \mathrm{PSL}(2, \mathbb{C})$. This manifold is identified with a neighborhood U of $(\infty, \ldots, \infty)$ in $(\mathbb{CP}^1)^n$, where $\mathbb{CP}^1$ is identified with $\mathbb{C} \cup \{\infty\}$.
2. Show that (possibly after shrinking U) every point $(z_1, \ldots, z_n) \in U$ such that each z_j lies in $\mathbb{Z}[i] \cup \{\infty\}$ corresponds to a representation of Γ whose image is a lattice in $\mathrm{PSL}(2, \mathbb{C})$.

The Dehn filling theorem in this paper can be thought of as giving a general tool for carrying out the analog of step 2. above, when Γ is an arbitrary relatively hyperbolic group included into an arbitrary semisimple Lie group in a "geometrically finite" way (to be precise, when $\Gamma \hookrightarrow G$ is an EGF representation). We note that, although our proof does not follow anything resembling Thurston's original approach, our result can still be used to recover this part of the original hyperbolic Dehn filling theorem (see Example 1.7 below).

Remark 1.1. It is difficult to envision a general theorem giving an analog of step 1. above for an arbitrary EGF representation $\Gamma \to G$. Analysis of the character variety $X(\Gamma, G)$ can already be challenging for specific examples of a relatively hyperbolic group Γ and semisimple Lie group G, even in a neighborhood of an explicit EGF representation $\Gamma \to G$.

Our main theorem is stated using the language of EGF representations and *extended Dehn filling spaces*. We give all of the details in Section 3, but the rough idea of these spaces is the following. If Γ is a relatively hyperbolic group, relative to a collection $\mathcal{P}$ of peripheral subgroups, we can equip the set of representations $\Gamma \to G$ with the topology of *relative geometric convergence*; this space is denoted $\mathrm{Hom}_{\mathrm{geom}}(\Gamma, G; \mathcal{P})$. A sequence of representations ρ_n converges in $\mathrm{Hom}_{\mathrm{geom}}(\Gamma, G; \mathcal{P})$ to some representation ρ if and only if ρ_n converges to ρ in the compact-open topology, *and* if, for each $P \in \mathcal{P}$, the restrictions $\rho_n|_P$ converge *strongly* or *geometrically* to the restriction $\rho|_P$ (see Section 3.4). In particular, if ρ is discrete and faithful, then for large n, the restrictions $\rho_n|_P$ are discrete (but not necessarily faithful) for all $P \in \mathcal{P}$.

Each element $\rho \in \mathrm{Hom}_{\mathrm{geom}}(\Gamma, G; \mathcal{P})$ also determines an action of each $P \in \mathcal{P}$ on the various flag manifolds associated to G. An *extended Dehn filling space* is a certain subset of $\mathrm{Hom}_{\mathrm{geom}}(\Gamma, G; \mathcal{P})$ where the images of the orbit maps associated to these peripheral actions deform in a "uniformly continuous" way.

Remark 1.2. The precise definition of an extended Dehn filling space (given in Section 3.6) is rather technical, but in special cases there can be a more concrete description. In fact, in some situations, it turns out that the whole space $\mathrm{Hom}_{\mathrm{geom}}(\Gamma, G; \mathcal{P})$ already satisfies the definition (see e.g. Theorem 1.10).

For the formal statement of the main theorem, we need a bit more notation.

Definition 1.3. Let $(\Gamma, \mathcal{P})$ be a relatively hyperbolic pair, and let $\sigma : \Gamma \to G$ be a homomorphism. We let Γ^σ denote the quotient group Γ / N^σ, where $N^\sigma = \langle\langle \bigcup_{P \in \mathcal{P}} \ker(\sigma|_P) \rangle\rangle$, and we let $\mathcal{P}^\sigma$ denote the collection of groups $\{P / \ker(\sigma|_P)\}_{P \in \mathcal{P}}$.

Our main theorem is then as follows:

Theorem 1.4. *Let $(\Gamma, \mathcal{P})$ be a relatively hyperbolic pair, let Q be a symmetric parabolic subgroup of a semisimple Lie group G, let $\rho : \Gamma \to G$ be a Q-EGF representation, and let $W \subseteq \mathrm{Hom}_{\mathrm{geom}}(\Gamma, G; \mathcal{P})$ be an extended Dehn filling space for ρ.*

Then there is a neighborhood O of ρ in $\mathrm{Hom}_{\mathrm{geom}}(\Gamma, G; \mathcal{P})$ such that, if $\sigma \in O \cap W$, then $(\Gamma^\sigma, \mathcal{P}^\sigma)$ is a relatively hyperbolic pair, and σ induces a Q-EGF representation $\Gamma^\sigma \to G$.

Remark 1.5.

(a) Our proof of Theorem 1.4 also gives us some control over the "limit sets" of the deformed EGF representations σ, as well as geometric convergence of σ as it approaches ρ in $\mathrm{Hom}_{\mathrm{geom}}(\Gamma, G; \mathcal{P})$. For the precise statements, see Propositions 5.1 and 5.2.

(b) It is not obvious from the definition that the inclusions $P \hookrightarrow \Gamma$ induce injective maps $P / \ker(\sigma|_P) \to \Gamma^\sigma$. This follows from the *relatively hyperbolic Dehn filling theorem* of Osin [Osi07] and Groves–Manning [GM08] (see Section 2.6).

(c) Theorem 1.4 can be viewed as a generalization of the main result in [Wei22], which also gives a criterion determining when a deformation of an EGF representation is still EGF (see also [MMW24] for an analogous result in a different context). Crucially, the deformed actions considered in both [Wei22] and [MMW24] have kernel no larger than the original action, which means they do not come from representations of nontrivial Dehn fillings of the original relatively hyperbolic group.

1.2.1. *Obtaining Anosov representations by filling.* By applying an Anosov relativization theorem (see [Wei22, Section 4]), we can use Theorem 1.4 to describe situations where a non-Anosov subgroup occurs as a limit of Anosov subgroups. This generalizes the way that the original hyperbolic Dehn filling theorem describes situations in which nonuniform lattices in $\mathrm{PSL}(2,\mathbb{C})$ occur as limits of uniform lattices. The precise statement is the following.

Corollary 1.6. *In the context of Theorem 1.4, if, for each $P \in \mathcal{P}$, the group $P/\ker(\sigma|_P)$ is hyperbolic and σ induces a Q-Anosov representation of $P/\ker(\sigma|_P)$, then Γ^σ is hyperbolic and σ induces a Q-Anosov representation of Γ^σ.*

Example 1.7 (Dehn filling in $\mathbb{H}^3$)**.** Consider the case where Γ is the fundamental group of a finite-volume hyperbolic 3-manifold M, and $\mathcal{P}$ is the collection of cusp subgroups. Then $(\Gamma, \mathcal{P})$ is relatively hyperbolic, and the holonomy representation $\rho : \Gamma \to \mathrm{PSL}(2,\mathbb{C})$ is geometrically finite (in particular, EGF). Below we explain how to apply Theorem 1.4 towards hyperbolic Dehn fillings of M.

It turns out that in this situation, the entire space $\mathrm{Hom}_{\mathrm{geom}}(\Gamma, \mathrm{PSL}(2,\mathbb{C}); \mathcal{P})$ is an extended Dehn filling space about ρ (see Theorem 1.10 below). For simplicity, assume that M has one cusp C. Then, to apply Theorem 1.4, one must check that, if $\rho_{p,q} : \Gamma \to \mathrm{PSL}(2,\mathbb{C})$ is a sequence of representations parameterized by Dehn filling slopes (p,q) tending to infinity, then the restrictions $\rho_{p,q}|_{\pi_1 C}$ converge *strongly* to the restriction $\rho|_{\pi_1 C}$. This essentially boils down to considering the behavior of the developing maps for the corresponding sequence of complex affine structures on a torus cross-section of C.

Once this has been verified, one can use Corollary 1.6 to see that all but finitely many $\rho_{p,q}$ descend to Anosov (i.e. convex cocompact) representations $\pi_1 M_{p,q} \to \mathrm{PSL}(2,\mathbb{C})$, where $M_{p,q}$ is the (p,q) Dehn filling of M. From here, to establish that $\rho_{p,q}(\Gamma)$ is actually a lattice (without circularly relying on the original Dehn filling theorem), one can for example employ [GMS19] to see that the limit set of $\rho_{p,q}(\Gamma)$ is eventually homeomorphic to a 2-sphere, and therefore equal to $\partial\mathbb{H}^3$. (An alternative approach might use an analog of [MMW24, Lem. 6.10].)

We emphasize that none of the steps above necessarily rely on anything like a triangulation of M, a well-behaved "thick-thin" decomposition of $\mathbb{H}^3/\Gamma$, or a nice fundamental domain for the Γ-action on $\mathbb{H}^3$. This points towards the applicability of Theorem 1.4 in settings where such tools might not be available (in particular, in higher rank).

1.3. **Relatively Anosov representations.** An important special case of Q-EGF representations are the relatively Q-Anosov representations originally defined by by Kapovich–Leeb [KL23] and Zhu [Zhu21] (see also [ZZ22]). Our main Theorem 1.4 already applies to relatively Anosov representations, but in this special case, we also get better control over the convergence $\rho_n \to \rho$ and over the convergence of the corresponding Q-limit sets.

Theorem 1.8. *Let $(\Gamma, \mathcal{P})$ be a relatively hyperbolic pair, and let $\rho : \Gamma \to G$ be a relatively Q-Anosov representation with Q-limit set Λ. Suppose that ρ_n converges to ρ in an extended Dehn filling space $W \subseteq \mathrm{Hom}_{\mathrm{geom}}(\Gamma, G; \mathcal{P})$. Then:*

(1) ρ_n converges strongly to ρ, and
(2) for sufficiently large n, ρ_n induces a Q-EGF representation of Γ^{ρ_n}.

If, in addition, $\rho_n(\Gamma)$ is Q-divergent (in particular, if it is relatively Q-Anosov) for every n, then the Q-limit sets Λ_n of ρ_n converge to Λ with respect to Hausdorff distance on G/Q.

Note that Corollary 1.6 applies in this context, meaning that the theorem shows how to obtain (non-relatively) Anosov representations as fillings of relatively Anosov representations.

Remark 1.9. It is typically not difficult to check the extra hypothesis in the last part of the theorem, since a result of Wang [Wan23] implies that it can often be reduced to a peripheral criterion (see Section 6.1). Also, as Q-EGF representations themselves have a (weaker) notion of "limit set," a version of this part of the theorem still holds even without the added assumptions on the Dehn fillings ρ_n, but we defer the statement until Section 6.

1.4. **The rank one case.** When $(\Gamma, \mathcal{P})$ is a relatively hyperbolic pair with every $P \in \mathcal{P}$ virtually nilpotent, then a representation ρ from Γ to some rank one Lie group G is EGF precisely when ρ has finite kernel and geometrically finite image (see [GW24]). We can assume in this case that the peripheral subgroups $\mathcal{P}$ are a maximal collection of pairwise non-conjugate *ρ-horospherical* subgroups (see Definition 7.1).

In this situation, the notion of an extended Dehn filling space in $\mathrm{Hom}_{\mathrm{geom}}(\Gamma, G; \mathcal{P})$ is conceptually much simpler.

Theorem 1.10. *Let G be a rank-one semisimple Lie group, let $\rho : \Gamma \to G$ be a geometrically finite representation, and let $\mathcal{P}$ be a maximal collection of pairwise-nonconjugate ρ-horospherical subgroups of Γ. Then $\mathrm{Hom}_{\mathrm{geom}}(\Gamma, G; \mathcal{P})$ is an extended Dehn filling space about ρ.*

By applying the result above, we obtain a general rank-one version of our main Dehn filling theorem. To state this, let $\mathrm{Hom}_{\mathrm{geom}}(\Gamma, G)$ denote the space of representations $\Gamma \to G$ equipped with the topology of geometric (or *strong*) convergence.

Theorem 1.11. *Suppose that G is rank-one and $\rho : \Gamma \to G$ is geometrically finite. Let $\mathcal{P}$ be a maximal collection of pairwise non-conjugate ρ-horospherical subgroups of Γ, and let ρ_n be a sequence of representations converging to ρ in the compact-open topology on $\mathrm{Hom}(\Gamma, G)$. Then the following are equivalent:*

(1) ρ_n converges to ρ strongly*, i.e. in the space $\mathrm{Hom}_{\mathrm{geom}}(\Gamma, G)$.*
(2) ρ_n converges to ρ relatively strongly*, i.e. in the space $\mathrm{Hom}_{\mathrm{geom}}(\Gamma, G; \mathcal{P})$.*
(3) The set $\{\rho\} \cup \{\rho_n\}_{n \in \mathbb{N}}$ is an extended Dehn filling space about ρ.

When this holds, then $\rho_n(\Gamma)$ is geometrically finite for all sufficiently large n, and the limit sets $\Lambda_n := \Lambda(\rho_n(\Gamma))$ converge to the limit set $\Lambda(\rho(\Gamma))$ with respect to Hausdorff distance.

As mentioned previously, this theorem applies directly to the case of Dehn filling in real hyperbolic 3-space.

Remark 1.12. When G is the isometry group of d-dimensional real hyperbolic space, then Theorem 1.11 is equivalent to an earlier result of McMullen [McM99]. We remark that McMullen's proof implicitly assumes that each group $\rho_n(\Gamma)$ is torsion-free, which avoids some technicalities. This is partially justified by using Selberg's lemma to replace Γ with a finite-index subgroup Γ' so that $\rho(\Gamma')$ is torsion-free. However, torsion cannot be entirely sidestepped this way: it is possible to construct examples where Theorem 1.11 applies, but each $\rho_n(\Gamma)$ contains torsion elements whose orders tend to infinity in n; in fact this is already possible when $G = \mathrm{PSL}(2, \mathbb{R})$. In such situations, $\rho_n(\Gamma')$ eventually has nontrivial torsion elements for any fixed finite-index $\Gamma' < \Gamma$. Even in these cases, however, our proof of Theorem 1.11 goes through with no modification needed.

Note also that if one assumes that ρ and all of the representations ρ_n are *faithful* (meaning they do *not* descend to nontrivial Dehn fillings of Γ), then Theorem 1.11 follows from the recent paper [GW24], which uses the special case of the main result mentioned in Remark 1.5(c).

1.5. **Examples.** Below we describe a few interesting situations where Theorem 1.4 applies. In all of these examples, we start with a representation ρ which is the holonomy of a manifold (or orbifold) O_∞ with a (G, X)-*structure*; then ρ is deformed to obtain a sequence of holonomy representations ρ_n for (G, X)-orbifolds O_n converging to O_∞, obtained by a form of Dehn filling on O_∞.

In several of the examples below, it is also possible to obtain a *topological* description of the Dehn filled orbifolds O_n, and indeed such a topological description is an important part of Thurston's original $\mathbb{H}^3$ Dehn filling theorem. As stated, our main theorem really only gives group-theoretic information about O_n, but it would be very interesting to explore what topological statements can be derived from our result in various specific cases.

1.5.1. *Spherical CR structures on 3-manifolds.* In [Sch07], Schwartz considered a form of Dehn filling for 3-manifolds with *spherical CR structures*, i.e. (G, X)-structures where $G = \mathrm{SU}(2, 1)$ and X is the 3-sphere forming the ideal boundary of the complex hyperbolic plane $\mathbb{H}^2_{\mathbb{C}}$. The main result in [Sch07] is a version of Theorem 1.11 for "horotube representations," which are a certain class of discrete representations $\rho : \Gamma \to \mathrm{SU}(2, 1)$. Schwartz additionally analyzed the topology of the quotients $\Omega/\rho(\Gamma)$, where Ω is the domain of discontinuity in $\partial\mathbb{H}^2_{\mathbb{C}}$ for a horotube representation ρ. By considering perturbations of a particular geometrically finite horotube representation ρ, Schwartz constructed infinitely many examples of closed hyperbolic 3-manifolds admitting spherical CR structures.

Acosta [Aco16; Aco19] later proved a related spherical CR Dehn filling theorem, and applied it to perturbations of some other geometrically finite representations in $\mathrm{SU}(2, 1)$. The specific examples considered by both Acosta and Schwartz fall into the framework of Theorem 1.11 (for sufficiently small perturbations).

1.5.2. *Coxeter group actions on real projective space.* The projective model for real hyperbolic d-space realizes $\mathbb{H}^d$ as a convex ball in real projective space $\mathbb{P}(\mathbb{R}^{d+1})$, invariant under the subgroup $\mathrm{PO}(d, 1) < \mathrm{PGL}(d + 1, \mathbb{R})$. Thus any real hyperbolic orbifold O automatically has a *convex projective structure*; in particular O is a (G, X)-orbifold with $G = \mathrm{PGL}(d+1, \mathbb{R})$ and $X = \mathbb{P}(\mathbb{R}^{d+1})$. If the holonomy group of the orbifold is geometrically finite in $\mathrm{PO}(d, 1)$, then its inclusion into $\mathrm{PGL}(d + 1, \mathbb{R})$ is a relatively Anosov representation.

In [CLM20], Choi–Lee–Marquis used the theory of Coxeter polytopes to show that certain finite-volume hyperbolic orbifolds O_∞ in dimensions 4-7 can be realized as a limit of a sequence O_n of *projective Dehn fillings*: convex projective orbifolds whose orbifold fundamental groups $\pi_1 O_n$ are (abstractly) nontrivial Dehn fillings of $\pi_1 O_\infty$. In their examples, each holonomy representation $\pi_1 O_n \to \mathrm{PGL}(d + 1, \mathbb{R})$ is an EGF representation, and the sequence converges to the (relatively Anosov) holonomy of O.

One can verify that the hypotheses of Theorem 1.8 are satisfied by these examples. Notably, the holonomies of the O_n are *not* relatively Anosov, so this is one situation where the more general EGF framework is useful.

1.5.3. *Exotic convex projective structures on 3-manifolds.* Similarly to the above, upcoming work of Ballas–Danciger–Lee–Marquis (communicated to the author, to appear as [BDLM]) shows that certain finite-volume hyperbolic 3-manifolds can be Dehn filled to yield closed convex projective 3-manifolds, whose holonomy in $\mathrm{PGL}(d+1,\mathbb{R})$ converges to the holonomy of the original manifold. The closed 3-manifolds in question admit hyperbolic structures, but the projective structure yielded by the Dehn filling is *not* projectively equivalent to the hyperbolic one (unique by Mostow rigidity). For these examples, Corollary 1.6 also applies, and the Dehn-filled representations are actually Anosov.

1.5.4. *Exotic geometrically finite $\mathbb{H}^3_{\mathbb{C}}$-manifolds.* In work in progress with Jeff Danciger, we will show how Theorem 1.11 can be applied to obtain "exotic" Dehn fillings of 3-manifold groups in $\mathrm{PU}(3,1)$, yielding new examples of geometrically finite and convex cocompact group actions on complex hyperbolic 3-space. These examples are closely related to the ones in [BDLM], and we expect that many of them cannot be obtained as faithful deformations of previously known examples.

1.5.5. *Rigidity obstructions; other notions of Dehn filling.* When $G = \mathrm{PSL}(2,\mathbb{C})$, Thurston's Dehn filling theorem can be applied to construct examples of *lattices* in G, but it turns out that this case is rather special. Due to seminal rigidity theorems of Garland–Raghunathan [GR70] and Margulis [Mar84], whenever Γ is an irreducible lattice in a semisimple Lie group G not locally isomorphic to $\mathrm{PSL}(2,\mathbb{C})$ or $\mathrm{PSL}(2,\mathbb{R})$, then the inclusion $\Gamma \hookrightarrow G$ is locally rigid in $\mathrm{Hom}(\Gamma, G)$. So we should only hope to find applications of Theorem 1.4 to (G,X)-orbifolds in cases where either X is *not* a symmetric space, or else the initial orbifold O_∞ has infinite volume.

We mention that in [MR18], Martelli–Riolo showed that a form of hyperbolic Dehn surgery nevertheless exists for certain finite-volume hyperbolic 4-manifolds. Their examples are not covered by Theorem 1.4; they do not violate the rigidity theorems mentioned above because the finite-volume manifolds in question are first *drilled* before their hyperbolic structures are deformed (and subsequently surgered). Thus one deforms the holonomy representation for an *incomplete* hyperbolic 4-manifold, and the initial representation has infinite kernel while its deformations may not. This contrasts with the framework of Theorem 1.4, where the original EGF representation always has *finite* kernel.

1.6. **Proof strategy.** Our proof of Theorem 1.4 uses two different combinatorial tools. The first is the *cusped space* $X = X(\Gamma, \mathcal{P})$ associated to any relatively hyperbolic pair $(\Gamma, \mathcal{P})$. This is a locally finite graph originally defined by Groves–Manning [GM08], which allows us to study the interplay between the coarse geometry of a relatively hyperbolic group and its *group-theoretic Dehn fillings* (see Section 2.6). The second tool is a *relative quasi-geodesic automaton* $\mathcal{G}$ which gives a "relatively locally finite" encoding of the action of a relatively hyperbolic group on a flag manifold G/Q.

Given a Q-EGF representation $\rho : \Gamma \to G$, we use a construction from [Wei22] to build such a relative automaton $\mathcal{G}$, which has finitely many vertices in its underlying graph. Each vertex of the automaton is associated to an open subset of G/Q. The edges of the automaton correspond to rules describing when these open subsets must map inside of each other under translation by certain elements of Γ.

In [Wei22] (see also [MMW24]), it is shown that if one deforms ρ in $\mathrm{Hom}(\Gamma, G)$ in a way which respects all of these rules, then one can actually recover a form of "geometrical finiteness" from the combinatorial picture. We take a similar approach here, but the key difference is that we allow ourselves to deform ρ in a way which does *not* respect all of the conditions imposed by the automaton. Specifically, if the deformation descends to a representation of a group-theoretic Dehn filling Γ/N, then we are allowed to *ignore* conditions related to elements that are close to N. We construct "limit points" in G/Q of geodesics in the cusped space $X(\Gamma/N)$ for the Dehn-filled group by relating those geodesics to paths in the automaton $\mathcal{G}$ that only use valid rules. These limit points can be organized into a *boundary extension* for the deformed representation, and ultimately this shows that the deformation descends to an EGF representation of Γ/N.

1.7. **Outline of the paper.** We provide necessary background on relatively hyperbolic Dehn filling and EGF representations in Sections 2 and 3. For technical reasons, the definition of an EGF representation in this paper is slightly different than the one that appeared in the original version of [Wei22], but Section 3 also contains a result (proved in the appendix) resolving some of these issues.

In Section 4, we review some properties of the relative automaton mentioned above. Then we put everything together and prove our main theorem in Section 5. We discuss applications towards relatively Anosov representations in Section 6, and towards rank-one geometrical finiteness in Section 7.

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2. Relative hyperbolicity and group-theoretic Dehn filling

In this section we give some reminders about the coarse geometry of relatively hyperbolic groups and their *Dehn fillings*, group-theoretic analogs of Dehn fillings of hyperbolic 3-manifolds. Our main aim is to establish some basic relationships between the various metric spaces associated to a relatively hyperbolic group and its Dehn fillings. We refer to [Bow12], [GM08] for background; see also Section 2 of [Wei22] for a brief introduction to the theory of relatively hyperbolic groups, with an eye towards the convergence group viewpoint (relevant in this paper).

2.1. **Setup.** Throughout this section, we fix a relatively hyperbolic pair $(\Gamma, \mathcal{P})$. Here Γ is a finitely generated group, and $\mathcal{P}$ is the finite collection of peripheral subgroups. We will assume throughout this paper that all of our relatively hyperbolic groups are finitely generated; this ensures that every group in $\mathcal{P}$ is finitely generated also (see [Ger09]).

We do *not* assume that each $P \in \mathcal{P}$ is an infinite group. This differs from the convention in the earlier paper [Wei22], and introduces some minor extra technicalities in the definition of the Bowditch boundary of $(\Gamma, \mathcal{P})$ (Section 2.3) and of an EGF representation $\Gamma \to G$ (Section 3). Throughout, we let $\mathcal{P}_\infty$ denote the set of infinite subgroups in $\mathcal{P}$. It is always true that $(\Gamma, \mathcal{P}_\infty)$ is also a relatively hyperbolic pair whenever $(\Gamma, \mathcal{P})$ is.

The Bowditch boundary of $(\Gamma, \mathcal{P})$ (see Section 2.3) is denoted $\partial(\Gamma, \mathcal{P})$. Each $P \in \mathcal{P}$ is the stabilizer of a unique parabolic point $p \in \partial(\Gamma, \mathcal{P})$. We let $\Pi \subset \partial(\Gamma, \mathcal{P})$ denote the (finite) set of parabolic points fixed by the groups in $\mathcal{P}$, and let $\Pi_\infty \subset \partial(\Gamma, \mathcal{P}_\infty)$ denote the parabolic points fixed by groups in $\mathcal{P}_\infty$.

We also fix a finite generating set S for Γ which is *compatible* with $\mathcal{P}$, in the sense that $S \cap P$ generates P for each $P \in \mathcal{P}$. Cayley graphs of Γ and each $P \in \mathcal{P}$ will always be defined with respect to S or $S \cap P$.

2.2. **Metrics on relatively hyperbolic groups.** In general, a relatively hyperbolic group comes equipped with several natural (but inequivalent) metrics. In this paper we will frequently work with three of them:

(1) The *word metric*: we use $|\gamma|_\Gamma$ to denote the word-length of γ with respect to the (implicit) finite generating set S. Then the distance $d_\Gamma(h, g)$ is equal to $|h^{-1}g|_\Gamma$; equivalently, d_Γ is the path metric on $\mathrm{Cay}(\Gamma) = \mathrm{Cay}(\Gamma, S)$.
(2) The "electrified" or *coned-off* metric: for this metric, we consider the *relative Cayley graph* $\widehat{\mathrm{Cay}}(\Gamma; \mathcal{P})$, obtained by adding one vertex to $\mathrm{Cay}(\Gamma)$ for each coset of each group in $\mathcal{P}$, and then adding an edge from each element of the coset to that vertex. The metric $d_{\hat{\Gamma}}$ is the path metric on $\widehat{\mathrm{Cay}}(\Gamma; \mathcal{P})$ (restricting to a metric on Γ), and the *relative length* $|\gamma|_{\hat{\Gamma}}$ denotes the $d_{\hat{\Gamma}}$-distance from γ to id.
(3) The *cusped* metric: this metric, denoted d_X, is obtained by gluing *combinatorial horoballs* to cosets of parabolic subgroups in $\mathrm{Cay}(\Gamma)$, rather than coning them off. We describe the construction (originally due to Groves–Manning [GM08]) in more detail below, since we will need to work closely with it at several points in the paper.

Definition 2.1 (Combinatorial horoballs; see [GM08, Def. 3.1])**.** Let Y be a connected graph. The *combinatorial horoball* over Y, denoted $\mathcal{H}(Y)$, is a graph defined as follows:

- The vertex set of $\mathcal{H}(Y)$ is the set $Y^{(0)} \times (\{0\} \cup \mathbb{N})$, where $Y^{(0)}$ is the set of vertices of Y.
- There are two types of edges in $\mathcal{H}(Y)$:
  - For any $k \geq 0$ and any vertices u, v of Y such that $0 < d_Y(u, v) \leq 2^k$, there is an edge joining (u, k) to (v, k). Such edges are called *horizontal*.
  - For any $k \geq 0$ and any vertex $u \in Y$, there is an edge joining (u, k) to $(u, k+1)$. Such edges are called *vertical*.

Note that there is always an embedding of graphs $Y \hookrightarrow \mathcal{H}(Y)$ induced by the mapping of vertices $y \mapsto (y, 0)$. Due to [GM08], there is a uniform $\delta > 0$ so that any combinatorial horoball $\mathcal{H}(Y)$ is a δ-hyperbolic metric space (independent of Y). This hyperbolic metric space always has a unique point in its ideal boundary, corresponding to any ray in $\mathcal{H}(Y)$ consisting entirely of vertical edges.

The *cusped space* $X(\Gamma, \mathcal{P})$ for the relatively hyperbolic pair $(\Gamma, \mathcal{P})$ is the space obtained by the following procedure: for each coset gP of each $P \in \mathcal{P}$, identify gP with P, and glue a copy of the horoball $\mathcal{H}(P) := \mathcal{H}(\mathrm{Cay}(P, S \cap P))$ to $gP \subset \mathrm{Cay}(\Gamma, S)$ along the inclusion $\mathrm{Cay}(P, S \cap P) \hookrightarrow \mathcal{H}(\mathrm{Cay}(P, S \cap P))$.

Observe that Γ acts isometrically and properly discontinuously on $X(\Gamma, \mathcal{P})$. By [GM08, Thm. 3.25], there exists some $\delta > 0$ so that $X(\Gamma, \mathcal{P})$ is δ-hyperbolic, and in fact this property characterizes relative hyperbolicity of $(\Gamma, \mathcal{P})$.

The *cusped metric*, denoted d_X, is the induced path metric on $X(\Gamma, \mathcal{P})$. This restricts to a metric on $\mathrm{Cay}(\Gamma)$ via the inclusion $\mathrm{Cay}(\Gamma) \hookrightarrow X(\Gamma, \mathcal{P})$. We use $|\gamma|_X$ to denote $d_X(\mathrm{id}, \gamma)$.

2.3. **The Bowditch boundary.** The *Bowditch boundary* $\partial(\Gamma, \mathcal{P})$ of the relatively hyperbolic pair $(\Gamma, \mathcal{P})$ may be defined as the Gromov boundary of the the δ-hyperbolic metric space $X = X(\Gamma, \mathcal{P})$. The isometric action of Γ on X induces a homeomorphic action of Γ on $\partial(\Gamma, \mathcal{P})$. Since we assume that Γ is finitely generated, $\partial(\Gamma, \mathcal{P})$ is compact and metrizable. We will often work with a fixed metric d_∂ on $\partial(\Gamma, \mathcal{P})$. This can be taken to be a visual metric, but none of our arguments are sensitive to the choice. By adjoining $\partial(\Gamma, \mathcal{P}) = \partial X$ to X, we obtain the *compactified cusped space* $\overline{X} = \overline{X}(\Gamma, \mathcal{P})$. The *Bowditch compactification* $\overline{\Gamma}$ of Γ is the closure of Γ in $\overline{X}$.

If all of the groups in $\mathcal{P}$ are infinite, then every point in ∂X is an accumulation point of Γ, meaning that the Bowditch compactification can be identified (as a Γ-set) with the disjoint union $\Gamma \sqcup \partial(\Gamma, \mathcal{P})$. This is no longer true if some group $P \in \mathcal{P}$ is finite. In this case, the combinatorial horoball $\mathcal{H}(P)$ over P is quasi-isometric to a ray, the single point in the ideal boundary of $\mathcal{H}(P) \subset X$ is identified with an isolated point in ∂X, and this point is not an accumulation point of any sequence in Γ.

The intersection $\overline{\Gamma} \cap \partial X$ is canonically identified with the Bowditch boundary $\partial(\Gamma, \mathcal{P}_\infty)$, and it is always compact and Γ-invariant. Whenever $(\Gamma, \mathcal{P})$ is non-elementary (meaning that $\partial(\Gamma, \mathcal{P}_\infty)$ contains at least three points), then $\partial(\Gamma, \mathcal{P}_\infty)$ is precisely the set of *non-isolated* points in ∂X, and the Γ-action on $\partial(\Gamma, \mathcal{P}_\infty)$ is minimal.

2.4. **Convergence actions.** In this paper, we link relative hyperbolicity to geometrical finiteness through the theory of *convergence group actions*. A group Γ *acts as a (discrete) convergence group* on a compact metrizable space M if it acts properly discontinuously on the space of triples of pairwise distinct points in M. An equivalent characterization, due to Bowditch [Bow99], is the following: for every sequence γ_n of pairwise distinct elements in Γ, after extracting a subsequence, there are points $a, b \in M$ such that γ_n converges to the constant map b on $M \setminus \{a\}$, uniformly on compact subsets of $M \setminus \{a\}$. If Y is a proper hyperbolic metric space, then any discrete group of isometries acts on both ∂Y and $\overline{Y}$ as a discrete convergence group [Tuk94].

When Γ acts on a general space M as a convergence group, a point $z \in M$ is a *conical limit point* if there exists some sequence γ_n and *distinct* points $a, b \in M$ so that $\gamma_n^{-1} z \to a$ and $\gamma_n^{-1} y \to b$ for every $y \in M$ distinct from z. A point $p \in M$ is *parabolic* if it has infinite stabilizer in Γ, and every infinite-order element in its stabilizer uniquely fixes p. A parabolic point is *bounded* if its stabilizer acts cocompactly on $M \setminus \{p\}$. A key fact we need throughout this paper is:

Theorem 2.2 ([Bow12])**.** *If $(\Gamma, \mathcal{P})$ is a relatively hyperbolic pair, then Γ acts on the Bowditch boundary $\partial(\Gamma, \mathcal{P})$ as a convergence group, and every point in $\partial(\Gamma, \mathcal{P})$ is either a conical limit point or a bounded parabolic point for the Γ-action. The set of stabilizers of parabolic points are precisely the conjugates of the groups in $\mathcal{P}$.*

A theorem of Yaman [Yam04] shows that the behavior in the theorem actually characterizes relative hyperbolicity of the pair $(\Gamma, \mathcal{P})$, but we will not use that fact in this paper.

2.5. **Geometry of cusped spaces.** Below, we recall a few more features of the cusped space $X = X(\Gamma, \mathcal{P})$, and prove some routine lemmas which allow us to use hyperbolicity of X to deduce properties of certain paths in the group Γ.

2.5.1. *Horoballs and regular geodesics.* The *depth* of a vertex in X is its distance to $\mathrm{Cay}(\Gamma) \subset X$; this extends affinely on edges in X to a function $D : X \to \mathbb{R}_{\geq 0}$, also called depth. For any $k \geq 0$, the intersection of $D^{-1}([k, \infty))$ with a combinatorial horoball $\mathcal{H} \subset X$ is a connected set called a *k-horoball*, or simply a *horoball*. Due to [GM08, Lem. 3.26], if X is δ-hyperbolic, then any k-horoball for $k \geq \delta$ is isometrically embedded in X. In particular, this implies that any horoball $\mathcal{H}$ contains a horoball $\mathcal{H}'$, isometrically embedded in X, such that the Hausdorff distance between $\mathcal{H}$ and $\mathcal{H}'$ is at most δ.

If $\mathcal{H}$ is a horoball, a segment in $\mathcal{H}$ is called *regular* if it consists of a (possibly degenerate) vertical segment, followed by a horizontal segment of length at most 3, followed by a (possibly degenerate) vertical segment. Lemma 3.10 in [GM08] says that every pair of points in a combinatorial horoball $\mathcal{H}$ can be joined by a regular segment which is a geodesic for the path metric on $\mathcal{H}$.

2.5.2. *Paths and quasi-geodesics in X.* In this paper, a *path* in a graph Y (that is, a finite, infinite, or bi-infinite sequence of vertices in Y, such that consecutive vertices are adjacent) will always be implicitly identified with its geometric realization, which extends the sequence to a continuous map c from an interval $I \subseteq \mathbb{R}$ to the geometric realization of Y.

Any geodesic in Y is a path, but the same need not be true for a quasi-geodesic. However, for technical convenience, we will nearly always work with quasi-geodesic *paths* rather than arbitrary quasi-geodesics. A *ray* $r : [0, \infty) \to Y$ or a *segment* $c : [a, b] \to Y$, quasi-geodesic or otherwise, is always assumed to be a path. When we refer to the "distance" or "Hausdorff distance" between two paths, or between a path and some other subset of Y, we always mean this with respect to images of geometric realizations.

2.5.3. *Comparing d_Γ and d_X.* We have the following comparison estimate for the cusped metric and word metric on Γ.

Lemma 2.3. *Let $u, v \in \Gamma$. Then we have*

$$d_X(u, v) \le d_\Gamma(u, v) \le d_X(u, v)\sqrt{2}^{d_X(u,v)}.$$

The right-hand side of this estimate is rather weak. Note, however, that it allows us to bound d_Γ in terms of d_X, in a way which does not depend at all on Γ or $\mathcal{P}$.

Proof. Let $X = X(\Gamma)$. The left-hand inequality is immediate since the inclusion $\mathrm{Cay}(\Gamma) \hookrightarrow X$ is an embedding of graphs, and d_Γ and d_X are respectively defined as path metrics on $\mathrm{Cay}(\Gamma)$ and X.

For the second inequality, consider a geodesic $c : I \to X$ between u and v, and suppose that the length of this geodesic is L. Suppose (a, b) is a connected component of $c^{-1}(X \setminus \mathrm{Cay}(\Gamma))$, meaning the restriction of c to (a, b) is contained in a single combinatorial horoball $\mathcal{H}$. Consider a regular geodesic in $\mathcal{H}$ between $c(a)$ and $c(b)$. The length of this geodesic is at most L, and the vertical parts of this regular geodesic have length at most $L/2$, which means that the d_Γ-distance between the endpoints of the regular geodesic is at most $3\sqrt{2}^L$.

Observe that the restriction of c to each component of $c^{-1}(X \setminus \mathrm{Cay}(\Gamma))$ is a sub-path containing at least two vertical edges and at least one horizontal edge, meaning it must have length at least 3. So the number of components of $c^{-1}(X \setminus \mathrm{Cay}(\Gamma))$ is most $L/3$. Thus, if we we replace all of the sub-paths in $c^{-1}(X \setminus \mathrm{Cay}(\Gamma))$ with geodesics in $\mathrm{Cay}(\Gamma)$, the argument from the previous paragraph shows that the overall length of the path increases by a multiplicative factor of at most $1/3 \cdot 3\sqrt{2}^L = \sqrt{2}^L$ and we are done. □

2.5.4. *Coarse geometry of paths in X.* Below, we prove several results which formalize the idea that intersections of geodesics and quasi-geodesics with horoballs in the cusped space X are coarsely well-defined. We start with the following lemma, which quantifies the fact that any geodesic whose endpoints lie near some horoball $\mathcal{H}$ must enter "deep into" $\mathcal{H}$.

Lemma 2.4. *Let $X = X(\Gamma, \mathcal{P})$ be the cusped space for $(\Gamma, \mathcal{P})$, and suppose that X is δ-hyperbolic. Let $x, y \in X$ be points lying within X-distance C of some horoball $\mathcal{H}$ in X, and let ℓ be a geodesic segment in X from x to y. For any $z \in \ell$, we have*

$$d_X(z, \{x, y\}) \le d_X(z, X \setminus \mathcal{H}) + 3C + 7\delta.$$

Proof. Let $\mathcal{H}' \subset X$ be an isometrically embedded horoball within Hausdorff distance δ of $\mathcal{H}$, and let x', y' be points in $\mathcal{H}'$ satisfying

$$d_X(x, x') \le C + \delta, \qquad d_X(y, y') \le C + \delta.$$

Let ℓ' be a regular geodesic segment between x' and y'. Then the Hausdorff distance between ℓ and ℓ' is at most $C + 3\delta$.

Let z' be a point on ℓ' so that $d_X(z, z') \le C + 3\delta$. Since ℓ' is a regular geodesic between $x', y' \in \mathcal{H}'$, we have

$$d_X(z', X \setminus \mathcal{H}) \ge \min(d_X(z', x'), d_X(z', y')) - 3. \tag{1}$$

On the other hand we have

$$\begin{aligned} d_X(z', x') &\ge d_X(z, x) - d_X(x, x') - d_X(z', z) \\ &\ge d_X(z, x) - 2C - 4\delta. \end{aligned}$$

Similarly, we have $d_X(z', y') \ge d_X(z, y) - 2C - 4\delta$, so putting these together we get

$$\min(d_X(z', x'), d_X(z', y')) \ge d_X(z, \{x, y\}) - 2C - 4\delta.$$

Then applying this with (1) we get

$$\begin{aligned} d_X(z, X \setminus \mathcal{H}) &\ge d_X(z', X \setminus \mathcal{H}) - d_X(z, z') \\ &\ge d_X(z, \{x, y\}) - 3C - 7\delta \end{aligned}$$

which is equivalent to the desired inequality. □

For the statement of the next result (and for the rest of the paper), when Y is a subset of some space equipped with a metric d, and $R > 0$, then $\mathcal{N}_R(Y; d)$ denotes the R-neighborhood of Y with respect to the metric d. If the metric is understood from context, then we will write $\mathcal{N}_R(Y) = \mathcal{N}_R(Y; d)$.

Lemma 2.5. *Let $X = X(\Gamma, \mathcal{P})$ be the cusped space for $(\Gamma, \mathcal{P})$, and suppose X is δ-hyperbolic. Given constants $K \ge 1$ and $A, D \ge 0$, there exists $R \ge 0$ (depending only on K, A, D, δ) so that the following holds. If $c : I \to X$ is a (K, A)-quasi-geodesic path, and each (non-ideal) endpoint of c lies in* $\mathrm{Cay}(\Gamma)$*, then the pairwise Hausdorff distances between the three sets*

$$c(I) \cap \mathrm{Cay}(\Gamma), \quad \mathcal{N}_D(c(I); d_\Gamma) \cap \mathrm{Cay}(\Gamma), \quad \mathcal{N}_D(c(I); d_X) \cap \mathrm{Cay}(\Gamma)$$

with respect to any of the metrics d_Γ, $d_{\hat{\Gamma}}$, or d_X are bounded by R.

Remark 2.6. We allow the quasi-geodesic c in the statement of the lemma to have two, one, or zero non-ideal endpoints (meaning c could be a segment, a ray, or a bi-infinite path in X). Since interiors of combinatorial horoballs are pairwise disjoint, our assumption on endpoints only disallows the cases where c is a ray entirely contained in the interior of a single combinatorial horoball, or a finite segment which either starts or ends in the interior of a horoball. This also applies to Corollary 2.7 below.

Proof of Lemma 2.5. Because of the right-hand inequality in Lemma 2.3, and because of the fact that $d_{\hat{\Gamma}}(u, v) \le d_\Gamma(u, v)$ for all $u, v \in \mathrm{Cay}(\Gamma)$, it suffices to prove the bound on Hausdorff distance with respect to the metric d_X. We will show that there is a uniform constant R so that $\mathcal{N}_D(c(I); d_X) \cap \mathrm{Cay}(\Gamma))$ lies in an R-neighborhood of $c(I) \cap \mathrm{Cay}(\Gamma)$, with respect to d_X. This is sufficient, because the left-hand inequality in Lemma 2.3 tells us that

$$c(I) \subseteq \mathcal{N}_D(c(I); d_\Gamma) \subseteq \mathcal{N}_D(c(I); d_X).$$

So, suppose that $y \in \mathcal{N}_D(c(I); d_X) \cap \mathrm{Cay}(\Gamma)$, and let z be a point on $c(I)$ such that $d_X(y, z) \le D$. If z lies in $\mathrm{Cay}(\Gamma)$ then w already lies in the D-neighborhood of $c(I) \cap \mathrm{Cay}(\Gamma)$, so assume that z lies in $c((a, b))$, where $(a, b) \subset I$ is a component of $c^{-1}(X \setminus \mathrm{Cay}(\Gamma))$. Since distinct combinatorial horoballs have disjoint interiors, the set $c((a, b))$ lies in a single combinatorial horoball $\mathcal{H}$.

It is possible that either $a = -\infty$ or $b = +\infty$. However, we cannot have $(a, b) = (-\infty, +\infty)$, since $\mathcal{H}$ is a quasi-isometrically embedded subset of X with a unique point in its ideal boundary. Thus, up to reversing the direction of c, we can assume that a is finite, and that $d_X(c(a), z) \le d_X(c(b), z)$. Our assumption on the endpoints of c means that $c(a)$ must lie in $\partial\mathcal{H} \subset \mathrm{Cay}(\Gamma)$. We can also choose some finite $b' \in [a, b]$ so that $c(b) \in \mathcal{H}$ and $d_X(c(a), z) \le d_X(c(b'), z)$; if b is already finite then we can take $b = b'$, and otherwise any sufficiently large $b' > a$ will do.

Let ℓ be a geodesic segment in X in joining $c(a)$ to $c(b')$. By the Morse lemma, there is a uniform constant M (depending only on K, A, δ) and a point w on ℓ so that $d_X(z, w) \le M$. From Lemma 2.4 we know that

$$d_X(w, \{c(a), c(b')\}) \le d_X(w, X \setminus \mathcal{H}) + 7\delta.$$

Now, since $y \in \mathrm{Cay}(\Gamma)$, we have

$$d_X(w, X \setminus \mathcal{H}) \le d_X(w, z) + d_X(z, y) \le M + D.$$

We also know that

$$d_X(z, c(a)) = d_X(z, \{c(a), c(b')\}) \le d_X(w, \{c(a), c(b)\}) + M,$$

so we conclude that $d_X(z, c(a)) \le 2M + D$. Thus $d_X(y, c(I) \cap \mathrm{Cay}(\Gamma)) < 2M + 2D$, which is what we wanted to show. ☐

Corollary 2.7. *For any constants $K \ge 1$ and $A, D \ge 0$, there exists a constant $R \ge 0$ satisfying the following. If $c : I \to X$ and $c : I' \to X$ are a pair of (K, A)-quasi-geodesic paths in X whose non-ideal endpoints all lie in $\mathrm{Cay}(\Gamma)$, and the Hausdorff distance between c and c' with respect to d_X is at most D, then the Hausdorff distance between $c(I) \cap \mathrm{Cay}(\Gamma)$ and $c'(I') \cap \mathrm{Cay}(\Gamma)$ with respect to d_Γ is at most R.*

Proof. If the Hausdorff distance in X between $c(I)$ and $c'(I')$ is at most D, then $c'(I')$ is contained in $\mathcal{N}_D(c(I); d_X)$, and thus $c'(I') \cap \mathrm{Cay}(\Gamma)$ is contained in $\mathcal{N}_D(c(I); d_X) \cap \mathrm{Cay}(\Gamma)$. By the previous lemma, this set is in turn contained in the R-neighborhood of $c(I) \cap \mathrm{Cay}(\Gamma)$ for a uniform constant $R > 0$ depending only on K, A, D, with respect to the metric d_Γ. So, $c(I') \cap \mathrm{Cay}(\Gamma)$ is contained in the $D + R$ neighborhood of $c(I) \cap \mathrm{Cay}(\Gamma)$. Arguing symmetrically completes the proof, using $D + R$ for the constant R in the statement of the corollary. ☐

2.5.5. *Approximating elements by rays.* Below, we record one more useful property of the cusped space $X = X(\Gamma, \mathcal{P})$.

Proposition 2.8. *Suppose that $\mathcal{P} \ne \emptyset$ and $\mathcal{P} \ne \{\Gamma\}$, and that the cusped space $X = X(\Gamma, \mathcal{P})$ is δ-hyperbolic. Then, for any $\gamma \in \Gamma$, there is a geodesic ray $r : [0, \infty) \to X$ so that $r(0) = \mathrm{id}$ and $d_X(r(t), \gamma) \le 8 + 21\delta$ for some $t \ge 0$.*

Proof. Fix $\gamma \in \Gamma$ and let $P \in \mathcal{P}$ be a proper subgroup of Γ. There is some s in our compatible generating set S so that the cosets P, sP are distinct, which in turn means that the cosets γP, γsP are distinct. Let $\mathcal{H}_1, \mathcal{H}_2$ be isometrically embedded horoballs in X which are within Hausdorff distance δ of the combinatorial horoballs attached at γP, γsP, respectively, and for $i = 1, 2$, let p_i be the unique point in the ideal boundary of $\mathcal{H}_i$.

For $i = 1, 2$, let $x_i \in \partial\mathcal{H}_i$ be a point such that $d_X(\gamma, x_i) \le 1 + \delta$, and let Q_i be a partly ideal geodesic quadrilateral with vertices $(\mathrm{id}, \gamma, x_i, p_i)$ in that order. We may choose Q_i so that the side between x_i and p_i is a vertical geodesic ray $[x_i, p_i) \subset \mathcal{H}_i$. The side of Q_i between id and p_i is parameterized by a unit-speed ray $r_i : [0, \infty) \to X$, which eventually lies in the horoball $\mathcal{H}_i$. We will show that γ is within distance $8 + 21\delta$ of at least one of the rays r_1, r_2.

Let $y_i \in \partial\mathcal{H}_i$ be the first point at which r_i enters the horoball $\mathcal{H}_i$. We know Q_i is 3δ-thin, so y_i is contained in a 3δ-neighborhood of either one of the geodesic segments $[\mathrm{id}, \gamma]$ or $[\gamma, x_i]$, or else the ray $[x_i, p_i)$.

If y_i is contained in a 3δ-neighborhood of $[\gamma, x_i]$, then $d_X(y_i, \gamma) \le 1 + 4\delta$. Alternatively, if y_i is contained in a 3δ-neighborhood of some point $z_i \in [x_i, p_i)$, then $d_X(z_i, \partial\mathcal{H}_i) \le 3\delta$, hence $d_X(z_i, x_i) \le 3\delta$ and therefore $d_X(y_i, \gamma) \le 1 + 7\delta$. So we are done unless there are points $z_1, z_2 \in [\mathrm{id}, \gamma]$ so that

$$d_X(y_1, z_1) \le 3\delta, \qquad d_X(y_2, z_2) \le 3\delta.$$

Without loss of generality, the points z_1, z_2 appear in that order on the geodesic $[\mathrm{id}, \gamma]$. By considering a 2δ-thin geodesic quadrilateral with vertices γ, x_1, y_1, z_1, we see that the sub-geodesic $[z_1, \gamma] \subseteq [\mathrm{id}, \gamma]$ is contained in the $(1 + 5\delta)$-neighborhood of a regular geodesic $[x_1, y_1] \subset \mathcal{H}_1$. Arguing similarly, a regular geodesic $[x_2, y_2] \subset \mathcal{H}_2$ is contained in the $(1+5\delta)$-neighborhood of $[z_2, \gamma] \subseteq [z_1, \gamma]$, hence in the $(2 + 10\delta)$-neighborhood of $[x_1, y_1]$.

In particular, $[x_2, y_2]$ is contained in the $(2+10\delta)$-neighborhood of $X \backslash \mathcal{H}_2$, and since $[x_2, y_2]$ is a regular geodesic in $\mathcal{H}_2$, this means that it has length at most $2(2 + 10\delta) + 3 = 7 + 20\delta$. Thus $d_X(y_2, \gamma) \le d_X(y_2, x_2) + d_X(x_2, \gamma) \le 8 + 21\delta$. □

Remark 2.9. A version of Proposition 2.8 still holds even if $\mathcal{P}$ is empty (see e.g. [Bog97]). However, for some applications of the proposition, we will need to use the fact that the distance between r and γ depends only on δ, and not on the space X itself. It is easier to obtain this control when $\mathcal{P}$ is nonempty.

2.6. **Dehn fillings.** Below we recall the definition of a *Dehn filling* of the relatively hyperbolic pair $(\Gamma, \mathcal{P})$.

Definition 2.10. Let $\mathcal{P} = \{P_1, \dots, P_k\}$, and let $(N_1, \dots, N_k)$ be a set of subgroups of Γ with $N_i \trianglelefteq P_i$ for each i. Let $N = \langle\!\langle \bigcup_{i=1}^k N_i \rangle\!\rangle$. The quotient map $\pi : \Gamma \to \Gamma/N$ is called a *Dehn filling* of $(\Gamma, \mathcal{P})$, and the subgroups $N_i \trianglelefteq P_i$ are called *Dehn filling kernels*.

When $\pi : \Gamma \to \Gamma/N$ is a Dehn filling as above, then we use $\mathcal{P}^\pi$ to denote the collection of subgroups $\{\pi(P_1), \dots \pi(P_k)\}$ of Γ/N. A property of $(\Gamma, \mathcal{P})$ is said to hold for *sufficiently long fillings* if there exists some finite set $B \subset \Gamma$ so that, whenever $(N_1, \dots, N_k)$ is a set of Dehn filling kernels satisfying $N_i \cap B = \emptyset$ for all i, and $\pi : \Gamma \to \Gamma/N$ is the corresponding Dehn filling, then the pair $(\Gamma/N, \mathcal{P}^\pi)$ has this property.

We have the following important theorem regarding Dehn fillings of relatively hyperbolic groups, due to Osin and Groves–Manning. This theorem can be thought of as a group-theoretic analog of the hyperbolic Dehn filling theorem for Kleinian groups.

Theorem 2.11 ([Osi07], [GM08])**.** *Let F be a finite subset of Γ. Then for any sufficiently long filling $\pi : \Gamma \to \Gamma/N$ of the relatively hyperbolic pair $(\Gamma, \mathcal{P} = \{P_1, \dots, P_k\})$, all of the following hold:*

1. *For each $P_i \in \mathcal{P}$, the induced map $\pi : P_i/N_i \to \Gamma/N$ is injective. Thus $\mathcal{P}^\pi$ is identified with the set $\{P_1/N_1, \dots, P_k/N_k\}$.*
2. *The pair $(\Gamma/N, \mathcal{P}^\pi)$ is relatively hyperbolic.*
3. *The restriction of π to F is injective.*

2.6.1. *Geometry of Dehn fillings.* Throughout this paper, we will need a basic understanding of the relationship between the coarse geometry of a relatively hyperbolic group Γ and the geometry of its Dehn fillings $\pi : \Gamma \to \Gamma/N$. Here we make note of several results in this direction.

Consider a Dehn filling $\pi : \Gamma \to \Gamma/N$. The subgroup N acts by automorphisms on all three graphs $\mathrm{Cay}(\Gamma)$, $\widehat{\mathrm{Cay}}(\Gamma, \mathcal{P})$, and $X(\Gamma, \mathcal{P})$ (recall that these are defined with respect to a fixed compatible generating set S). The resulting quotient graphs are *almost* isomorphic to the graphs $\mathrm{Cay}(\Gamma/N)$, $\widehat{\mathrm{Cay}}(\Gamma/N, \mathcal{P}^\pi)$, and $X(\Gamma/N, \mathcal{P}^\pi)$, defined with respect to the finite compatible generating set $\pi(S)$; there is only "almost" an isomorphism because the latter graphs are obtained from the former by deleting self-loops. Following e.g. [GM21], since such self-loops do not affect the metric on the zero-skeleton of any of these graphs, we will frequently employ a slight abuse of notation and assume that the loops are present in each of $\mathrm{Cay}(\Gamma/N)$, $\widehat{\mathrm{Cay}}(\Gamma/N, \mathcal{P}^\pi)$, and $X(\Gamma/N, \mathcal{P}^\pi)$.

This means that the Dehn filling $\pi : \Gamma \to \Gamma/N$ induces equivariant surjective maps of graphs

$$\begin{aligned} \pi &: \mathrm{Cay}(\Gamma) \to \mathrm{Cay}(\Gamma/N), \\ \hat{\pi} &: \widehat{\mathrm{Cay}}(\Gamma, \mathcal{P}) \to \widehat{\mathrm{Cay}}(\Gamma/N, \mathcal{P}^\pi), \\ \pi_X &: X(\Gamma, \mathcal{P}) \to X(\Gamma, \mathcal{P}^\sigma). \end{aligned}$$

The maps $\hat{\pi}$ and π_X both restrict to π on the copies of $\mathrm{Cay}(\Gamma)$ respectively embedded in $\widehat{\mathrm{Cay}}(\Gamma, \mathcal{P})$ and $X(\Gamma, \mathcal{P})$. Moreover, π_X sends combinatorial horoballs to combinatorial horoballs, and preserves vertical and horizontal edges; vertical edges are never sent to loops. Finally, since $\pi, \hat{\pi}$, and π_X are all surjective, and the metric on each graph is a path metric, each of $\pi, \hat{\pi}$, and π_X is 1-Lipschitz.

A straightforward consequence of Theorem 2.11 is the following:

Proposition 2.12. *Let $R > 0$. For all sufficiently long fillings $\pi : \Gamma \to \Gamma/N$, the restriction of π_X to any ball of radius R in $X(\Gamma, \mathcal{P})$ centered at a point in* $\mathrm{Cay}(\Gamma)$ *is an isometric embedding whose image is a metric ball.*

The lemma above can be used to prove the following, which sharpens part of Theorem 2.11. This result is originally due to Agol–Groves–Manning [AGM09]; the precise version here is stated (with the given level of generality) in e.g. [GMS19].

Proposition 2.13 ([AGM09, Prop. 2.3])**.** *There exists a constant $\delta \geq 0$ satisfying the following: the cusped space $X = X(\Gamma, \mathcal{P})$ is δ-hyperbolic, and for all sufficiently long fillings $\pi : \Gamma \to \Gamma/N$, the cusped space $X(\Gamma/N, \mathcal{P}^\pi)$ is δ-hyperbolic.*

2.6.2. *Lifts.* Whenever $\pi : \Gamma \to \Gamma/N$ is a sufficiently long filling, it restricts to an injection on the generating set S and thus the induced map $\mathrm{Cay}(\Gamma) \to \mathrm{Cay}(\Gamma/N)$ is a local homeomorphism of locally finite graphs, hence a covering map. In this case paths in $\mathrm{Cay}(\Gamma)$ lift to paths in $\mathrm{Cay}(\Gamma/N)$, uniquely after a choice of basepoint in $\mathrm{Cay}(\Gamma/N)$.

On the other hand, while the map $\pi_X : X(\Gamma, \mathcal{P}) \to X(\Gamma/N, \mathcal{P}^\pi)$ is typically *not* a covering map, we can still work with lifts.

Definition 2.14. Let $\pi : \Gamma \to \Gamma/N$ be a Dehn filling. If $c : I \to X(\Gamma/N, \mathcal{P}^\pi)$ is a path in the quotient cusp space, a *lift* of c is any path $\tilde{c} : I \to X(\Gamma, \mathcal{P})$ such that $\pi_X \circ \tilde{c} = c$.

Since π_X is a surjective map of (combinatorial) graphs, any path in $X(\Gamma/N, \mathcal{P}^\pi)$ can be lifted edge-by-edge to obtain a path in X. However, while lifts exist, they need not be unique, even after fixing a basepoint. We observe:

Proposition 2.15. *Let $c : I \to X(\Gamma, \mathcal{P})$ be a path, and let $\tilde{c}$ be a lift of c. If c is a (K, A)-quasi-geodesic, then so is $\tilde{c}$.*

Proof. This is immediate from the fact that π_X is 1-Lipschitz with respect to the metrics d_X, d_X^π on $X(\Gamma, \mathcal{P})$ and $X(\Gamma/N, \mathcal{P}^\pi)$. ☐

In the opposite direction, we also have the following:

Proposition 2.16. *For any $R > 0$, and any $K > 1$, any sufficiently long Dehn filling $\pi : \Gamma \to \Gamma/N$ satisfies the following. Suppose that $\tilde{c} : I \to X(\Gamma, \mathcal{P})$ is a geodesic, and the maximum depth of any point on $\tilde{c}$ is at most R. Then $c = \pi_X \circ \tilde{c}$ is a $(K, 2\delta)$-quasi-geodesic in $X(\Gamma/N, \mathcal{P}^\pi)$.*

Proof. Fix δ as in Proposition 2.13 so that the cusped space for all sufficiently long fillings of $(\Gamma, \mathcal{P})$ is δ-hyperbolic, and let $k > 8\delta$. By Proposition 2.12, for any sufficiently long filling $\pi : \Gamma \to \Gamma/N$, if B is any ball of radius $R+k$ centered at a point in $\mathrm{Cay}(\Gamma)$, then π_X restricts to an isometric embedding on B. By assumption, every point in $\tilde{c}$ is contained in the ball of radius R centered at some point in $\mathrm{Cay}(\Gamma)$, so this implies that $\pi_X \circ \tilde{c}$ is a k-local geodesic in $X(\Gamma/N, \mathcal{P}^\pi)$.

It now follows from the local-to-global principle in δ-hyperbolic metric spaces that $\pi_X \circ \tilde{c}$ is a (global) quasi-geodesic. For a precise estimate, we can use e.g. [BH99, III.H.1.13] to see that it is a $\left(\dfrac{k + 4\delta}{k - 4\delta}, 2\delta\right)$-quasigeodesic. Since $k > 8\delta$ could be chosen arbitrarily large, this completes the proof. ☐

3. Extended geometrically finite representations

In this section, we recall the definition of an *extended geometrically finite* (EGF) representation of a relatively hyperbolic group Γ into a semisimple Lie group G. EGF representations were originally defined in [Wei22], and we refer to that paper for further detail. See also Section 2 of [Wei23] for a brief introduction, focusing on the case where $G = \mathrm{PGL}(d, \mathbb{R})$.

When G has rank one, then EGF representations are closely related to (but not exactly the same as) representations with finite kernel and geometrically finite image. We give a precise statement connecting the two notions in Section 7 of this paper. See also Section 2 of [GW24] for a detailed account of the relationship.

3.1. Extended convergence actions.

The definition of an EGF representation is based on the notion of an *extended convergence action*. Extended convergence actions generalize the notion of a convergence action of a group Γ on a compact metrizable space M (see Section 2.4), in a way which accomodates many discrete group actions on the flag manifold G/Q associated to a semisimple Lie group G and a parabolic subgroup $Q < G$.

Definition 3.1 (See [Wei22, Def. 1.2])**.** Let $(\Gamma, \mathcal{P})$ be a relatively hyperbolic pair, with Γ acting on a compact metrizable space M by homeomorphisms. Let $\Lambda \subset M$ be a closed Γ-invariant set. We say that a Γ-equivariant surjective map $\phi : \Lambda \to \partial(\Gamma, \mathcal{P}_\infty)$ *extends the convergence group action* of Γ on $\partial(\Gamma, \mathcal{P}_\infty)$ if there exists an assigment $z \mapsto C_z$ of points $z \in \partial(\Gamma, \mathcal{P}_\infty)$ to open subsets $C_z \subset M$ satisfying the following properties:

(C1) For every compact subset $K \subset \partial(\Gamma, \mathcal{P}_\infty)$, the intersection $\bigcap_{z \in K} C_z$ contains an open neighborhood of $\phi^{-1}(\partial(\Gamma, \mathcal{P}_\infty) \setminus K)$.

(C2) For every sequence $\gamma_n \in \Gamma$ such that (in the Bowditch compactification $\overline{\Gamma}$) we have $\gamma_n \to z_+$ and $\gamma_n^{-1} \to z_-$ for $z_\pm \in \partial(\Gamma, \mathcal{P}_\infty)$, every compact subset $K \subset C_{z_-}$, and every open set $U \subset M$ containing $\phi^{-1}(z_+)$, we have $\gamma_n K \subset U$ for all sufficiently large n.

The map ϕ is called a *boundary extension*, the set Λ is called a *boundary set*, and the sets C_z are called the *repelling strata*.

Note that, in the context of Definition 3.1, if the equivariant map $\phi : \Lambda \to \partial(\Gamma, \mathcal{P}_\infty)$ is a homeomorphism, and if each repelling stratum C_z is equal to $M \setminus \phi^{-1}(z)$, then the second part of the definition says that Γ acts on M as a convergence group with limit set Λ.

Remark 3.2. The original version of Definition 3.1 given in [Wei22] differs from the definition above in several respects. Most significantly, the definition in [Wei22] replaces condition (C1) with the weaker condition:

(C1*) for every $z \in \partial(\Gamma, \mathcal{P}_\infty)$, the set C_z contains $\partial(\Gamma, \mathcal{P}_\infty) \setminus \{z\}$.

However, the stronger assumption (C1) is actually used implicitly in some of the general constructions in [Wei22], so we correct the definition by including it here. In Section 3.3 below, when we consider Q-extended geometrically finite representations ρ into a semisimple Lie group G, we will see that condition (C1) is equivalent to the combination of (C1*) with a natural additional assumption on ρ called *relative Q-divergence*.

We also comment that the version of Definition 3.1 in [Wei22] assumes that $\mathcal{P} = \mathcal{P}_\infty$; however, this difference in the definitions is superficial, since if $(\Gamma, \mathcal{P})$ is a relatively hyperbolic pair then so is $(\Gamma, \mathcal{P}_\infty)$.

Remark 3.3. It is also possible to define extended convergence group actions for groups which are not necessarily relatively hyperbolic, but we will not address this here.

3.2. **Semisimple Lie groups, parabolic subgroups, and flag manifolds.** Let G be a semisimple Lie group, let $K < G$ be a maximal compact subgroup, and let $\mathbb{X}$ be the associated nonpositively curved Riemannian symmetric space G/K. A subgroup $Q < G$ is *parabolic* if it is the stabilizer in G of some point $z \in \partial\mathbb{X}$. Two parabolic subgroups Q_+, Q_- are *antipodal* if they are respectively the stabilizers of points z_+, z_- in $\partial\mathbb{X}$ joined by a bi-infinite geodesic in $\mathbb{X}$. A parabolic subgroup Q is *symmetric* if it has an antipodal subgroup Q' which is conjugate to Q in G.

For any parabolic subgroup $Q < G$, the homogeneous G-space G/Q is called a *flag manifold*. Two flags $\xi_1 \in G/Q_1$ and $\xi_2 \in G/Q_2$ are *antipodal* if their stabilizers in G are antipodal parabolic subgroups. If Q_+, Q_- are antipodal parabolic subgroups, and $\xi_+ \in G/Q_+$, then $\mathrm{Opp}(\xi_+) \subset G/Q_-$ denotes the set of flags antipodal to ξ_+. It is an open dense subset of G/Q_-.

Example 3.4. If G has rank one, then $\mathbb{X}$ is negatively curved, so any two distinct points in $\partial\mathbb{X}$ are joined by a geodesic in $\mathbb{X}$. In addition G acts transitively on $\partial\mathbb{X}$, so there is only one conjugacy class of parabolic subgroup in G and all parabolic subgroups are symmetric. All of the flag manifolds for G thus have a unique G-equivariant identification with the visual boundary of $\mathbb{X}$, and two flags in $\partial\mathbb{X}$ are antipodal if and only if they are distinct.

Example 3.5. Let $G = \mathrm{PGL}(d, \mathbb{R})$. Then $\mathrm{PO}(d)$ is a maximal compact subgroup of G. For each $k \in \{1, \ldots, d-1\}$, the stabilizer of a k-plane in $\mathbb{R}^d$ is a maximal parabolic subgroup of G, meaning that each Grassmannian of k-planes can be identified with a flag manifold G/Q. More generally, all of the different flag manifolds for G can be uniquely identified with *partial flag spaces* associated to nonempty subsets of $\{1, \ldots, d-1\}$; the partial flag space associated to a set of integers $\{i_1 < \ldots < i_k\}$ is the space of *partial flags*

$$V_{i_1} \subset \ldots \subset V_{i_k},$$

where each V_{i_j} is a subspace of $\mathbb{R}^d$ with dimension i_j.

A pair of partial flags $V_{i_1} \subset \ldots \subset V_{i_k}$ and $W_{j_1} \subset \ldots \subset W_{j_k}$ are antipodal if they are *transverse*, meaning that for any $i \in \{i_1, \ldots, i_k\}$, V_i is defined if and only if W_{d-i} is defined, and $V_i + W_{d-i} = \mathbb{R}^d$. A parabolic subgroup Q is symmetric if it is the stabilizer of a partial flag of the form

$$V_{i_1} \subset \ldots \subset V_{i_k} \subseteq V_{d-i_k} \subset \ldots \subset V_{d-i_1}.$$

3.2.1. *Q-divergence.* Returning to the context of a general semisimple Lie group G, let $Q < G$ be a symmetric parabolic subgroup, and let Λ be a subset of the flag manifold G/Q. We will say that a sequence g_n in G is *Q-divergent* if, for every subsequence of g_n, there exists an open set $U \subset G/Q$ and a further subsequence g_m so that $g_m U$ converges to a singleton $\{\xi\}$ in G/Q. In this case ξ is said to be a *Q-limit point* of the sequence g_n. Note that Q-divergence depends only on the conjugacy class of Q in G. Also, since we have assumed that Q is a symmetric parabolic subgroup, a sequence g_n is Q-divergent if and only if the sequence g_n^{-1} of inverses is Q-divergent.

A subgroup $\Gamma < G$ is Q-divergent if all sequences of pairwise distinct elements in Γ are Q-divergent. The *Q-limit set* of a sequence or subgroup in G is the set of all of its Q-limit points. The Q-limit set of a group $\Gamma < G$ is a Γ-invariant subset of G/Q, and it is closed if Γ is Q-divergent.

If G has rank one, a sequence g_n is Q-divergent (for some, equivalently any, parabolic Q) if and only if it is divergent in the sense that it eventually leaves every compact subset of G. Thus in the rank-one case a group $\Gamma < G$ is Q-divergent if and only if it is discrete, and its Q-limit set is precisely the limit set in the usual sense.

There are several notions equivalent to Q-divergence which will not appear in this paper, but arise frequently in the literature; usually these definitions are stated in terms of the behavior of a *Cartan projection* for G [GGKW17] or the *vector-valued distance* on the Riemannian symmetric space G/K [KLP17]. For the equivalence of the definitions, see e.g. [KLP17, Lem. 4.23(i)] and the appendix of [Wei22].

We record one more property of Q-divergent sequences, which we will use several times throughout the paper:

Lemma 3.6 ([KLP17, Lem. 4.19])**.** *Let g_n be a sequence in G, and let $\xi^{\pm} \in G/Q$. The following are equivalent:*

1. *The sequence g_n converges to the constant map ξ^+ on $\mathrm{Opp}(\xi^-)$, uniformly on compact sets.*
2. *The sequence g_n is Q-divergent, the unique Q-limit point of g_n is ξ^+, and the unique Q-limit point if g_n^{-1} is ξ^-.*

Note that every Q-divergent sequence in G has a subsequence satisfying the conditions of the lemma.

3.3. **Extended geometrically finite representations.** We are now ready to give the main definition of this section. Below, let $(\Gamma, \mathcal{P})$ be a relatively hyperbolic pair, let G be a semisimple Lie group, and let $Q < G$ be a symmetric parabolic subgroup.

Definition 3.7 (See [Wei22, Def. 1.3])**.** A representation $\rho : \Gamma \to G$ is *Q-extended geometrically finite* (with respect to the peripheral structure $\mathcal{P}$) if there exists a compact ρ-invariant subset $\Lambda \subset G/Q$ and a surjective antipodal map $\phi : \Lambda \to \partial(\Gamma, \mathcal{P}_\infty)$ extending the convergence action of Γ on $\partial(\Gamma, \mathcal{P}_\infty)$ (Definition 3.1).

Here, a map $\phi : \Lambda \to Z$ is *antipodal* if, for any distinct $z_1, z_2 \in Z$, every flag in $\phi^{-1}(z_1)$ is antipodal to every flag in $\phi^{-1}(z_2)$, i.e. if

$$\phi^{-1}(z_1) \subseteq \bigcap_{\xi \in \phi^{-1}(z_2)} \mathrm{Opp}(\xi).$$

When G has rank one, this condition is vacuous.

As mentioned in Remark 3.2, it is not clear that Definition 3.7 agrees exactly with the version originally given in [Wei22], since it is possible that condition (C1) in Definition 3.1 is actually stronger than condition (C1*). Note, however, that we are frequently in the situation where the repelling strata C_z are all equal to the sets $\mathrm{Opp}(\phi^{-1}(z))$, and in this case both (C1) and (C1*) follow automatically from the antipodality assumption on ϕ and compactness of Λ.

Even when the above does not hold, we can still obtain (C1) from (C1*) if we also assume that ρ is *relatively Q-divergent* in the following sense:

Definition 3.8. A representation $\rho : \Gamma \to G$ is *relatively Q-divergent* if, for every sequence $\gamma_n \in \Gamma$ satisfying $|\gamma|_{\hat{\Gamma}} \to \infty$, the sequence $\rho(\gamma_n)$ is Q-divergent.

The proof of Lemma 8.1 in [Wei22] shows that any representation which is Q-EGF in the sense of Definition 3.7 is also relatively Q-divergent (see also Remark 5.14 in the present paper). In an appendix to this paper, we prove:

Proposition 3.9. *Let $\rho : \Gamma \to G$ be a representation, and suppose that there exists a compact ρ-invariant set $\Lambda \subset G/Q$, an equivariant surjective antipodal map $\phi : \Lambda \to \partial(\Gamma, \mathcal{P}_\infty)$, and open sets $C_z \subset G/Q$ for each $z \in \partial(\Gamma, \mathcal{P}_\infty)$ satisfying conditions (C1*) and (C2) above.*

Then, if ρ is relatively Q-divergent, there exists a (possibly different) equivariant surjective antipodal map $\hat{\phi} : \hat{\Lambda} \to \partial(\Gamma, \mathcal{P}_\infty)$ and open sets $\{\hat{C}_z\}_{z \in \partial(\Gamma, \mathcal{P}_\infty)}$ satisfying conditions (C1) and (C2) in Definition 3.1; in other words, ρ is Q-EGF.

3.4. **The Chabauty topology and geometric convergence.** Below we give some basic reminders about the *Chabauty topology* on the space of closed subgroups of a Lie group G. We refer to [BHK09] for further background. This material is required to develop the notion of an *extended Dehn filling space* for an EGF representation $\rho : \Gamma \to G$.

Definition 3.10. Let G be a Lie group, and let $\mathcal{C} = \mathcal{C}(G)$ denote the set of closed subgroups of G. We equip $\mathcal{C}$ with the *Chabauty topology*, defined in terms of a basis as follows. For a compact subset $K \subset G$, a neighborhood U of the identity in G, and $C \in \mathcal{C}$, the basic open subset $V_{K,U,C} \subset \mathcal{C}(G)$ is

$$V_{K,U,C} := \{D \in \mathcal{C} : D \cap K \subset CU \text{ and } C \cap K \subset DU\}.$$

Then the Chabauty topology on $\mathcal{C}(G)$ is the topology generated by the basis of all open sets $V_{K,U,C}$. This makes $\mathcal{C}(G)$ into a compact space.

Using some fixed metrization of G, the Chabauty topology on a Lie group G can also be viewed as the topology of *local Hausdorff convergence* of closed subgroups in G: a sequence of closed subgroups $C_n < G$ converges to some closed subgroup $C < G$ in $\mathcal{C}(G)$ if and only if, for every bounded open subset $U \subset G$, the intersections $C_n \cap U$ converge to the intersection $C \cap U$, with respect to Hausdorff distance.

We also have the following alternative characterization of convergence in $\mathcal{C}(G)$:

Proposition 3.11. *Let G be a Lie group, let $C \in \mathcal{C}(G)$, and let C_n be a sequence in $\mathcal{C}(G)$. Then C_n converges to C in $\mathcal{C}(G)$ if and only if both of the following hold:*

(A) For every $g \in C$, there exists a sequence $g_n \in C_n$ so that $g_n \to g$.
(B) If a sequence $g_n \in C_n$ converges to some $g \in G$, then $g \in C$.

This is the description of the Chabauty topology we will typically use in this paper.

3.4.1. *Geometric convergence.* Let Γ be a finitely generated group and let G be a Lie group. Typically, the set $\mathrm{Hom}(\Gamma, G)$ is equipped with the compact-open topology, where Γ is viewed as an abstract topological group with the discrete topology. Sometimes this is called the topology of *algebraic convergence*, since in this space a representation ρ is the limit of a sequence of representations ρ_n if and only if, for each s in a finite generating set S for Γ, we have $\rho_n(s) \to \rho(s)$. In this paper, unless explicitly stated otherwise, "convergence in $\mathrm{Hom}(\Gamma, G)$" always means convergence with respect to this topology.

However, there is also a natural finer topology on $\mathrm{Hom}(\Gamma, G)$, which we define below.

Definition 3.12. Let Γ be a finitely generated group and let G be a Lie group. Then $\mathrm{Hom}_{\mathrm{geom}}(\Gamma, G)$ is the set $\mathrm{Hom}(\Gamma, G)$ equipped with the topology of *geometric* or *strong* convergence. Precisely, the topology on $\mathrm{Hom}_{\mathrm{geom}}(\Gamma, G)$ is defined to be the coarsest refinement of the topology on $\mathrm{Hom}(\Gamma, G)$ so that the map $\mathrm{Hom}_{\mathrm{geom}}(\Gamma, G) \to \mathcal{C}(G)$ given by $\rho \mapsto \overline{\rho(\Gamma)}$ is continuous.

Equivalently, a sequence of representations ρ_n converges to ρ in $\mathrm{Hom}_{\mathrm{geom}}(\Gamma, G)$ if and only if ρ_n converges to ρ in $\mathrm{Hom}(\Gamma, G)$, and $\overline{\rho_n(\Gamma)}$ converges to $\overline{\rho(\Gamma)}$ in $\mathcal{C}(G)$. In this situation we say that ρ_n converges *strongly* to ρ.

3.5. **Relative geometric convergence.** In practice, checking that a sequence of representations $\rho_n : \Gamma \to G$ converges strongly to some ρ can be challenging, since one needs some form of uniform control on the behavior of every element in every group $\rho_n(\Gamma)$. The process is often much easier with some additional assumptions on Γ—for instance if Γ is assumed to be virtually nilpotent or virtually abelian.

Now suppose that Γ is a relatively hyperbolic group, relative to some collection $\mathcal{P}$ of peripheral subgroups, and let $\rho_n : \Gamma \to G$ be a sequence of representations. As above, if the groups in $\mathcal{P}$ are e.g. virtually nilpotent, then it can be much easier to directly control geometric convergence of the *restrictions* $\rho_n|_P$ for each $P \in \mathcal{P}$ than it is to control Chabauty convergence of the full group $\rho_n(\Gamma)$; one part of the main theorem in this paper essentially gives a method for upgrading Chabauty convergence of the subgroups $\rho_n(P)$ to Chabauty convergence of the full group $\rho_n(\Gamma)$.

To that end, we recall the following definition from the introduction of the paper:

Definition 3.13. Let Γ be a finitely generated group, let $\mathcal{P}$ be a collection of finitely generated subgroups of Γ, and let G be a Lie group. We let $\mathrm{Hom}_{\mathrm{geom}}(\Gamma, G; \mathcal{P})$ denote the space of representations $\Gamma \to G$ with the topology of *relative geometric convergence*: the coarsest refinement of the compact-open topology on $\mathrm{Hom}(\Gamma, G)$ so that for every $P \in \mathcal{P}$, the restriction map $\mathrm{Hom}_{\mathrm{geom}}(\Gamma, G; \mathcal{P}) \to \mathrm{Hom}_{\mathrm{geom}}(P, G)$ is continuous.

Thus, a sequence of representations $\rho_n : \Gamma \to G$ converges to ρ if and only if ρ_n converges to ρ in $\mathrm{Hom}(\Gamma, G)$, and for each $P \in \mathcal{P}$, $\overline{\rho_n(P)}$ converges to $\overline{\rho(P)}$ in $\mathcal{C}(G)$.

3.6. **Extended Dehn filling spaces.** With the notation and terminology developed above, we can give the precise definition of an extended Dehn filling space. For the below, fix a relatively hyperbolic pair $(\Gamma, \mathcal{P})$, a symmetric parabolic subgroup Q of a semisimple Lie group G, and a Q-EGF representation $\rho : \Gamma \to G$ with boundary extension $\phi : \Lambda \to \partial(\Gamma, \mathcal{P}_\infty)$

and repelling strata $\{C_z\}_{z\in\partial(\Gamma,\mathcal{P}_\infty)}$. Let $\Pi_\infty \subset \partial(\Gamma,\mathcal{P}_\infty)$ be the finite set of points fixed by the groups $P \in \mathcal{P}_\infty$, and for each $p \in \Pi_\infty$ let $\Gamma_p \in \mathcal{P}$ be the Γ-stabilizer of p.

Definition 3.14. A subspace $W \subseteq \mathrm{Hom}_{\mathrm{geom}}(\Gamma, G; \mathcal{P})$ is an *extended Dehn filling space* if the following holds. For any $p \in \Pi_\infty$, any open set $U \subset G/Q$ containing $\phi^{-1}(p)$, any finite subset $F \subseteq \Gamma_p$, and any compact set $K \subset C_p$ such that $\rho(\Gamma_p \setminus F)K \subset U$, there exists an open neighborhood $O \subset \mathrm{Hom}_{\mathrm{geom}}(\Gamma, G; \mathcal{P})$ such that for any $\sigma \in O \cap W$, we have

$$(\sigma(\Gamma_p) \setminus \sigma(F))K \subset U. \tag{2}$$

Remark 3.15. If we replace (2) above with the more restrictive condition

$$\sigma(\Gamma_p \setminus F)K \subset U, \tag{3}$$

we recover the definition of a *peripherally stable subspace* given in [Wei22]. Note that (3) implies that for each $P \in \mathcal{P}$, the kernels of the restrictions $\sigma|_P$ are no larger than the kernels of $\rho|_P$, if σ is sufficiently close to ρ in $\mathrm{Hom}(\Gamma, G)$. Weakening from (3) to (2) allows the deformation σ to potentially acquire a larger kernel on peripherals, allowing it to descend to a representation of a nontrivial Dehn filling of Γ.

4. Relative quasi-geodesic automata

In this section, we review a key tool we need for the proof of the main theorem in this paper: a *relative quasi-geodesic automaton*, originally developed in [Wei22] in order to prove a less general version of our main theorem. We recall the basic definitions and main results concerning these automata here, and refer to sections 5 and 6 of [Wei22] for further details.

At the end of the section, we prove several results relating the combinatorics of the automaton to the coarse geometry of the corresponding relatively hyperbolic group; our main aim is to show that there is a reasonably nice correspondence between the "codings" given by the automaton and quasi-geodesics (see Corollary 4.16). Similar results were proved previously in [Wei22] and [MMW22; MMW24], but we also provide proofs in this paper since we sometimes need slightly different statements.

For this section of the paper, fix a non-elementary relatively hyperbolic pair $(\Gamma, \mathcal{P})$. We assume that $\mathcal{P} \neq \emptyset$. In this section we also assume that every group in $\mathcal{P}$ is infinite, i.e. that $\mathcal{P} = \mathcal{P}_\infty$. Let $\Pi \subset \partial(\Gamma, \mathcal{P})$ denote the finite set of fixed points of groups in P.

Definition 4.1. A $(\Gamma, \mathcal{P})$*-graph* is a finite directed graph $\mathcal{G}$ with vertex set $V(\mathcal{G})$ and vertex labels $v \mapsto T_v$ for T_v a subset of Γ, subject to the following condition: for every vertex $v \in V(\mathcal{G})$, either:

- (i) T_v is a singleton $\{\alpha_v\}$, or
- (ii) T_v is a set of the form $gP \setminus F_v$, where $g \in \Gamma$, $P \in \mathcal{P}$ and $F_v \subset gP$ is finite.

If v is a vertex of the second kind above, so $T_v = gP \setminus F_v$ for $P \in \mathcal{P}$, then we say the vertex v is a *parabolic vertex*. In this case (since P is infinite) the coset gP is uniquely determined by the label set T_v, so we let q_v denote the *associated parabolic point* gp_v, where $p_v \in \Pi$ is the unique point fixed by P.

Definition 4.2. Let $\mathcal{G}$ be a $(\Gamma, \mathcal{P})$-graph. A $\mathcal{G}$*-path* is a sequence of pairs $\{(v_i, \alpha_i)\}_{i=1}^N$, with $N \in \mathbb{N} \cup \{\infty\}$, such that the sequence $\{v_i\}_{i=1}^N$ is a vertex path in $\mathcal{G}$, and each α_i lies in the label set T_{v_i}.

Definition 4.3. If $z \in \partial(\Gamma, \mathcal{P})$, we say that a $\mathcal{G}$-path $\{(v_i, \alpha_i)\}_{i=1}^N$ is a $\mathcal{G}$-path *limiting to* z if one of the following holds:

- z is a conical limit point, $N = \infty$, and the infinite sequence $\{g_n\}_{n=1}^{\infty}$ defined by
$$g_n = \alpha_1 \cdots \alpha_n$$
is unbounded and lies within a uniform neighborhood of a ray $r : [0, \infty) \to X(\Gamma, \mathcal{P})$ with ideal endpoint z.
- z is a parabolic point, N is finite, v_N is a parabolic vertex corresponding to a parabolic point q_v, and
$$z = \alpha_1 \cdots \alpha_{N-1} q_v.$$

In this paper, we use a $(\Gamma, \mathcal{P})$-graph $\mathcal{G}$ to encode "convergence-like" actions of Γ on certain compact metrizable spaces M; in particular we want to encode *extended convergence actions* as in Section 3.1. We express the connection between Γ-actions and $(\Gamma, \mathcal{P})$-graphs with the definition below.

Definition 4.4. Let $\mathcal{G}$ be a $(\Gamma, \mathcal{P})$-graph, and suppose that Γ acts by homeomorphisms on a compact metric space M. A *$\mathcal{G}$-compatible system of subsets* of M is an assignment $v \mapsto U_v$ of vertices of $\mathcal{G}$ to open subsets of M such that, for every edge $v \to w$ in $\mathcal{G}$, there is some $\varepsilon > 0$ so that $\overline{\mathcal{N}_\varepsilon(U_w)} \neq M$, and
$$\alpha \cdot \overline{\mathcal{N}_\varepsilon(U_w)} \subset U_v$$
for every $\alpha \in T_v$.

4.1. **Automata for EGF representations.** One of the main results of [Wei22] is that, given a Q-EGF representation $\rho : \Gamma \to G$, it is always possible to construct a $(\Gamma, \mathcal{P})$-graph $\mathcal{G}$ and $\mathcal{G}$-compatible systems of subsets of both $\partial(\Gamma, \mathcal{P})$ and G/Q, which satisfy certain desirable properties. Then, the finiteness properties of $\mathcal{G}$ then can be used to deduce local stability properties of the representation ρ.

The key construction in [Wei22] can be stated as follows. Recall that we have fixed a relatively hyperbolic pair $(\Gamma, \mathcal{P})$ with $\mathcal{P} = \mathcal{P}_\infty$. Below, and for the rest of the paper, we fix an auxiliary metric on the flag manifold G/Q; neighborhoods of points and sets in G/Q will be with respect to this metric, while neighborhoods in $\partial(\Gamma, \mathcal{P})$ are taken with respect to a fixed choice of metric d_∂.

Proposition 4.5 ([Wei22, Proposition 6.1])**.** *Let $\rho : \Gamma \to G$ be a Q-EGF representation of Γ with boundary extension $\phi : \Lambda \to \partial(\Gamma, \mathcal{P})$ and repelling strata $\{C_z : z \in \partial(\Gamma, \mathcal{P})\}$.*

Then, for every $\varepsilon > 0$, there is a $(\Gamma, \mathcal{P})$-graph $\mathcal{G}$ and a pair of $\mathcal{G}$-compatible systems $\{U_v \subset G/Q\}_{v \in V(\mathcal{G})}$, $\{W_v \subset \partial(\Gamma, \mathcal{P})\}_{v \in V(\mathcal{G})}$ satisfying the following conditions:

(G1) *For every $z \in \partial(\Gamma, \mathcal{P})$, there is a $\mathcal{G}$-path limiting to z.*

(G2) *For every vertex $v \in V(\mathcal{G})$, there is some $z \in W_v$ so that*
$$\phi^{-1}(W_v) \subset U_v \subset \mathcal{N}_\varepsilon(\phi^{-1}(z)).$$

(G3) *Every vertex of $\mathcal{G}$ has at least one outgoing edge.*

(G4) *For every $p \in \Pi$, there is a parabolic vertex v with $q_v = p$. Moreover, for every parabolic vertex w with $q_w = gp$, there is an edge $v \to b$ in $\mathcal{G}$ if and only if there is also an edge $w \to b$.*

(G5) *If v is a parabolic vertex with $q_v = gp$ for $p \in \Pi$, and (v, w) is an edge of $\mathcal{G}$, then $q_v \in W_v$ and $U_w \subset C_p$.*

Remark 4.6.

(a) In [Wei22], this proposition was originally stated for representations satisfying the weaker condition (C1*) discussed in the previous section. However, the proof actually relies on the stronger condition (C1) in Definition 3.1.

(b) Property (G3) is not explicitly stated as part of Propositon 6.1 in [Wei22], but it is a consequence of the proof (see the discussion above Proposition 6.9 in [Wei22]).

For the rest of the paper, we will refer to any $(\Gamma, \mathcal{P})$-graph as in the proposition above as a *relative automaton* adapted to the EGF representation ρ. The precise construction yielding the automaton and $\mathcal{G}$-compatible systems given by Proposition 4.5 is rather technical, so we will not discuss any details here. However, we will take advantage of a few additional properties of the automaton, which follow from either the properties above or arguments in [Wei22].

The first two properties are straightforward consequence of compactness of the sets $\Lambda, \partial(\Gamma, \mathcal{P})$.

Proposition 4.7. *If the parameter $\varepsilon > 0$ in Proposition 4.5 is sufficiently small, then:*

(G6) For each set W_v in the system $\{W_v\}_{v \in V(\mathcal{G})}$, the set $\partial(\Gamma, \mathcal{P}) \setminus \overline{W_v}$ is nonempty.

(G7) For each set U_v in the system $\{U_v\}_{v \in V(\mathcal{G})}$, the set $\mathrm{Opp}(\overline{U_v}) = \bigcap_{\xi \in \overline{U_v}} \mathrm{Opp}(\xi)$ is nonempty.

Proof. Note that for every compact subset $K \subset \partial(\Gamma, \mathcal{P})$, the set $\phi^{-1}(K)$ is a compact subset of flags satisfying $\phi^{-1}(\partial(\Gamma, \mathcal{P}) \setminus K) \subset \mathrm{Opp}(\phi^{-1}(K))$. In particular, if there is some $z \in \partial(\Gamma, \mathcal{P}) \setminus K$, then $\phi^{-1}(z) \in \mathrm{Opp}(\phi^{-1}(K))$. Then, since antipodality is an open condition, for each proper compact subset $K \subsetneq \partial(\Gamma, \mathcal{P})$, we can find some $\varepsilon > 0$ and some $z \in \partial(\Gamma, \mathcal{P})$ so that $\mathrm{Opp}(\mathcal{N}_\varepsilon(\phi^{-1}(K)))$ contains $\phi^{-1}(z)$.

We can cover $\partial(\Gamma, \mathcal{P})$ with finitely many compact proper subsets, and then use the above to find a uniform $\varepsilon > 0$ so that for each $z \in \partial(\Gamma, \mathcal{P})$, there is some $z' \in \partial(\Gamma, \mathcal{P})$ so that $\phi^{-1}(z')$ is contained in $\mathrm{Opp}(\mathcal{N}_\varepsilon(\phi^{-1}(z)))$. Then both (G6) and (G7) above follow from (G2). ☐

The next property says that we can choose the $\mathcal{G}$-compatible system of sets $\{U_v\}_{v \in V(\mathcal{G})}$ so that, if a pair of points $z_1, z_2 \in \partial(\Gamma, \mathcal{P})$ are well-separated, then the lifts of z_1 and z_2 in G/Q lie in uniformly antipodal neighborhoods U_{v_1}, U_{v_2}.

Proposition 4.8 (See [Wei22, Proposition 9.6]). *For any given $\Delta > 0$, the automaton and sets in Proposition 4.5 can also be chosen to satisfy:*

(G8) If $z_1, z_2 \in \partial(\Gamma, \mathcal{P})$ satisfy $d_\partial(z_1, z_2) > \Delta$, and $\phi^{-1}(z_1) \subset U_{v_1}$, $z_2 \in \phi^{-1}(z_2) \subset U_{v_2}$, then $\overline{U_{v_1}}$ and $\overline{U_{v_2}}$ are pairwise antipodal, i.e.

$$\overline{U_{v_i}} \subseteq \bigcap_{\xi \in \overline{U_{v_j}}} \mathrm{Opp}(\xi).$$

for $\{i, j\} = \{1, 2\}$.

The following property says that if a point z in $\partial(\Gamma, \mathcal{P})$ is "far from" some parabolic point $p \in \Pi$, then there is a $\mathcal{G}$-path limiting to z which has a certain special form. This property is useful for showing that the action of $\rho(\Gamma_p)$ on $\phi^{-1}(z)$ is stable under certain perturbations of ρ, where Γ_p is the stabilizer of p.

Proposition 4.9 ([Wei22, Proposition 6.15]). *For each parabolic point $p \in \Pi$, let K_p be a compact subset of $\partial(\Gamma, \mathcal{P}) \setminus \{p\}$. The automaton and sets in Proposition 4.5 can further be chosen to satisfy:*

(G9) For every parabolic vertex w with $p_w = g \cdot p$ for $p \in \Pi$, and every $z \in K_p$, there is a $\mathcal{G}$-path $\{(v_i, \alpha_i)\}_{i=1}^N$ limiting to z, whose first vertex v_1 is connected to w by an edge (w, v_1) in $\mathcal{G}$.

The final property says the automaton can be chosen so that, for any compact subset $Z \subseteq \partial(\Gamma, \mathcal{P})$, the initial open subsets appearing in $\mathcal{G}$-paths limiting to points in Z can be chosen to "approximate" $\phi^{-1}(Z)$ arbitrarily well.

Proposition 4.10 ([Wei22, Prop. 6.14])**.** *Let Z be a compact subset of $\partial(\Gamma, \mathcal{P})$, and let $\varepsilon > 0$ be given. The automaton and sets in Proposition 4.5 can also be chosen to satisfy:*

(G10) For every $z \in Z$, there exists a $\mathcal{G}$-path limiting to z, with initial vertex v_1, such that $U_{v_1} \subset \mathcal{N}_\varepsilon(\phi^{-1}(Z))$.

4.2. **$\mathcal{G}$-paths and the geometry of X.** For the rest of the section, suppose that $\rho : \Gamma \to G$ is a Q-EGF representation, and that $\mathcal{G}$ is a relative automaton with $\mathcal{G}$-compatible systems of sets $\mathcal{U} = \{U_v \subset G/Q\}_{v \in V(\mathcal{G})}$ and $\mathcal{W} = \{W_v \subset \partial(\Gamma, \mathcal{P})\}_{v \in V(\mathcal{G})}$ satisfying properties (G1)-(G7) above. (Since properties (G8)-(G10) depend on additional extrinsic parameters, we will ignore them until the next section.)

It turns out that the relative automaton $\mathcal{G}$ and the system $\mathcal{W}$ are also well-adapted to the metric geometry of the relatively hyperbolic group Γ, in the sense that $\mathcal{G}$-paths determine sequences in Γ which are closely related to geodesic rays and segments in the cusped space $X(\Gamma, \mathcal{P})$. The next several lemmas make this precise. We note any analogies with similar results in [MMW24] or [Wei22] where they occur.

First, we make the following observation:

Lemma 4.11. *There is a constant $k > 0$ satisfying the following: for any $\mathcal{G}$-path $\{(v_i, \alpha_i)\}_{i=1}^N$, if g_n is the sequence $g_n = \alpha_1 \cdots \alpha_n$, then no element in Γ can appear more than k times in g_n.*

Proof. We know by (G7) that each $\overline{U_v}$ is a proper nonempty subset of G/Q. Then, since the flag manifold G/Q is connected, and $\overline{\alpha_i U_{v_{i+1}}} \subset U_{v_i}$ for all i, we must have $\alpha_i U_{v_{i+1}} \subsetneq U_{v_i}$ for all i. It follows that for any $m < n$, we have

$$g_n U_{v_{n+1}} \subsetneq g_m U_{v_{m+1}}. \tag{4}$$

Now we can prove the lemma by arguing as in [MMW24, Lem. 4.10]. Let k be the number of vertices in the automaton $\mathcal{G}$. If there is some g such that $g_n = g$ for more than k different indices n, then there are indices m, n with $v_{m+1} = v_{n+1}$ and $g_m = g_n$, but this contradicts (4), proving the claim. □

Lemma 4.12 (See [Wei22, Prop 5.11] or [MMW24, Lem 4.7])**.** *There exists a constant $D > 0$ such that, for any (finite or infinite) $\mathcal{G}$-path $\{(v_i, \alpha_i)\}_{i=1}^N$, any point $z \in \bigcap_{i=1}^N \alpha_1 \cdots \alpha_{i-1} W_{v_{i+1}}$, and any geodesic ray $c : [0, \infty) \to X = X(\Gamma, \mathcal{P})$ from the identity to z, the set of points*

$$\{g_n = \alpha_1 \cdots \alpha_n\}_{n=0}^N$$

lies within a D-neighborhood of a geodesic ray in X from id *to z, with respect to the metric d_X.*

Proof. First, fix $r > 0$ so that for every edge $u \to v$ in $\mathcal{G}$, and every $\alpha \in T_u$, the inclusion

$$\alpha \cdot \mathcal{N}_r(W_v) \subset W_u. \tag{5}$$

By property (G6), we may also assume that r is sufficiently small so that $\mathcal{N}_r(W_v) \neq \partial(\Gamma, \mathcal{P})$ for every $W_v \in \mathcal{W}$.

The vertex sequence $v_1, v_2, \ldots,$ of the $\mathcal{G}$-path determines a sequence of sets $W_i := W_{v_i}$. Fix a point $z_- \in \partial(\Gamma, \mathcal{P}) \setminus \mathcal{N}_r(W_1)$. Since $z \in \overline{\alpha_1 W_2} \subset W_1$, we know that, with respect to our fixed metric d_∂ on $\partial(\Gamma, \mathcal{P})$, we have $d_\partial(z, z_-) \geq \varepsilon$. Therefore, there is a bi-infinite geodesic $c : (-\infty, \infty) \to X$ passing within distance R of the identity, for R depending only on r and the metric d_∂.

For each n, we know that $g_n^{-1}c$ is a bi-infinite geodesic joining $g_n^{-1}z_-$ to $g_n^{-1}z$. By assumption, $g_n^{-1}z$ lies in W_{n+1}, and (5) guarantees that $\mathcal{N}_r(W_{n+1}) \subset g_n^{-1}W_1$, hence

$$\partial(\Gamma, \mathcal{P}) \setminus g_n^{-1}W_1 \subset \partial(\Gamma, \mathcal{P}) \setminus g_n^{-1}\mathcal{N}_r(W_{n+1}).$$

It follows that $d_\partial(g_n^{-1}z_-, g_n^{-1}z) > r$ and so $g_n^{-1}c$ also passes within distance R of the identity. Therefore, the sequence g_n lies within an R-neighborhood of c.

The argument above actually shows that g_n lies within distance R of *any* bi-infinite geodesic joining z to some point z_- in $\partial(\Gamma, \mathcal{P}) \setminus \mathcal{N}_r(W_1)$. Since $\partial(\Gamma, \mathcal{P})$ is perfect and compact, we can choose r small enough so that for every vertex v in the automaton, the complement $\partial(\Gamma, \mathcal{P}) \setminus \mathcal{N}_r(W_v)$ contains at least two points whose distance is at least r. So, by choosing two points z_-, z'_- in $\mathcal{N}_r(W_1)$ with $d_\partial(z_-, z'_-) > r$, we see that g_n lies in the intersection of R-neighborhoods of bi-infinite geodesics c, c' joining z_-, z'_- respectively to z. By e.g. [MMW24, Lemma 4.6], this means that g_n also lies in a uniform neighborhood (in X) of a geodesic ray based at the identity with ideal endpoint z. ☐

As a consequence, we have:

Lemma 4.13 (see [MMW24, Cor. 4.11])**.** *Any infinite-length $\mathcal{G}$-path $\{(v_i, \alpha_i)\}_{i=1}^\infty$ limits to a unique conical limit point z. Moreover, this point is the unique element contained in the infinite intersection*

$$\bigcap_{i=1}^{\infty} \alpha_1 \cdots \alpha_n W(v_{n+1}).$$

Proof. For each n, let $g_n = \alpha_1 \cdots \alpha_n$. Since $\partial(\Gamma, \mathcal{P})$ is compact, there is at least one point z in the infinite intersection $\bigcap_{i=1}^\infty \alpha_1 \cdots \alpha_n W(v_{n+1})$. It follows from Lemma 4.12 that g_n lies within a uniform neighborhood of some ray in X whose ideal endpoint is z. By Lemma 4.11, the sequence g_n is unbounded in Γ, so z must be a conical limit point (see e.g. [BH20, Prop. A.2]), and z is uniquely determined by g_n. ☐

Next we want to prove a more specific version of Lemma 4.12, but we need an intermediate result first.

Lemma 4.14. *For any vertex v in $\mathcal{G}$, there exists a constant $C > 0$ so that, if, α is any element in T_v, and ℓ is a geodesic segment in $X = X(\Gamma, \mathcal{P})$ between id and α, then for any $R > 0$, the Hausdorff distance (with respect to d_X) between $\mathcal{N}_R(\ell; d_X) \cap \mathrm{Cay}(\Gamma)$ and $\{\mathrm{id}, \alpha\}$ is at most $C + 2R$.*

For the proof below, all distances are taken with respect to the metric d_X.

Proof. If v is not a parabolic vertex, then T_v is a singleton $\{\alpha_v\}$ and we can just take $C = |\alpha_v|_X$. So, suppose that v is parabolic. Then the associated parabolic point q_v has the form $q_v = gp$ for some $g \in \Gamma$ and $p \in \Pi$. Let P be the stabilizer in Γ of p and let $\mathcal{H}$ be the combinatorial horoball in X based at gP, so that any $\alpha \in T_v$ is in $\mathcal{H}$. Thus, if ℓ is a geodesic segment from id to α, the endpoints of ℓ lie within distance $|g|_X$ of $\mathcal{H}$.

Now, if $u \in \mathcal{N}_R(\ell; d_X) \cap \mathrm{Cay}(\Gamma)$, then there is some $w \in \ell$ with $d_X(w, u) \le R$, hence $d_X(w, \mathrm{Cay}(\Gamma)) \le R$. So by Lemma 2.4, we have $d_X(w, \{\mathrm{id}, \alpha\}) \le R + 3|g|_X + 7\delta$, and therefore $d_X(u, \{\mathrm{id}, \alpha\}) \le 2R + 3|g|_X + 7\delta$. ☐

Now we strengthen Lemma 4.12. Compare the proof of the below to the proof of [MMW24, Lem. 4.13].

Lemma 4.15. *There exists a constant $D > 0$ satisfying the following. Suppose that $\{(v_i, \alpha_i)\}_{i=1}^N$ is a (finite or infinite) $\mathcal{G}$-path, let $g_n = \alpha_1 \cdots \alpha_n$, and let $z \in \bigcap_{n=1}^N g_{n-1} W_{v_n}$. If $c : [0, \infty) \to X$ is a geodesic ray from* id *to z, then there is a subinterval $I \subseteq [0, \infty)$ containing 0 such that the set $\{g_n\}_{n=0}^{N-1}$ lies within Hausdorff distance D of $c(I) \cap \mathrm{Cay}(\Gamma)$, with respect to d_X.*

Proof. We know from Lemma 4.12 that there is a constant D_0 so that the set $\mathcal{S} = \{g_n\}_{n=0}^{N-1}$ is contained in a D_0-neighborhood of $c([0, \infty))$, so let $I \subseteq [0, \infty)$ be a minimal interval containing 0 such that a D_0-neighborhood of $c(I)$ also contains $\mathcal{S}$. That is, $\mathcal{N}_{D_0}(c(I); d_X) \cap \mathrm{Cay}(\Gamma)$ contains $\mathcal{S}$, so because of Lemma 2.5 we just need to check that $c(I) \cap \mathrm{Cay}(\Gamma)$ lies in a uniform neighborhood of $\mathcal{S}$.

For each $1 \leq n < N - 1$, let ℓ_n be a geodesic segment in X joining g_{n-1} to g_n, and let Y be the union $\bigcup_{n=1}^{N-1} \ell_n$. Each ℓ_n is contained in a $(D_0 + 2\delta)$-neighborhood of $c(I)$, meaning that Y is a connected subset also lying in a $(D_0 + 2\delta)$-neighborhood of $c(I)$.

Let u be a point in $c(I) \cap \mathrm{Cay}(\Gamma)$, and let $R = D_0 + 2\delta$. Now, if u lies within distance $2R$ of an endpoint of $c(I)$, then by minimality of I, u must also be within distance $2R + D_0$ of some g_n and we are done. So, we may assume the metric ball $\mathcal{B}_{2R}(u; d_X)$ separates $c(I)$ into two nonempty connected components B_-, B_+. Since c is a geodesic, we have

$$d_X(B_-, B_+) \geq 4R. \tag{6}$$

Suppose for a contradiction that $Y \cap \mathcal{B}_{3R}(u; d_X)$ is empty. Then for any $w \in \mathcal{B}_{2R}(u; d_X)$ and any $y \in Y$, we have $d_X(y, w) > 3R - 2R = R$. Thus, since Y is in an R-neighborhood of $B_- \cup \mathcal{B}_{2R}(u; d_X) \cup B_+$, for any $y \in Y$, there is some $w \in B_- \cup B_+$ with $d_X(y, w) \leq R$. That is, for any $y \in Y$, we have

$$d_X(y, c(I)) = \min(d_X(y, B_-), d_X(y, B_+)) \leq R.$$

Since Y contains id, and I was chosen minimally, there are points $y_-, y_+ \in Y$ so that $d_X(y_-, B_-) \leq R$ and $d_X(y_+, B_+) \leq R$. By (6), we have $d_X(y_-, B_+) \geq 3R$ and $d_X(y_+, B_-) \geq 3R$. Then since Y is connected, continuity of the distance function implies that there is some point $y \in Y$ so that $d_X(y, B_-) = d_X(y, B_+)$, hence $d_X(y, B_\pm) \leq R$ and $d_X(B_-, B_+) \leq 2R$. But this contradicts (6).

We conclude that there is some $y \in Y$ so that $d_X(y, u) \leq 3R$. Since $y \in \ell_n$ for some n, and $u \in \mathrm{Cay}(\Gamma)$ by assumption, we may apply Lemma 4.14 to see that there is a uniform constant C such that $d_X(u, \{g_{n-1}, g_n\}) \leq C + 6R$. Thus $c(I) \cap \mathrm{Cay}(\Gamma)$ lies in a $(C + 6R)$-neighborhood of $\mathcal{S}$, completing the proof. ∎

We can further strengthen the previous result so that it also applies to *uniform quasi-geodesic* rays. For this result and the next, no exact analogy appears in either [MMW24] or [Wei22].

Corollary 4.16. *For any $K \geq 1, A \geq 0$, there exists a constant $D > 0$ satisfying the following. For any $z \in \partial(\Gamma, \mathcal{P})$, any (finite or infinite) $\mathcal{G}$-path $\{(v_i, \alpha_i)\}_{i=1}^N$ limiting to z, and any (K, A)-quasi-geodesic ray $r : [0, \infty) \to X$ from the identity to z, the set $\{g_n = \alpha_1 \cdots \alpha_n\}_{n=0}^{N-1}$ is within Hausdorff distance D of $r([0, \infty)) \cap \mathrm{Cay}(\Gamma)$, with respect to d_X.*

Proof. We first claim that, if the $\mathcal{G}$-path $\{(v_i, \alpha_i)\}_{i=1}^N$ limits to $z \in \partial(\Gamma, \mathcal{P})$, then z lies in the intersection of sets

$$\bigcap_{n=1}^{N} g_{n-1} W_{v_n}.$$

If $N = \infty$ this follows from Lemma 4.13. If N is finite, then v_N is a parabolic vertex, and by condition (G5), we know that the corresponding parabolic point $q = q_{v_N}$ lies in W_{v_N}. Since the given $\mathcal{G}$-path limits to z, by definition we have $z = \alpha_1 \cdots \alpha_{N-1} q$ and therefore $z \in \alpha_1 \cdots \alpha_{N-1} W_N$.

Now fix a geodesic ray $c : [0, \infty) \to X$ from the identity to z. By Corollary 2.7 and the Morse lemma, it suffices to show that the Hausdorff distance between the sets $\mathcal{S} = \{g_n\}_{n=0}^{N-1}$ and $c([0, \infty)) \cap \mathrm{Cay}(\Gamma)$ is uniformly bounded. By Lemma 4.15, there is a constant D_0 and a sub-interval $I \subseteq [0, \infty)$ containing 0 so that $\mathcal{S}$ lies within Hausdorff distance D_0 of $c(I) \cap \mathrm{Cay}(\Gamma)$, so we will be done if we can show that the Hausdorff distance between $c(I) \cap \mathrm{Cay}(\Gamma)$ and $c([0, \infty)) \cap \mathrm{Cay}(\Gamma)$ is uniformly bounded.

If $I = [0, \infty)$ then there is nothing to prove, so assume that $I = [0, t]$ for some $t < \infty$. This can only occur if the sequence g_n is bounded in $\mathrm{Cay}(\Gamma)$, meaning (by Lemma 4.11) that N is finite and z is a parabolic point in the Γ-orbit of some $p \in \Pi$. Let P be the stabilizer of p in Γ. Since our $\mathcal{G}$-path $\{(v_i, \alpha_i)\}_{i=1}^{N}$ limits to z by assumption, z must be the center of the combinatorial horoball $\mathcal{H}$ based on the coset $g_{N-1}P$.

Fix $t' \leq t$ so that $d_X(g_{N-1}, c(t')) < D_0$, and suppose that $c(s) \in \mathrm{Cay}(\Gamma)$ for some $s > t$. Since c is asymptotic to z, there is some t'', much larger than s, so that $c(t'') \in \mathcal{H}$ and $d_X(c(s), c(t')) \leq d_X(c(s), c(t''))$. Then the sub-segment of c between $c(t')$ and $c(t'')$ has endpoints within distance D_0 of $\mathcal{H}$, so by Lemma 2.4, we have

$$d_X(c(s), c(t')) \leq 3D_0 + 7\delta.$$

This shows that any point in $c((t, \infty)) \cap \mathrm{Cay}(\Gamma)$ lies within distance $3D_0 + 7\delta$ of $c(I)$, meaning that $\mathcal{N}_{3D_0+7\delta}(c(I); d_X) \cap \mathrm{Cay}(\Gamma)$ contains $c([0, \infty)) \cap \mathrm{Cay}(\Gamma)$. Thus by Lemma 2.5 there is a uniform neighborhood of $c(I) \cap \mathrm{Cay}(\Gamma)$ containing $c([0, \infty)) \cap \mathrm{Cay}(\Gamma)$, and therefore a uniform bound on the Hausdorff distance between $c([0, \infty)) \cap \mathrm{Cay}(\Gamma)$ and $c(I) \cap \mathrm{Cay}(\Gamma)$, as desired. ☐

4.3. $\mathcal{G}$-paths close to $\mathrm{Cay}(\Gamma)$.

The final lemma in this section says that, when the elements α_i appearing in a $\mathcal{G}$-path are all short, then any geodesic in X "tracking" the $\mathcal{G}$-path does not enter deeply into any horoball.

Lemma 4.17. *Let $\{(v_i, \alpha_i)\}_{i=1}^{\infty}$ be an infinite $\mathcal{G}$-path, and assume that the quantity $C = \max_{i \geq 1}\{|\alpha_i|_X\}$ is finite. Let $c : [0, \infty) \to X$ be a geodesic ray based at the identity, and suppose that the d_X-Hausdorff distance between $c([0, \infty)) \cap \mathrm{Cay}(\Gamma)$ and the set $\{g_n = \alpha_1 \cdots \alpha_n\}$ is at most D. Then $c([0, \infty))$ lies in a $(C + 3D)$-neighborhood of $\mathrm{Cay}(\Gamma)$, with respect to d_X.*

Proof. Since the $\mathcal{G}$-path is infinite, it follows from Lemma 4.13 that the diameter of $c([0, \infty)) \cap \mathrm{Cay}(\Gamma)$ is infinite. So, it suffices to show that for any $x \in c([0, \infty)) \cap \mathrm{Cay}(\Gamma)$, the sub-segment of c between id and x lies in a $(C + 3D)$-neighborhood of $\mathrm{Cay}(\Gamma)$.

Fix g_n so that $d_X(x, g_n) \leq D$. Then for each index $1 \leq j \leq n$, there is some point x_j on c so that $d_X(x_j, g_j) \leq D$. For every $1 \leq j < n$, we have $d_X(g_j, g_{j+1}) \leq C$ by assumption, meaning $d_X(x_j, x_{j+1}) \leq C + 2D$. Thus, the ball of radius $C + 3D$ centered at g_j contains the entire sub-segment of c between x_j and x_{j+1}. The union of all of these segments contains the segment of c between id and x, so this segment is in the $(C + 3D)$-neighborhood of $\mathrm{Cay}(\Gamma)$, as required. ☐

5. Extended Dehn filling

The aim of this section of the paper is to prove our main Dehn filling theorem, Theorem 1.4. At the same time, we will prove that the Dehn fillings given by Theorem 1.4 satisfy several additional desirable properties; in particular, we will show that the boundary sets of Dehn fillings vary (semi)continuously, and that, with a mild additional assumption on boundary extensions, convergence to an EGF representation ρ in an extended Dehn filling space is *strong*.

For the rest of the section, fix a non-elementary relatively hyperbolic pair $(\Gamma, \mathcal{P})$, with $\mathcal{P}$ a non-empty collection peripheral subgroups. We will assume that each group in $\mathcal{P}$ is infinite, but later in the section we will consider other relatively hyperbolic pairs where the peripheral subgroups are *not* necessarily infinite.

Let Q be a symmetric parabolic subgroup in a semisimple Lie group G, and let $\rho : \Gamma \to G$ be a Q-EGF representation with boundary extension $\phi : \Lambda \to \partial(\Gamma, \mathcal{P})$ and repelling strata $\{C_z\}_{z \in \partial(\Gamma, \mathcal{P})}$. For simplicity we will assume that ρ is faithful.

We will prove Proposition 5.1 below, which is a strengthening of Theorem 1.4. For the statement of the proposition, recall from the introduction that any representation $\sigma : \Gamma \to G$ determines a collection of (possibly trivial) Dehn filling kernels $\{\ker(\sigma|_P)\}_{P \in \mathcal{P}}$. There is an associated Dehn filling $\pi^\sigma : \Gamma \to \Gamma^\sigma$ and collection of groups $\mathcal{P}^\sigma = \{P/\ker(\sigma|_P)\}_{P \in \mathcal{P}}$.

Proposition 5.1. *Let $W \subseteq \mathrm{Hom}_{\mathrm{geom}}(\Gamma, G; \mathcal{P})$ be an extended Dehn filling space, and fix a constant $\varepsilon > 0$ and a compact subset $Z \subseteq \partial(\Gamma, \mathcal{P})$.*

Then there exists an open subset $O \subseteq \mathrm{Hom}_{\mathrm{geom}}(\Gamma, G; \mathcal{P})$ containing ρ so that, for every representation $\sigma \in O \cap W$, the pair $(\Gamma^\sigma, \mathcal{P}^\sigma)$ is relatively hyperbolic, and σ descends to a Q-EGF representation of Γ^σ. Moreover, the boundary set Λ_σ for σ is contained in an ε-neighborhood of Λ, and has nonempty intersection with the ε-neighborhood of $\phi^{-1}(Z)$.

The main ingredient of the proof of Proposition 5.1 will be the relative automaton $\mathcal{G}$ and a certain $\mathcal{G}$-compatible system $\mathcal{U} = \{U_v\}_{v \in V(\mathcal{G})}$ of open subsets of G/Q, as described in the previous section. The general idea is that the automaton $\mathcal{G}$ and system $\mathcal{U}$ give a "relatively locally finite" encoding of the extended convergence action of Γ on G/Q, in the sense that the data of the extended convergence action can be (partially) recovered from $\mathcal{G}$ and $\mathcal{U}$. We will be able to show that, if we perturb ρ in a manner which roughly preserves $\mathcal{G}$-compatibility of the system, we then get a new extended convergence action, hence a new EGF representation.

The outline here is similar to the proof of the special case of this theorem appearing in [Wei22]. The main conceptual difference is that here, we allow perturbations of ρ which do not actually preserve $\mathcal{G}$-compatibility of the system $\mathcal{U}$. Instead, we only ask for $\mathcal{G}$-compatibility to hold along paths which avoid the nontrivial parts of Dehn filling kernels.

The (semi)continuity properties of the set Λ expressed by the "moreover" part of Proposition 5.1 follow from the fact that the union of vertex sets $\bigcup_{v \in V(\mathcal{G})} U_v$ is an "approximation" of the boundary set Λ, both for the original representation ρ and its perturbations. So, we can guarantee that Λ_σ is close to Λ by picking all of the sets U_v to be small neighborhoods of $\phi^{-1}(z)$ for some $z \in \partial(\Gamma, \mathcal{P})$.

During the course of the proof of Proposition 5.1, we will also show the following:

Proposition 5.2. *Let $W \subseteq \mathrm{Hom}_{\mathrm{geom}}(\Gamma, G; \mathcal{P})$ be an extended Dehn filling space, and assume that for every parabolic point $q \in \partial(\Gamma, \mathcal{P})$, the set $\phi^{-1}(q)$ has empty interior as a subset of G/Q. If a sequence $\rho_n \in W$ converges* relatively strongly *to ρ (i.e. in $\mathrm{Hom}_{\mathrm{geom}}(\Gamma, G; \mathcal{P})$), then ρ_n converges strongly to ρ (i.e. in $\mathrm{Hom}_{\mathrm{geom}}(\Gamma, G)$).*

Remark 5.3. There exist examples of boundary extensions for EGF representations where the fiber over a parabolic point has *nonempty* interior; see [GW24, Example 2.6]. These boundary extensions do not satisfy the hypotheses of Proposition 5.2. However, in the cited examples, there is a different choice of boundary extension where each fiber *does* have empty interior, and it seems likely to us that it is always possible to choose such a boundary extension for any EGF representation.

5.1. **Choosing a neighborhood in the Dehn filling space.** We fix once and for all an extended Dehn filling space W about the EGF representation $\rho : \Gamma \to G$. To choose the open neighborhood O in $\mathrm{Hom}_{\mathrm{geom}}(\Gamma, G; \mathcal{P})$ as in Proposition 5.1, we need to construct a relative automaton for ρ in a way which depends on the coarse metric geometry of the pair $(\Gamma, \mathcal{P})$, as well as the parameters ε and Z appearing in the statement of the proposition. So, for the rest of the section, let $X = X(\Gamma, \mathcal{P})$ be the cusped space for $(\Gamma, \mathcal{P})$, and fix a compact set $Z \subseteq \partial(\Gamma, \mathcal{P})$ and a constant $\varepsilon > 0$.

Before defining the automaton, we need to set up a few more auxiliary parameters.

Definition 5.4. Recall that we have fixed a metric d_∂ on the Bowditch boundary $\partial(\Gamma, \mathcal{P})$. Define the quantity $\Delta > 0$ by:

$$\Delta := \min\{d_\partial(z_1, z_2) : z_1, z_2 \in \partial(\Gamma, \mathcal{P}) \text{ are joined by a geodesic in } X \text{ through id}\}.$$

For each point $p \in \Pi$, we also pick a compact set $K_p \subset \partial(\Gamma, \mathcal{P}) \setminus \{p\}$ as follows:

Definition 5.5. For each $p \in \Pi$, let $K_p \subset \partial(\Gamma, \mathcal{P})$ be the set

$$\{z \in \partial(\Gamma, \mathcal{P}) : \text{there exists a geodesic through id in } X \text{ joining } z \text{ to } p\}.$$

Note that K_p is compact because X is locally compact. Moreover, since every point in $\partial(\Gamma, \mathcal{P}) \setminus \{p\}$ is joined to p by a geodesic passing through the boundary of the combinatorial horoball with ideal boundary $\{p\}$, we have $\mathrm{Stab}_\Gamma(p) \cdot K_p = \partial(\Gamma, \mathcal{P}) \setminus \{p\}$.

Definition 5.6 (The relative automaton)**.** Let $\mathcal{G}$ be a relative automaton for $(\Gamma, \mathcal{P})$, and let $\mathcal{W} = \{W_v\}_{v \in V(\mathcal{G})}$ and $\mathcal{U} = \{U_v\}_{v \in V(\mathcal{G})}$ be $\mathcal{G}$-compatible systems of subsets adapted to the EGF representation ρ, satisfying each of the properties (G1)-(G5) in Proposition 4.5. The constant ε in condition (G2) is chosen to be smaller than our fixed $\varepsilon > 0$.

Shrinking ε if necessary, we also ensure that $\mathcal{G}$, $\mathcal{W}$, and $\mathcal{U}$ satisfy conditions (G6)-(G7). We additionally ensure that the relative automaton satisfies condition (G8) (with respect to the constant Δ in Definition 5.4), condition (G9) (with respect to the sets K_p in Definition 5.5), and condition (G10) (with respect to our fixed $\varepsilon > 0$ and compact set Z).

The automaton $\mathcal{G}$ and the systems $\mathcal{W}, \mathcal{U}$ determine a few additional constants, which we will use to define a neighborhood in the extended Dehn filling space W.

Definition 5.7. Define a constant C_1 by:

$$C_1 := \max_{v \in V(\mathcal{G})} \min_{\alpha \in T_v} |\alpha|_X.$$

Recall that for each parabolic vertex $v \in V(\mathcal{G})$, the label set T_v lies in a unique coset gP for $g \in \Gamma$ and $P \in \mathcal{P}$. For each of these vertices, fix such a $g = g_v$ to minimize $|g_v|_X$, and define

$$C_2 := \max\{|g_v|_X : v \in V(\mathcal{G}) \text{ parabolic}\}.$$

Finally, recall that for each parabolic vertex v, the set $F_v = gP \setminus T_v$ is finite. Define

$$C_3 := \max\{|\gamma|_X : \gamma \in F_v \text{ for some parabolic } v \in V(\mathcal{G})\}.$$

Now, fix $\delta > 0$ so that the cusped space $X = X(\Gamma, \mathcal{P})$ is δ-hyperbolic, and so that (by Proposition 2.13) the cusped space for any sufficiently long Dehn filling of $(\Gamma, \mathcal{P})$ is also δ-hyperbolic. For convenience, assume that $\delta > 1$ throughout.

Definition 5.8. Let $D > 0$ be a constant satisfying the following: if c is any $(2, 18\delta)$-quasi-geodesic ray in X from id to $z \in \partial(\Gamma, \mathcal{P})$, and $\{(v_i, \alpha_i)\}_{i=1}^N$ is a $\mathcal{G}$-path limiting to z, then the set of elements

$$g_n = \alpha_1 \cdots \alpha_n$$

lies within Hausdorff distance D of $c([0,\infty)) \cap \mathrm{Cay}(\Gamma)$, with respect to the metric d_X.

Note that such a D exists because of Corollary 4.16. The specific quasi-geodesic constants $(2, 18\delta)$ we have chosen will arise in later arguments.

As noted previously, any representation $\sigma : \Gamma \to G$ determines a Dehn filling $\pi^\sigma : \Gamma \to \Gamma^\sigma$ and a collection of subgroups $\mathcal{P}^\sigma = \{P/\ker(\sigma|_P)\}_{P \in \mathcal{P}}$. The kernel π^σ is a subgroup of $\ker(\sigma)$, so we will view σ as a representation of both Γ and Γ^σ.

Definition 5.9 (Metrics determined by a representation $\Gamma \to G$)**.** Recall that we have fixed a finite generating set S for Γ. For any representation $\sigma : \Gamma \to G$, and any $\gamma \in \Gamma^\sigma$, let $|\gamma|_\Gamma^\sigma$ denote the word length of γ with respect to the generating set $\pi^\sigma(S)$. Let d_Γ^σ denote the word metric $d_\Gamma^\sigma(\gamma_1, \gamma_2) = |\gamma_1^{-1}\gamma_2|_\Gamma^\sigma$. When $\gamma \in \Gamma$, we will usually write $|\gamma|_\Gamma^\sigma$ for $|\pi^\sigma(\gamma)|_\Gamma^\sigma$, and similarly for d_Γ^σ.

Let X^σ denote the cusped space $X(\Gamma^\sigma, \mathcal{P}^\sigma)$; recall from Section 2.6.1 that we view this as a quotient of the cusped space X by the action of the kernel of the map $\Gamma \to \Gamma^\sigma$. Let d_X^σ denote the path metric on X^σ (which restricts to a metric on Γ^σ), and let $|\gamma|_X^\sigma = d_X^\sigma(\mathrm{id}, \gamma)$.

We are now ready to describe the open neighborhood of σ in $\mathrm{Hom}_{\mathrm{geom}}(\Gamma, G; \mathcal{P})$ appearing in Proposition 5.1.

Proposition 5.10. *There exists an open subset $O \subset \mathrm{Hom}_{\mathrm{geom}}(\Gamma, G; \mathcal{P})$ containing ρ and a constant $r > 0$ so that every $\sigma \in O \cap W$ satisfies all of the following conditions:*

- *(O1) The cusped space X^σ is δ-hyperbolic (so $(\Gamma^\sigma, \mathcal{P}^\sigma)$ is a relatively hyperbolic pair). Further, if c is any geodesic in X lying in a $(C_1 + 3D)$-neighborhood of $\mathrm{Cay}(\Gamma)$, then its image under the quotient map $X \to X^\sigma$ is a $(2, 18\delta)$-quasi-geodesic.*
- *(O2) For every non-parabolic vertex v in the relative quasi-geodesic automaton $\mathcal{G}$, and every edge $v \to w$ in $\mathcal{G}$, we have*

$$\sigma(\alpha_v) \cdot \mathcal{N}_r(U_w) \subset U_v.$$

- *(O3) For every parabolic vertex v in the automaton $\mathcal{G}$, if $\alpha \in T_v$ satisfies*

$$|\alpha|_X \le 2C_2 + 4C_3 + 6D + 18\delta,$$

then for every edge $v \to w$ in $\mathcal{G}$, we have

$$\sigma(\alpha) \cdot \mathcal{N}_r(U_w) \subset U_v.$$

- *(O4) For every parabolic vertex v in the automaton $\mathcal{G}$, with corresponding parabolic point $q_v = gp$ for $p \in \Pi$, and every edge $v \to w$ in $\mathcal{G}$, we have*

$$(\sigma(g\Gamma_p) \setminus \sigma(F_v)) \cdot \mathcal{N}_r(U_w) \subset U_v.$$

Remark 5.11. The quantity $C_1 + 3D$ in (O1) is motivated by Lemma 4.17. The inequality in (O3) will arise later, in the proof of Lemma 5.12.

Proof of Proposition 5.10. First, note that when $\sigma = \rho$, condition (O1) holds by assumption, since ρ is faithful and thus $X = X^\rho$. In this case the existence of an r so that the other three conditions hold is equivalent to the $\mathcal{G}$-compatibility of the system $\mathcal{U}$ for the ρ-action of Γ on G/Q. So we need to show that all four of these conditions are relatively open in W, in the topology of relative strong convergence.

Condition (O2) only refers to finitely many elements of Γ, so it is already open in $\mathrm{Hom}(\Gamma, G)$; by Lemma 2.3, the same is true for Condition (O3). From Proposition 2.13 and Proposition 2.16, the conditions in (O1) hold whenever X^σ is the cusped space associated to a sufficiently long filling of Γ. Thus this condition holds as long as the nontrivial elements in $\ker(\sigma)$ avoid a fixed finite subset of Γ. We assumed that ρ was faithful, so this also corresponds to finitely many conditions on σ which are open in $\mathrm{Hom}(\Gamma, G)$.

On the other hand, the final condition (O4) is *not* necessarily open in $\mathrm{Hom}(\Gamma, G)$, but it *is* relatively open in the extended Dehn filling space $W \subset \mathrm{Hom}_{\mathrm{geom}}(\Gamma, G; \mathcal{P})$, essentially by definition. □

5.2. **Compatibility of the deformed automaton along lifts.** For the rest of the section, fix a neighborhood $O \subset \mathrm{Hom}_{\mathrm{geom}}(\Gamma, G; \mathcal{P})$ as in Proposition 5.10. The lemma below is in some sense the key step in the proof of Proposition 5.1: it says that, if σ is any deformation of ρ in $O \cap W$, then for certain special $\mathcal{G}$-paths, the $\mathcal{G}$-compatibility condition is preserved for the action of σ on G/Q.

We should *not* expect the $\mathcal{G}$-compatibility condition to be preserved for *arbitrary* $\mathcal{G}$-paths, since we need to worry about elements in the kernel of σ appearing along the path. However, the lemma below shows (roughly) that if we only consider $\mathcal{G}$-paths which stay close to *lifts* of quasi-geodesic paths in X^σ, then each element appearing in the path is uniformly far from the nontrivial part of the kernel of σ, and so $\mathcal{G}$-compatibility still holds. (Recall the definition of a *lift* in X of a path in X^σ from the end of Section 2.6.)

Lemma 5.12. *Let $\sigma \in W \cap O$, let $I \subseteq [0, \infty)$ be an interval, and let $c : I \to X^\sigma$ be a $(2, 18\delta)$-quasi-geodesic path baseed at* id *in X^σ with a lift $\tilde{c}$ based at* id *in X. Let $\{(v_i, \alpha_i)\}_{i=1}^N$ be a (finite or infinite) $\mathcal{G}$-path so that the sequence $g_n = \alpha_1 \cdots \alpha_n$ lies in a D-neighborhood of $\tilde{c}(I) \cap \mathrm{Cay}(\Gamma)$.*

Then, for any $1 \leq i < N$, we have

$$\sigma(\alpha_i) \cdot \mathcal{N}_r(U_{v_{i+1}}) \subset U_{v_i}. \tag{7}$$

Proof. Fix $1 \leq i < N$. Condition (O2) on our open subset O implies that the desired inclusion holds whenever v_i is a non-parabolic vertex. So, assume that v_i is parabolic.

Fix $\tilde{a}_j, \tilde{a}_k \in \Gamma$ lying along $\tilde{c}$ so that $d_X(\tilde{a}_j, g_i) \leq D$ and $d_X(\tilde{a}_k, g_{i+1}) \leq D$, and let $a_j = \pi^\sigma(\tilde{a}_j)$, $a_k = \pi^\sigma(\tilde{a}_k)$. Then, since π^σ is 1-Lipschitz with respect to the metrics d_X, d_X^σ, we have

$$\begin{aligned} d_X^\sigma(a_j, a_k) &\leq d_X^\sigma(g_i, g_{i+1}) + 2D \\ &= |\alpha_i|_X^\sigma + 2D. \end{aligned}$$

Now, if $\sigma(\alpha_i) \notin \sigma(F_{v_i})$, condition (O4) on O implies that the desired inclusion (7) holds. So assume that $\sigma(\alpha_i) = \sigma(f)$ for some $f \in F_{v_i}$. By way of Definition 5.7, we can write $\alpha_i = gh_i$ and $f = gh$, where h_i, h both lie in some parabolic subgroup $P \in \mathcal{P}$, and $g \in \Gamma$ satisfies $|g|_X \leq C_2$. We must have $\sigma(h_i) = \sigma(h)$, so $|h_i|_X^\sigma = |h|_X^\sigma$. Therefore,

$$\begin{aligned} |\alpha_i|_X^\sigma &\leq |g|_X^\sigma + |h_i|_X^\sigma = |h|_X^\sigma + |g|_X^\sigma \\ &\leq |gh|_X^\sigma + 2|g|_X^\sigma \\ &\leq |gh|_X + 2|g|_X \leq C_2 + 2C_3. \end{aligned}$$

This means that

$$d_X^\sigma(a_j, a_k) \leq C_2 + 2C_3 + 2D.$$

Since c is a $(2, 18\delta)$-quasi-geodesic path and $\tilde{c}$ is a lift of c, this implies

$$d_X(\tilde{a}_k, \tilde{a}_{k'}) \le 2C_2 + 4C_3 + 4D + 18\delta,$$

and therefore

$$|\alpha_i|_X = d_X(g_i, g_{i+1}) \le 2C_2 + 4C_3 + 6D + 18\delta.$$

Then we are done after applying condition (O3). □

5.3. **Strong convergence.** With the setup above, we are in position to prove Proposition 5.2, which says that convergence to ρ in $\mathrm{Hom}_{\mathrm{geom}}(\Gamma, G; \mathcal{P})$ (*relative* strong convergence) implies convergence to ρ in $\mathrm{Hom}_{\mathrm{geom}}(\Gamma, G)$ (strong convergence).

We first prove the following lemma, which will also be useful for the proof of Proposition 5.1.

Lemma 5.13. *Let ρ_n be a sequence in $O \cap W$, and let γ_n be a sequence in Γ. If the sequence of relative word-lengths $|\gamma_n|_{\hat{\Gamma}}^{\rho_n}$ tends to infinity, then $\rho_n(\gamma_n)$ is Q-divergent.*

In addition, if there is a sequence of ρ_n-invariant sets $\Lambda_n \subset G/Q$ such that $\Lambda_n \cap U_v \ne \emptyset$ for each $v \in V(\mathcal{G})$, then every Q-limit point of $\rho_n(\gamma_n)$ is an accumulation point of some sequence $\xi_n \in \Lambda_n$.

Proof. It suffices to show that every subsequence of $\gamma_n, \rho_n, \Lambda_n$ has a further subsequence which satisfies the desired statement, so we can extract subsequences throughout the proof below.

Let X_n be the sequence of cusped spaces X^{ρ_n}, and let d_X^n denote the induced metric $d_X^{\rho_n}$ on the quotient $\Gamma_n = \Gamma^{\rho_n}$. Condition (O1) ensures that each of these cusped spaces is δ-hyperbolic, so by Proposition 2.8, there is a geodesic ray $c_n : [0, \infty) \to X_n$ and $t_n \in [0, \infty)$ so that $d_X^n(c_n(t_n), \gamma_n) < 8 + 21\delta$.

Let $\tilde{c}_n$ be a lift of c_n in X based at the identity. The quotient map $X \to X_n$ defined in Section 2.6.1 takes combinatorial horoballs to combinatorial horoballs, so by lifting a geodesic in X_n from $c(t_n)$ to γ_n, we see that the point $\tilde{c}_n(t_n)$ lies d_X^n-distance at most $8 + 21\delta$ from some $\tilde{\gamma}_n \in \Gamma$, satisfying $\pi_n(\tilde{\gamma}_n) = \gamma_n$. Since the quotient $\Gamma \to \Gamma_n$ also induces a 1-Lipschitz map $\widehat{\mathrm{Cay}}(\Gamma) \to \widehat{\mathrm{Cay}}(\Gamma_n)$, the relative word lengths $|\tilde{\gamma}_n|_{\hat{\Gamma}}$ must tend to infinity.

For each n, let $\{(v_i^{(n)}, \alpha_i^{(n)})\}_{i=1}^N$ be a $\mathcal{G}$-path limiting to the endpoint of $\tilde{c}_n$ in $\partial(\Gamma, \mathcal{P})$, and for each finite $m \le N$ define

$$g_m^{(n)} = \alpha_1^{(n)} \cdots \alpha_m^{(n)}.$$

By Lemma 2.5 and Corollary 4.16, for each n, we can find some index $m_n < \infty$ so that the element $g_{m_n}^{(n)}$ lies within uniformly bounded $\mathrm{Cay}(\Gamma)$-distance of $\tilde{\gamma}_n$. Let $h_n = g_{m_n}^{(n)}$. We must have $|h_n|_{\hat{\Gamma}} \to \infty$, and since each $\alpha_i^{(n)}$ has uniformly bounded length in $\widehat{\mathrm{Cay}}(\Gamma)$, we have $m_n \to \infty$.

Now, after extracting a subsequence, we can assume that the vertices $v_1^{(n)}, v_{m_n+1}^{(n)}$ are both fixed (independent of n), respectively equal to vertices $v, w \in V(\mathcal{G})$. Consider the sequence of sets $\rho_n(h_n)U_w$. We have chosen the constant D (in Definition 5.8) in a way which ensures that the sequence h_n lies within Hausdorff distance D of $\tilde{c}_n \cap \mathrm{Cay}(\Gamma)$, so Lemma 5.12 ensures that the nesting condition

$$\rho_n(\alpha_i^{(n)})\mathcal{N}_r(U_{v_{i+1}^{(n)}}) \subset U_{v_i^{(n)}}$$

is satisfied for every $1 \le i \le m_n$. We also know from property (G7) that, if r is sufficiently small, each vertex set U_v satisfies $\mathrm{Opp}(\mathcal{N}_r(U_v)) \ne \emptyset$. Then, it follows from [Wei22, Cor. 712] that, for a particular choice of metric on U_{v_1} (compatible with the topology on G/Q), the diameter of the set $\rho_n(h_n)U_w$ tends to zero uniformly exponentially in m_n. In particular, since $\overline{U_1}$ is compact, these sets tend to a singleton $\{\xi\} \subset G/Q$. Since U_w is open, [Wei22, Prop. 3.7] shows that $\rho_n(h_n)$ is Q-divergent and ξ is the unique Q-limit point of the sequence. Since the Cay(Γ)-distance between h_n and $\tilde{\gamma}_n$ is uniformly bounded, [KLP17, Lem. 4.23(i)] implies that $\rho_n(\gamma_n) = \rho_n(\tilde{\gamma}_n)$ is also Q-divergent with unique Q-limit point ξ.

To finish the proof, we just need to show that, if $\Lambda_n \subset G/Q$ is a sequence of subsets as in the hypotheses, then ξ is an accumulation point of some sequence $\xi_n \in \Lambda_n$. However, we know that ξ is the limit of *any* sequence $\rho_n(h_n)\eta_n$ for any sequence $\eta_n \in U_1$. Since Λ_n is ρ_n-invariant and $\Lambda_n \cap U_1 \neq \emptyset$ by assumption, the result follows. □

Remark 5.14. Applying Lemma 5.13 to a constant sequence of representations shows that any representation in $O \cap W$ is relatively Q-divergent. In particular, this proves that ρ is relatively Q-divergent.

The next lemma says (modulo an additional assumption on the boundary extension ϕ) that the representations $\sigma \in O \cap W$ induce "uniformly proper" representations $\Gamma^\sigma \to G$. It implies in particular that every representation in $O \cap W$ is discrete.

Lemma 5.15. *Assume that for each parabolic point $q \in \partial(\Gamma, \mathcal{P})$, the set $\phi^{-1}(q)$ has empty interior. Let ρ_n be a sequence in $O \cap W$, and for each n, let $\gamma_n \in \Gamma^{\rho_n}$. If $|\gamma_n|_\Gamma^{\rho_n} \to \infty$, then $\rho_n(\gamma_n)$ leaves every compact subset of G.*

Proof. Since Q-divergent sequences necessarily leave every compact subset of G, by employing the previous lemma, we only need to consider the case where the relative lengths $|\gamma_n|_{\hat{\Gamma}}^{\rho_n}$ are bounded.

Let $\Gamma_n = \Gamma^{\rho_n}$, and let $\pi_n : \Gamma \to \Gamma_n$ be the quotient map. After extracting a subsequence, there is a fixed integer k and lifts $\tilde{\gamma}_n \in \Gamma$ of γ_n with the form

$$\tilde{\gamma}_n = \tilde{g}_1 \tilde{h}_1^{(n)} \ldots \tilde{g}_k \tilde{h}_k^{(n)} \tilde{g}_{k+1}, \tag{8}$$

where each $\tilde{g}_i \in \Gamma$ is fixed, and each $\tilde{h}_i^{(n)}$ is an element in some fixed $P_i \in \mathcal{P}$, such that $\pi_n(\tilde{h}_i^{(n)}) = h_i^{(n)}$ satisfies

$$\lim_{n\to\infty} |h_i^{(n)}|_\Gamma^{\rho_n} = \infty.$$

For simplicity we will assume $\tilde{g}_{k+1} = \mathrm{id}$, since the general case follows easily from this one. Let $p_i \in \Pi$ be the parabolic point fixed by P_i.

Note that for each $1 < j \leq k$, the fact that $|h_j^{(n)}|_\Gamma^{\rho_n} \to \infty$ as $n \to \infty$ means that for any finite subset $F \subset P_j$, we eventually have $\rho_n(h_j^{(n)}) \notin \rho_n(F)$. So, due to the extended Dehn filling assumption (Definition 3.14), for each j, if K is a compact subset of C_{p_j}, then the set $\rho_n(h_j^{(n)})K$ eventually lies in an arbitrarily small neighborhood of $\phi^{-1}(p_j)$. Since $\rho_n \to \rho$ in $\mathrm{Hom}(\Gamma, G)$, we also know that as $n \to \infty$, the set $\rho_n(\tilde{g}_j \tilde{h}_j^{(n)})K$ eventually lies in an arbitrarily small neighborhood of $\rho(\tilde{g}_j)\phi^{-1}(p_j)$.

We can induct on j to show that for any compact subset $K \subset C_{p_j}$, the set

$$\rho_n(\tilde{g}_1 \tilde{h}_1^{(n)} \cdots \tilde{g}_j \tilde{h}_j^{(n)})K$$

eventually lies in an arbitrarily small neighborhood of $\phi^{-1}(\tilde{g}_1 p_1)$. The case $j = 1$ is immediate from the previous paragraph. For the case $j > 1$, first observe that since $\tilde{g}_j p_j \neq p_{j-1}$, the group element $\rho(\tilde{g}_j)$ takes (the closure of) a neighborhood of $\phi^{-1}(p_j)$ to $C_{p_{j-1}}$. Again using algebraic convergence $\rho_n \to \rho$, the sequence $\rho_n(\tilde{g}_j)$ eventually takes some neighborhood of $\phi^{-1}(p_j)$ to land in a compact subset of $C_{p_{j-1}}$. Applying induction, we see that

$$\rho_n(\tilde{g}_1\tilde{h}_1^{(n)} \cdots \tilde{g}_j\tilde{h}_j^{(n)})K$$

eventually lies in an arbitrary neighborhood of $\phi^{-1}(\tilde{g}_j p_1)$.

Now, since C_{p_k} is an open subset of the flag manifold G/Q, it contains a compact subset K with nonempty interior. Suppose for a contradiction that $\rho_n(\gamma_n)$ does not leave every compact subset of G. Then, after extraction, $\rho_n(\gamma_n)$ converges to some $g \in G$, and $\rho_n(\gamma_n)K$ converges (with respect to Hausdorff distance on G/Q) to gK, which is a subset of G/Q with nonempty interior. However, from the argument above, we know that $\rho_n(\gamma_n)K$ eventually lies in every neighborhood of $\phi^{-1}(\tilde{g}_1 p_1)$, and therefore $gK \subseteq \phi^{-1}(\tilde{g}_1 p_1)$. This is a contradiction if $\phi^{-1}(\tilde{g}_1 p_1)$ has empty interior. ☐

Proof of Proposition 5.2. Suppose that a sequence $\rho_n \in W$ converges to ρ in $\mathrm{Hom}_{\mathrm{geom}}(\Gamma, G; \mathcal{P})$. We need to show that also $\rho_n \to \rho$ in $\mathrm{Hom}_{\mathrm{geom}}(\Gamma, G)$, or equivalently that the subgroups $\rho_n(\Gamma)$ converge to $\rho(\Gamma)$ in the Chabauty topology. We will employ the criteria in Proposition 3.11. Since $\rho_n \to \rho$ algebraically, criterion (A) in the proposition is immediate. For criterion (B), consider a sequence $\gamma_n \in \Gamma^{\rho_n}$ such that $\rho(\gamma_n) \to g$ for $g \in G$. By Lemma 5.15, the sequence $|\gamma_n|_\Gamma^{\rho_n}$ is bounded, so there is a sequence $\tilde{\gamma}_n \in \Gamma$ with $|\tilde{\gamma}_n|_\Gamma$ bounded, projecting to γ_n under the Dehn filling $\Gamma \to \Gamma^{\rho_n}$. After extraction $\tilde{\gamma}_n$ is a constant $\tilde{\gamma}$, so $\rho_n(\gamma_n) = \rho_n(\tilde{\gamma})$ converges to $\rho(\tilde{\gamma})$, and criterion (B) is satisfied. ☐

5.4. **Lifting points in boundaries of Dehn fillings.** For the rest of the section, we fix a representation $\sigma \in O \cap W$. Let $\mathcal{P}^\sigma_\infty$ denote the set of infinite subgroups in $\mathcal{P}^\sigma$ (note that this may be *smaller* than the set of images of groups in $\mathcal{P}_\infty$ in Γ^σ, since $\ker(\sigma|_P)$ could have finite index in some $P \in \mathcal{P}_\infty$). As in Section 2.3, we will identify the Bowditch boundary $\partial(\Gamma^\sigma, \mathcal{P}^\sigma_\infty)$ with a compact Γ^σ-invariant subset of $\partial(\Gamma^\sigma, \mathcal{P}^\sigma)$, which is the boundary of the cusped space $X^\sigma = X(\Gamma^\sigma, \mathcal{P}^\sigma)$.

Our aim now is to show that σ induces an EGF representation of Γ^σ. This means we need to construct a boundary extension $\phi_\sigma : \Lambda_\sigma \to \partial(\Gamma^\sigma, \mathcal{P}^\sigma_\infty)$ for σ, and then show that ϕ_σ satisfies all of the needed properties.

For the most part our strategy follows the proof of the main theorem in [Wei22], which is a special case of the result we are proving now. However, here we have a new technical challenge, since the Bowditch boundary $\partial(\Gamma^\sigma, \mathcal{P}^\sigma_\infty)$ is *not* necessarily homeomorphic to $\partial(\Gamma, \mathcal{P})$. In a sense $\partial(\Gamma^\sigma, \mathcal{P}^\sigma_\infty)$ is "smaller," since the quotient group Γ^σ contains "fewer" geodesic rays, but there is not a nice equivariant quotient map from a subset of $\partial(\Gamma, \mathcal{P})$ to $\partial(\Gamma^\sigma, \mathcal{P}^\sigma_\infty)$. However, the lifting operation on (quasi)-geodesic rays in X^σ (see Section 2.6.1) allows us to loosely associate points in $\partial(\Gamma^\sigma, \mathcal{P}^\sigma)$ to points in $\partial(\Gamma, \mathcal{P})$.

Definition 5.16. Let $z \in \partial(\Gamma^\sigma, \mathcal{P}^\sigma)$, and let $K \geq 1, A \geq 0$. A point $\tilde{z}$ is a (K, A)-*lift* of z if there is a (K, A)-quasi-geodesic ray $c : [0, \infty) \to X^\sigma$ from the identity to z and a lift $\tilde{c} : [0, \infty) \to X$ of c from the identity to $\tilde{z}$.

We say that $\tilde{z}$ is a *lift* of z if it is a (K, A)-lift of z for some K, A. If $\tilde{z}$ is a $(1, 0)$-lift of z then we say it is a *geodesic lift*.

Remark 5.17. Since every geodesic in X^σ lifts, every point $z \in \partial(\Gamma^\sigma, \mathcal{P}^\sigma)$ has at least one geodesic lift in $\partial(\Gamma, \mathcal{P})$; some points in $\partial(\Gamma^\sigma, \mathcal{P}^\sigma)$ may have a unique geodesic lift, while others may have many. The lifting operation also does *not* need to be equivariant in any sense. Moreover, if K', A' are different from K, A, then the set of (K, A)-lifts of a point $z \in \partial(\Gamma^\sigma, \mathcal{P}^\sigma)$ may in principle be very different from the set of (K', A')-lifts.

5.5. **Defining fibers of the boundary extension.** By considering lifts of points in $\partial(\Gamma^\sigma, \mathcal{P}^\sigma_\infty)$, we will define a map

$$\psi_\sigma : \partial(\Gamma^\sigma, \mathcal{P}^\sigma_\infty) \to \{\text{closed subsets of } G/Q\},$$

determining the fibers of ϕ_σ. As a first step, we relate lifts to $\mathcal{G}$-paths by the following lemma.

Lemma 5.18. *Let $z \in \partial(\Gamma^\sigma, \mathcal{P}^\sigma)$. For any lift $\tilde{z}$ of z, if $\{(v_i, \alpha_i)\}_{i=1}^N$ is a $\mathcal{G}$-path limiting to $\tilde{z}$, then for any quasi-geodesic ray $c : [0, \infty) \to X^\sigma$ with $c(\infty) = z$, the sequence*

$$\pi^\sigma(\alpha_1 \cdots \alpha_n)$$

lies within finite Hausdorff distance of $c([0, \infty)) \cap \mathrm{Cay}(\Gamma^\sigma)$, with respect to the metric d^σ_Γ on $\mathrm{Cay}(\Gamma^\sigma)$.

Proof. Let c' be a quasi-geodesic ray in X^σ from the identity to z, so that a lift $\tilde{c}'$ based at the identity has ideal endpoint $\tilde{z}$. By Corollary 4.16, if $\{(v_i, \alpha_i)\}_{i=1}^N$ is a $\mathcal{G}$-path limiting to $\tilde{z}$, the Hausdorff distance between $\tilde{c}'([0, \infty)) \cap \mathrm{Cay}(\Gamma)$ and the sequence $g_n = \alpha_1 \cdots \alpha_n$ is finite. Since π^σ is 1-Lipschitz on X, the Hausdorff distance between $c'([0, \infty)) \cap \mathrm{Cay}(\Gamma^\sigma)$ and $\pi^\sigma(g_n)$ with respect to d^σ_Γ is finite as well.

Since X^σ is δ-hyperbolic, the images of c and c' are also finite Hausdorff distance apart with respect to the metric on X^σ. Then by Corollary 2.7, the Hausdorff distance between $c'([0, \infty) \cap \mathrm{Cay}(\Gamma^\sigma)$ and $c([0, \infty)) \cap \mathrm{Cay}(\Gamma^\sigma)$ with respect to d^σ_Γ is also finite, and the result follows. ☐

5.5.1. *Conical fibers.* We will first define ψ_σ on the set of conical limit points in $\partial(\Gamma^\sigma, \mathcal{P}^\sigma)$, using the lemma below.

Lemma 5.19. *Let z be a conical limit point in $\partial(\Gamma^\sigma, \mathcal{P}^\sigma)$. There is a unique point $\psi_\sigma(z) \in G/Q$ satisfying the following: if $\tilde{z}$ is any $(2, 18\delta)$-lift of z in $\partial(\Gamma, \mathcal{P})$, and $\{(v_i, \alpha_i)\}_{i=1}^\infty$ is a $\mathcal{G}$-path limiting to $\tilde{z}$, then*

$$\{\psi_\sigma(z)\} = \bigcap_{i=1}^\infty \sigma(\alpha_1 \cdots \alpha_i) U_{v_{i+1}}.$$

Proof. By arguing as in the proof of Lemma 5.13, we can use property (G7) and employ [Wei22, Cor. 7.12] to see that the given intersection is always a singleton. We need to check that this singleton does not depend on the choice of $(2, 18\delta)$-lift $\tilde{z}$, or on the choice of $\mathcal{G}$-path limiting to $\tilde{z}$.

So, let $\tilde{z}$, $\tilde{z}'$ be a pair of $(2, 18\delta)$-lifts of z, and let c, c' be $(2, 18\delta)$-quasi-geodesic rays in X^σ from the identity to z so that lifts $\tilde{c}$, $\tilde{c}'$ tend from the identity to $\tilde{z}, \tilde{z}'$ respectively. We want to show that for any $\mathcal{G}$-paths

$$\{(v_i, \alpha_i)\}_{i=1}^\infty, \quad \{(w_j, \beta_j)\}_{j=1}^\infty$$

limiting to $\tilde{z}$, $\tilde{z}'$ respectively, we have

$$\bigcap_{i=1}^\infty \sigma(\alpha_1 \cdots \alpha_i) U_{v_{i+1}} = \bigcap_{j=1}^\infty \sigma(\beta_1 \cdots \beta_j) U_{w_{j+1}}. \tag{9}$$

Let $g_n = \alpha_1 \cdots \alpha_n$ and $h_n = \beta_1 \cdots \beta_n$. For any subsequence of g_n, we can choose a further subsequence so that $U_{v_{n+1}}$ is a fixed subset U_v for all n. Then, [Wei22, Prop. 3.6] implies that g_n is Q-divergent with Q-limit set equal to the singleton given by the left-hand side of (9). The same argument implies that h_n is also Q-divergent, with Q-limit set the right-hand side of (9).

By Lemma 5.18, both of $\pi^\sigma(g_n)$ and $\pi^\sigma(h_n)$ are within bounded Hausdorff distance of $r([0,\infty)) \cap \mathrm{Cay}(\Gamma^\sigma)$, with respect to d_Γ^σ. So the sequences $\sigma(g_n)$ and $\sigma(h_n)$ are bounded distance apart in G. By [KLP17, Lemma 4.19], this means that $\sigma(g_n)$ and $\sigma(h_n)$ have the same Q-limit sets, so the intersections in (9) are equal. □

Lemma 5.20. *The assignment $z \mapsto \psi_\sigma(z)$ determines a σ-equivariant map from the set of conical limit points in $\partial(\Gamma^\sigma, \mathcal{P}^\sigma)$ to G/Q.*

Proof. The map ψ_σ is well-defined on conical points by the previous lemma, so the only thing we need to check is equivariance. In fact, we just need to show that for any s in the generating set S, we have $\psi_\sigma(\pi^\sigma(s)z) = \sigma(s)\psi_\sigma(z)$. Fix a conical point $z \in \partial(\Gamma^\sigma, \mathcal{P}^\sigma)$, let $\tilde{z}$ be a geodesic lift of z, and let $\tilde{c}$ be a geodesic ray from id to $\tilde{z}$, lifting a geodesic from id to z. Let $\{(v_i, \alpha_i)\}_{i=1}^\infty$ be a $\mathcal{G}$-path limiting to $\tilde{z}$.

Now fix $s \in S$, and consider the path c^s in X^σ obtained by concatenating the edge $(\mathrm{id}, \pi^\sigma(s))$ with the $\pi^\sigma(s)$-translate of c. Since s has length 1 and we assume $\delta \geq 1$, this path is a $(1, 2\delta)$-quasi-geodesic, with endpoint $\pi^\sigma(s)z \in \partial(\Gamma^\sigma, \mathcal{P}^\sigma)$. By (O1), π^σ is injective on S, so $(\mathrm{id}, \pi^\sigma(s))$ lifts uniquely to the edge (id, s) in $\mathrm{Cay}(\Gamma)$. Then, concatenating this edge with the s-translate of $\tilde{c}$, we obtain a $(1, 2\delta)$-quasi-geodesic lift $\tilde{c}^s$ of c^s, whose endpoint is $s\tilde{z}$, a $(1, 2\delta)$-lift of $\pi(s)z$.

Let $\{(w_j, \beta_j)\}_{j=1}^\infty$ be a $\mathcal{G}$-path limiting to $s\tilde{z}$. By Corollary 4.16, the sequence

$$h_n = \beta_1 \cdots \beta_n$$

lies within finite Hausdorff distance of $\tilde{c}^s([0,\infty)) \cap \mathrm{Cay}(\Gamma)$, with respect to the metric d_X. The same corollary also implies that $g_n = \alpha_1 \cdots \alpha_n$ lies within finite Hausdorff distance of $\tilde{c}([0,\infty)) \cap \mathrm{Cay}(\Gamma)$, which means that the sequence sg_n lies within finite Hausdorff distance of h_n, again with respect to the metric d_X; by Lemma 2.3, the Hausdorff distance is also finite with respect to d_Γ.

Then, arguing as in the previous lemma, we see that the intersections

$$\psi_\sigma(\pi^\sigma(s)z) = \bigcap_{i=1}^\infty \sigma(h_n) U_{w_{n+1}}, \quad \bigcap_{i=1}^\infty \sigma(s)\sigma(g_n) U_{v_{n+1}} = \sigma(s)\psi_\sigma(z)$$

must agree. □

We have now defined ψ_σ as an equivariant map on conical points in $\partial(\Gamma^\sigma, \mathcal{P}^\sigma)$. Note that these are precisely the same as the conical points in $\partial(\Gamma^\sigma, \mathcal{P}_\infty^\sigma)$, when viewing $\partial(\Gamma^\sigma, \mathcal{P}_\infty^\sigma)$ as a subset of $\partial(\Gamma^\sigma, \mathcal{P}^\sigma)$.

5.5.2. *Parabolic fibers.* Next we want to extend the map ψ_σ to an equivariant map

$$\psi_\sigma : \partial(\Gamma^\sigma, \mathcal{P}_\infty^\sigma) \to \{\text{closed subsets of } G/Q\}.$$

Let Π_∞^σ be the (finite) set of fixed points of the groups in $\mathcal{P}_\infty^\sigma$; we will first define ψ_σ on Π_∞^σ. As in the analogous construction in [Wei22], for each vertex v of the relative automaton $\mathcal{G}$, define an open set $B_v \subset G/Q$ by

$$B_v = \bigcup_w U_w,$$

where the union is taken over all vertices $w \in V(\mathcal{G})$ such that there is an edge $v \to w$ in $\mathcal{G}$.

Each point p in $\Pi_\infty^\sigma \subseteq \Pi^\sigma$ is uniquely identified with some parabolic point in $\Pi \subset \partial(\Gamma, \mathcal{P})$. We leave this identification implicit, so we can view the points in Π_∞^σ as parabolic points in $\partial(\Gamma, \mathcal{P})$.

By condition (G4) on the automaton $\mathcal{G}$, for each $p \in \Pi^\sigma_\infty$, there is a parabolic vertex v of $\mathcal{G}$ so that the associated parabolic point q_v is p. Define the set $\psi_\sigma(p)$ to be the closure of the set of accumulation points of sequences of the form $\sigma(\gamma_n)\xi_n$ where:

- ξ_n is a sequence in B_v, and
- γ_n is a sequence of pairwise distinct elements in Γ^σ, all lying in $\Gamma^\sigma_p = \mathrm{Stab}_{\Gamma^\sigma}(p)$.

Extend ψ_σ to all parabolic points in $\partial(\Gamma^\sigma, \mathcal{P}^\sigma_\infty)$ by equivariance. We observe:

Lemma 5.21. *Let $p \in \Pi^\sigma_\infty$, and suppose that a parabolic vertex $v \in V(\mathcal{G})$ satisfies $q_v = gp$ for $g \in \Gamma$. Then $\sigma(g)\psi_\sigma(p) \subset U_v$.*

Proof. Let $\xi \in \psi_\sigma(p)$; we will show that $\sigma(g)\xi \in U_v$. Let $P = \mathrm{Stab}_\Gamma(p)$, and let $P^\sigma = \mathrm{Stab}_{\Gamma^\sigma}(p)$, so $P^\sigma = P/\ker(\sigma|_P)$. Fix a vertex $w \in V(\mathcal{G})$ with $q_w = p$. Then by definition ξ is the limit of a sequence $\sigma(\gamma_n)\xi_n$, where γ_n is a sequence of pairwise distinct elements of P^σ and ξ_n is a sequence in B_w.

By the definition of P^σ, the sequence γ_n must eventually satisfy

$$\sigma(g)\sigma(\gamma_n) \notin \sigma(F_v),$$

where F_v is the finite set $gP\backslash T_v$. Condition (O4) then implies that $\sigma(g)\sigma(\gamma_n)B_v$ is eventually a subset of U_v. Since $\lim_{n\to\infty} \sigma(\gamma_n)\xi_n = \xi$, we must have $\sigma(g)\xi \in U_v$, as required. ☐

We have now defined ψ_σ as an equivariant map on all of the conical points and parabolic point in $\partial(\Gamma^\sigma, \mathcal{P}^\sigma_\infty)$, hence on all of $\partial(\Gamma^\sigma, \mathcal{P}^\sigma_\infty)$.

Remark 5.22. The definition of ψ_σ on parabolic points makes it clear why we have only defined ψ_σ on $\partial(\Gamma^\sigma, \mathcal{P}^\sigma_\infty)$, rather than $\partial(\Gamma^\sigma, \mathcal{P}^\sigma)$: by definition, there are no infinite sequences of pairwise distinct elements in the Γ^σ-stabilizer of a point in $\partial(\Gamma^\sigma, \mathcal{P}^\sigma) \setminus \partial(\Gamma^\sigma, \mathcal{P}^\sigma_\infty)$.

5.6. **Defining the boundary extension.** Define a σ-invariant subset $\Lambda_\sigma \subset G/Q$ by

$$\Lambda_\sigma = \bigcup_{z \in \partial(\Gamma^\sigma, \mathcal{P}^\sigma_\infty)} \psi_\sigma(z).$$

We wish to define a surjective equivariant antipodal map $\Lambda_\sigma \to \partial(\Gamma^\sigma, \mathcal{P}^\sigma_\infty)$. To do so, we need to show that, as z ranges over $\partial(\Gamma^\sigma, \mathcal{P}^\sigma_\infty)$, the sets $\psi_\sigma(z)$ partition Λ_σ.

We will continue to imitate the proof of the special case in [Wei22], but now we run into yet another complication: the proof in [Wei22] takes advantage of the fact that a relatively hyperbolic group acts cocompactly on pairs of distinct points in its Bowditch boundary. We have two groups to work with here, which may not be isomorphic and which may not have homeomorphic Bowditch boundaries, so we need the slightly more intricate statement below.

Lemma 5.23. *For any pair of distinct points $z_1, z_2 \in \partial(\Gamma^\sigma, \mathcal{P}^\sigma)$, there is some $g \in \Gamma^\sigma$ and a pair of geodesic lifts $\tilde{z}_1$, $\tilde{z}_2$ of gz_1, gz_2 respectively such that $d_\partial(\tilde{z}_1, \tilde{z}_2) > \Delta$, where Δ is the constant from Definition 5.4.*

Proof. Since z_1 and z_2 are distinct, and X^σ is hyperbolic, there is a bi-infinite geodesic $c : (-\infty, \infty) \to X^\sigma$ such that $c(-\infty) = z_1$ and $c(\infty) = z_2$. This geodesic cannot lie in a single combinatorial horoball (since each such horoball has just one point in its ideal boundary) so it must pass through $\mathrm{Cay}(\Gamma^\sigma)$ at some point $g \in \Gamma^\sigma$; after reparameterization we may assume $g = c(0)$.

Consider the bi-infinite geodesic $c' : (-\infty, \infty) \to X^\sigma$ given by $c'(t) = g^{-1}c(t)$. The endpoints of this geodesic are $g^{-1}z_1$ and $g^{-1}z_2$, respectively.

Let $\tilde{c}$ be a lift of c' in X, chosen so that $\tilde{c}(0) = \mathrm{id}$. Since $\tilde{c}$ is a lift of a geodesic, it is a bi-infinite geodesic in X, and its forward and backward ideal endpoints are by definition lifts of $g^{-1}z_1$ and $g^{-1}z_2$. These lifts are at the opposite ends of a bi-infinite geodesic in X passing through the identity, so by definition of Δ, the distance between them (with respect to d_∂) is at least Δ. □

The lemma below similarly shows that we have a "uniform" version of boundedness of the parabolic points in $\partial(\Gamma^\sigma, \mathcal{P}^\sigma)$, via lifts in $\partial(\Gamma, \mathcal{P})$.

Lemma 5.24. *Let $p \in \Pi$, and let P be the stabilizer of p in Γ. If $K_p \subset \partial(\Gamma, \mathcal{P}) \setminus \{p\}$ is the compact set given by Definition 5.5, then for any $z \in \partial(\Gamma^\sigma, \mathcal{P}^\sigma) \setminus \{p\}$, there is some $h \in P^\sigma = \pi^\sigma(P)$ so that a geodesic lift of hz lies in K_p.*

Note that in the statement of the lemma, we have again implicitly identified Π^σ with Π.

Proof. Fix z in $\partial(\Gamma^\sigma, \mathcal{P}^\sigma) - \{p\}$, and let $c : (-\infty, \infty) \to X^\sigma$ be a bi-infinite geodesic joining z to p. This geodesic must eventually enter the combinatorial horoball $\mathcal{H}$ centered at p, i.e. the combinatorial horoball attached to P^σ. Let h be the last point in $P^\sigma = \partial\mathcal{H}$ reached by c, so that after reparameterization we have $c(0) = h$ and $c(t) \in \mathcal{H}$ for all $t > 0$. Then we may assume that the positive direction of c only traverses vertical edges in the combinatorial horoball $\mathcal{H}$.

The bi-infinite geodesic $h^{-1}c$ has ideal endpoints $c^{-1}p = p$ and $c^{-1}z$. Let $\tilde{c}$ be a lift of $h^{-1}c$ in X. The positive direction of $\tilde{c}$ must only traverse vertical edges lying in the combinatorial horoball attached to P, so $\tilde{c}(+\infty) = p$. On the other hand, $\tilde{c}(-\infty)$ is by definition a geodesic lift of $h^{-1}z$, which lies in K_p by definition. □

The next lemma says that we can use the vertex sets $U_v \in \mathcal{U}$, as well as their iteratively nested translates under certain $\mathcal{G}$-paths, to "approximate" the sets $\psi_\sigma(z)$.

Lemma 5.25. *If z is a point in $\partial(\Gamma^\sigma, \mathcal{P}^\sigma_\infty)$, $\tilde{z}$ is any $(2, 18\delta)$-lift of z, and $\{(v_i, \alpha_i)\}_{i=1}^N$ is a $\mathcal{G}$-path limiting to $\tilde{z}$, then for any $0 \le n < N$ we have*

$$\psi_\sigma(z) \subset \sigma(\alpha_1 \cdots \alpha_n) U_{v_{n+1}}.$$

In particular, $\psi_\sigma(z) \subset U_{v_1}$.

Proof. If z is a conical limit point, this is immediate from the definition of ψ_σ, so suppose z is parabolic. Let $c : [0, \infty) \to X^\sigma$ be a $(2, 18\delta)$-quasi-geodesic ray limiting to z. Since z is the center of a horoball in X^σ, there is some $a \ge 0$ such that $c(a) \in \Gamma^\sigma$ and $c(t)$ lies in a combinatorial horoball centered at z for all $t > a$. This horoball is attached to X^σ at the coset $c(a)P^\sigma$ for $P^\sigma \in \mathcal{P}^\sigma$. If P^σ is the stabilizer of $p \in \Pi^\sigma_\infty$, then $z = c(a)p$.

Fix $P \in \mathcal{P}$ so $\pi^\sigma(P) = P^\sigma$, and let $\tilde{c}$ be a lift of c. Since the combinatorial horoballs lift to combinatorial horoballs, for all $t > a$, $\tilde{c}(t)$ must lie in the combinatorial horoball attached to the coset $\tilde{c}(a)P$, and the ideal endpoint $\tilde{z}$ of $\tilde{c}$ is $\tilde{c}(a)p$, with p now viewed as a point in Π.

Let $\{(v_i, \alpha_i)\}_{i=1}^{n+1}$ be a $\mathcal{G}$-path limiting to $\tilde{z}$, so that v_{n+1} is a parabolic vertex. Let $q \in \partial(\Gamma, \mathcal{P})$ be the parabolic point corresponding to v_{n+1}. Since the $\mathcal{G}$-path limits to a parabolic point in the orbit of p, q must also be in this orbit, so write $q = g'p$ for $g' \in \Gamma$. The fact that the $\mathcal{G}$-path limits to $\tilde{z}$ means precisely that

$$\alpha_1 \cdots \alpha_n q = \tilde{z},$$

and thus $\tilde{z} = \alpha_1 \cdots \alpha_n g'p$. This means that $\alpha_1 \cdots \alpha_n g'$ lies in the coset $\tilde{c}(a)P$, so write $\alpha_1 \cdots \alpha_n g'h = \tilde{c}(a)$ for $h \in P$.

Then, by definition of ψ_σ, we have $\psi_\sigma(z) = \sigma(c(a))\psi_\sigma(p)$, and since $\tilde{c}$ lifts c this means

$$\begin{aligned}\psi_\sigma(z) &= \sigma(\alpha_1 \cdots \alpha_n)\sigma(g'h)\psi_\sigma(p)\\ &= \sigma(\alpha_1 \cdots \alpha_n)\sigma(g')\psi_\sigma(p).\end{aligned}$$

Now, we know that the sequence of elements $g_j = \alpha_1 \cdots \alpha_j$ for $1 \le j \le n+1$ lies within Hausdorff distance D of $\tilde{c}([0,\infty)) \cap \mathrm{Cay}(\Gamma)$ (see Definition 5.8). So, the $\mathcal{G}$-path satisfies the $\mathcal{G}$-compatibility property in Lemma 5.12. And, by Lemma 5.21, we have $\sigma(g')\psi_\sigma(p) \subset U_{v_{n+1}}$ and the desired inclusion follows. □

Lemma 5.26. *For any distinct $z_1, z_2 \in \partial(\Gamma^\sigma, \mathcal{P}^\sigma_\infty)$, we have $\psi_\sigma(z_2) \subset \mathrm{Opp}(\psi_\sigma(z_1))$. In particular, $\psi_\sigma(z_1)$ and $\psi_\sigma(z_2)$ are disjoint.*

Proof. By Lemma 5.23 and σ-equivariance of ψ_σ, it suffices to show the statement in the case where there are geodesic lifts $\tilde{z}_1$, $\tilde{z}_2$ of z_1 and z_2 satisfying $d_\partial(\tilde{z}_1, \tilde{z}_2) > \Delta$. Fix $\mathcal{G}$-paths $\{(v_i, \alpha_i)\}_{i=1}^N$ and $\{(w_j, \beta_j)\}_{j=1}^M$ limiting to $\tilde{z}_1$ and $\tilde{z}_2$ respectively. Then by property (G8), we have $\overline{U_{w_1}} \subset \mathrm{Opp}(U_{v_1})$. Lemma 5.25 implies that $\psi_\sigma(z_1) \subset U_{v_1}$ and $\psi_\sigma(z_2) \subset U_{w_1}$, so the conclusion follows. □

Definition 5.27. Define the map $\phi_\sigma : \Lambda_\sigma \to \partial(\Gamma^\sigma, \mathcal{P}^\sigma)$ by the rule:

$$\phi_\sigma(w) = z \iff w \in \psi_\sigma(z).$$

Lemma 5.26 ensures that this determines a well-defined antipodal map. It is surjective by construction, and it is equivariant because ψ_σ is equivariant.

5.7. **Compactness and continuity.** We have defined a map $\phi_\sigma : \Lambda_\sigma \to \partial(\Gamma^\sigma, \mathcal{P}^\sigma_\infty)$, which is our candidate boundary extension for an extended convergence action of Γ^σ on G/Q. The next step is to show that the domain Λ_σ of ϕ_σ is compact, and that ϕ_σ is continuous. We will need to directly get our hands on the topologies on both Bowditch boundaries $\partial(\Gamma, \mathcal{P})$ and $\partial(\Gamma^\sigma, \mathcal{P}^\sigma_\infty)$; our approach uses the Gromov product $(\cdot|\cdot)_x$ on the compactified cusped space $\overline{X^\sigma}$. First, we prove a general lemma:

Lemma 5.28. *Let Y be a δ-hyperbolic metric space and let $x \in Y$. For any infinite geodesic ray $c : [0,\infty) \to Y$ with $c(0) = x$ and $c(\infty) = z \in \partial Y$, if $z' \in \partial Y$ satisfies $(z|z')_x \ge N + 5\delta$, then there is a $(1, 18\delta)$-quasi-geodesic ray $c' : [0,\infty) \to X$ with $c'(\infty) = z'$ and $c'|_{[0,N]} = c|_{[0,N]}$.*

Proof. Denote the image of the geodesic ray c by $[x, z)$, and let $[x, z')$ be a geodesic ray from x to z'. Fix sequences y_n on $[x, z)$ and y'_m on $[x, z')$ tending to infinity. Then the Gromov product $(z|z')_x$ can be coarsely estimated as $\liminf_{m,n\to\infty}(y_n|y'_m)_x$. Explicitly, we have the estimate $\liminf_{n,m\to\infty}(y_n|y'_m)_x \ge (z|z')_x - 2\delta$ (see e.g. [BH99, III.H.3.17(5)]), which means that for all sufficiently large n, m, we have

$$(z|z')_x \le (y_n|y'_m)_x + 3\delta.$$

Now, consider a geodesic triangle with two sides equal to subsegments $[x, y_n] \subset [x, z)$ and $[x, y'_m) \subset [x, z')$. There is a point w on the side joining y_n to y'_m which lies within distance δ of both some $u \in [x, y_n]$ and some $u' \in [x, y'_n]$. Thus we have

$$\begin{aligned}d(y_n, u) - d(y_n, w) &\le \delta,\\ d(y'_m, u') - d(y'_n, w) &\le \delta.\end{aligned}$$

Moreover we also have $d(u, u') \le 2\delta$, hence

$$d(x, u') \le d(x, u) + 2\delta.$$

Therefore,

$$\begin{aligned}2(y_n|y'_m)_x &= d(x, y_n) + d(x, y'_m) - d(y_n, y'_n)\\ &= d(x, u) + d(u, y_n) + d(x, u') + d(u', y'_m) - d(y_n, w) - d(w, y'_m)\\ &\leq 2d(x, u) + 4\delta.\end{aligned}$$

Thus $d(x, u) \geq (z|z')_x - 5\delta$. Now let $[u, z')$ be a geodesic ray from u to z', and consider the ray c' in Y obtained by concatenating $[x, u]$ to $[u, z')$. We claim that c' is a $(1, 18\delta)$-quasi-geodesic.

To see this, let $a \in [x, u]$ and let $b \in [u, z')$. Then since $d(u, u') \leq 2\delta$, by considering a δ-thin triangle with vertices x, u, u', we see that there is a point a' on $[x, u']$ with $d(a, a') \leq 3\delta$. Similarly, by considering a 2δ-thin (partly ideal) triangle with vertices u, u', z', there is a point $b' \in [u', z')$ with $d(b, b') \leq 4\delta$. Therefore

$$d(a, u) + d(u, b) \leq d(a', u') + d(u', b') + (3 + 2 + 2 + 4)\delta.$$

Since u' lies on a geodesic between a' and b', this says that the sub-path of c' between a and b is at most

$$d(a', b') + 11\delta \leq d(a, b) + 18\delta.$$

Thus c' is a $(1, 18\delta)$-quasi-geodesic which agrees with c on a segment of length at least $(z|z')_x - 5\delta$. □

The lemma above gives us a tool for relating convergence in $\partial(\Gamma, \mathcal{P})$ to convergence in $\partial(\Gamma^\sigma, \mathcal{P}^\sigma_\infty)$.

Corollary 5.29. *Let z be a point in ∂X^σ, let z_n be a sequence converging to z, and let $\tilde{z}$ be a geodesic lift of z. There exist $(1, 18\delta)$-lifts $\tilde{z}_n$ of z_n so that $\tilde{z}_n$ converges to $\tilde{z}$.*

Proof. Let c be a geodesic ray from id to z and let $\tilde{c}$ be a lift of c from id to $\tilde{z}$. Using the previous lemma, we can find $N_n \to \infty$ and $(1, 18\delta)$-quasi-geodesic rays c_n so that $c_n(\infty) = z_n$, and c_n agrees with c on the interval $[0, N_n]$. So, we can choose lifts $\tilde{c}_n$ of c_n so that $\tilde{c}_n$ agrees with $\tilde{c}$ on $[0, N_n]$ as well. The ideal endpoints of these lifts necessarily converge to $\tilde{z}$. □

We can now show:

Proposition 5.30. *The set Λ_σ is compact.*

Proof. The proof is similar to the proof of Lemma 9.14 in [Wei22]. Let y_n be a sequence in Λ_σ, and let $x_n = \phi_\sigma(y_n)$. Up to subsequence x_n converges to $x \in \partial(\Gamma^\sigma, \mathcal{P}^\sigma_\infty)$. We want to show that (possibly after further extraction) y_n converges to a point in $\phi_\sigma^{-1}(x)$. We consider two cases:

Case 1: x is a parabolic point: Write $x = gp$ for $g \in \Gamma^\sigma$ and $p \in \Pi$, and let $x'_n = g^{-1}x_n$. By σ-equivariance, it suffices to show that $y'_n = \sigma(g^{-1})y_n$ converges to a point in in $\phi_\sigma^{-1}(p)$.

Let P be the stabilizer of p in Γ, and let $P^\sigma = \sigma(P)$. By Lemma 5.24, for each n we can find $h_n \in P^\sigma$ so that a geodesic lift $\tilde{x}'_n$ of $h_n x'_n$ lies in the compact set $K_p \subset \partial(\Gamma, \mathcal{P}) \setminus \{p\}$ in Definition 5.5. Let v be a parabolic vertex of the automaton $\mathcal{G}$ such that $q_v = p$.

We know that our automaton $\mathcal{G}$ satisfies condition (G9) in Proposition 4.9, with K_p as above. So, there is an edge $v \to w$ in $\mathcal{G}$ and a $\mathcal{G}$-path limiting to $\tilde{x}'_n$ whose first vertex is w. Lemma 5.25 implies that $\phi_\sigma^{-1}(h_n x'_n)$ lies in U_w, therefore in the set B_v defined in Section 5.5.2. So, for every n, we have $\sigma(h_n)y'_n \in B_v$.

The proof of Lemma 5.24 shows that for every n, there is a bi-infinite geodesic through id in X^σ joining $h_n x'_n$ to p, so $h_n x'_n$ cannot converge to p. Since x'_n converges to p, a subsequence of h_n^{-1} must consist of pairwise distinct elements. So, by definition of the set $\phi_\sigma^{-1}(p) = \psi_\sigma(p)$, the sequence $y_n' = \sigma(h_n^{-1})\sigma(h_n)y_n'$ accumulates in $\phi_\sigma^{-1}(p)$, hence in Λ_σ.

Case 2: x is a conical limit point: Fix a geodesic ray $c : [0, \infty) \to X^\sigma$ limiting to x, so that the ideal endpoint $\tilde{x}$ of a lift $\tilde{c}$ of c is a geodesic lift of x. Let $\{(v_i, \alpha_i)\}_{i=1}^\infty$ be a $\mathcal{G}$-path limiting to $\tilde{x}$. By Corollary 5.29, there are $(1, 18\delta)$-lifts $\tilde{x}_n$ of x_n so that $\tilde{x}_n$ converges to $\tilde{x}$.

Let $r > 0$ be given. As in the proof of Lemma 5.13, we can employ Lemma 5.12, property (G7), and [Wei22, Cor. 7.12] to see that there is a constant N and a metric $d_{U_{v_1}}$ on an open subset U_{v_1}' of G/Q containing $\overline{U_{v_1}}$ such that, for any $\mathcal{G}$-path $\{(w_j, \beta_j)\}_{j=1}^m$ with $w_1 = v_1$ and $m \geq N$, the diameter of

$$\sigma(\beta_1 \cdots \beta_m)U_{w_{m+1}}$$

with respect to the metric $d_{U_{v_1}}$ is less than r. It is also true that this metric $d_{U_{v_1}}$ induces the subspace topology on $U_{v_1} \subset G/Q$.

Now, by [Wei22, Lem. 5.15], for all sufficiently large n, we can find a $\mathcal{G}$-path

$$\{(w_j^{(n)}, \beta_j^{(n)})\}_{j=1}^{M_n}$$

limiting to $\tilde{x}_n$ such that $M_n \geq N$ and $w_i^{(n)} = v_i$, $\beta_i^{(n)} = \alpha_i$ for all $i \leq N$. Then, by Lemma 5.25, we have

$$\phi_\sigma^{-1}(x_n) \subset \sigma(\alpha_1 \cdots \alpha_N)U_{v_{N+1}}.$$

Since $\phi_\sigma^{-1}(x)$ must also lie in this set by definition, we must have $d_{U_{v_1}}(y_n, y) < r$ for all sufficiently large n. Since r was arbitrary this completes the proof.

□

Proposition 5.31. *The map $\phi_\sigma : \Lambda_\sigma \to \partial(\Gamma^\sigma, \mathcal{P}^\sigma)$ is continuous.*

Proof. We follow the proof of Lemma 9.15 in [Wei22]. Let $y \in \Lambda_\sigma$ and let y_n approach y. We want to show that $\phi_\sigma(y_n)$ approaches $\phi_\sigma(y) = x$. Suppose otherwise. Then since $\partial(\Gamma, \mathcal{P}_\infty)$ is compact, up to subsequence $z_n = \phi_\sigma(y_n)$ approaches some $z \neq x$. By Lemma 5.23, there is some $g \in \Gamma^\sigma$ so that for geodesic lifts $\tilde{x}$ of gx and $\tilde{z}$ of gz, we have $d(\tilde{x}, \tilde{z}) > \Delta$. Since gz_n approaches gz, Corollary 5.29 tells us that we can find $(1, 18\delta)$-lifts $\tilde{z}_n$ of gz_n converging to $\tilde{z}$. So, for sufficiently large n, we have $d(\tilde{x}, \tilde{z}_n) > \Delta$.

Property (G8) of the automaton then means that for any $\mathcal{G}$-paths $\{(v_i, \alpha_i)\}_{i=1}^N$ and $\{(w_j, \beta_j)\}_{j=1}^M$ limiting to $\tilde{x}$ and $\tilde{z}_n$ respectively, we have $\overline{U}_{v_1} \cap \overline{U}_{w_1} = \emptyset$. Lemma 5.25 implies that $\phi_\sigma^{-1}(x) \subset U_{v_1}$ and $\phi_\sigma^{-1}(z_n) \subset U_{w_1}$. Since $y \in \phi_\sigma^{-1}(x)$ and $y_n \in \phi_\sigma^{-1}(z_n) \subset U_{w_1}$, this contradicts the fact that $y_n \to y$. □

5.8. **Defining the repelling strata.** To finish showing that σ is an EGF representation, we need to construct the repelling stratum C_z^σ for each point $z \in \partial(\Gamma^\sigma, \mathcal{P}_\infty^\sigma)$. We do this separately for conical points and parabolic points:

- If z is a conical limit point, set $C_z^\sigma = \mathrm{Opp}(\phi_\sigma^{-1}(z))$.
- If z is a parabolic point, equal to gp for $p \in \Pi^\sigma$, $g \in \Gamma^\sigma$, then let P^σ be the stabilizer of p in Γ^σ. Let v be the parabolic vertex of $\mathcal{G}$ with $q_v = p$, and set

$$C_z^\sigma = \mathrm{Opp}(\phi_\sigma^{-1}(z)) \cap \sigma(gP^\sigma)B_v,$$

where B_v is the open subset defined in Section 5.5.2.

The next lemma says that these strata satisfy property (C1*) from Section 3.

Lemma 5.32. *For each $z \in \partial(\Gamma^\sigma, \mathcal{P}_\infty^\sigma)$, we have $\Lambda_\sigma \setminus \phi_\sigma^{-1}(z) \subset C_z^\sigma$.*

Proof. If z is a conical limit point, then the claim follows directly from the antipodality of ϕ_σ. So we can suppose that z is a parabolic point. Moreover, if $z = gp$ for $g \in \Gamma^\sigma$ and $p \in \Pi^\sigma$, then $C_z^\sigma = \sigma(g)C_p^\sigma$, so by σ-equviance of ϕ_σ we can assume that $z = p \in \Pi^\sigma$. Let w be a vertex in $\mathcal{G}$ with $q_w = p$ (see property (G4)).

Let $y \in \partial(\Gamma^\sigma, \mathcal{P}_\infty^\sigma) \setminus \{p\}$. By Lemma 5.24, we can find $h \in P^\sigma$ so that a geodesic lift $\tilde{y}$ of hy lies in the fixed compact set $K_p \subset \partial(\Gamma, \mathcal{P}) \setminus \{p\}$. By property (G9), there is a $\mathcal{G}$-path limiting to $\tilde{y}$ whose first vertex v_1 is connected to w by an edge $w \to v_1$ in $\mathcal{G}$.

Lemma 5.25 tells us that $\phi_\sigma^{-1}(hy) \subset U_{v_1}$, and we know that $U_{v_1} \subset B_w$ by definition. So, since $\phi_\sigma^{-1}(hy)$ is also in $\mathrm{Opp}(\phi_\sigma^{-1}(p))$ by antipodality, we conclude that

$$\phi_\sigma^{-1}(hy) \subset C_p^\sigma.$$

Since C_p^σ is P^σ-invariant this holds for $\phi_\sigma^{-1}(y)$ as well. □

Next we want to prove that the strata also satisfy property (C2), but first we need:

Lemma 5.33. *For every vertex $v \in V(\mathcal{G})$, the set $\Lambda_\sigma \cap U_v$ is nonempty.*

Proof. Fix a vertex $v = v_1$ in $V(\mathcal{G})$. Property (G3) says that every vertex of $\mathcal{G}$ has at least one outgoing edge, so there is an infinite $\mathcal{G}$-path $\{(v_i, \alpha_i)\}_{i=1}^\infty$ whose first vertex is v_1. We can construct this $\mathcal{G}$-path by choosing each element $\alpha_i \in T_{v_i}$ to be minimal with respect to $|\cdot|_X$, which means that $|\alpha_i|_X \le C_1$ for every index i (by definition of C_1).

By Lemma 4.13, this $\mathcal{G}$-path limits to some conical point $\tilde{z} \in \partial(\Gamma, \mathcal{P})$. Fix a geodesic $\tilde{c} : [0, \infty) \to X$ from id to $\tilde{z}$, so that (from Definition 5.8) the sequence $g_n = \alpha_1 \cdots \alpha_n$ has Hausdorff distance at most D from $\tilde{c}([0,\infty)) \cap \mathrm{Cay}(\Gamma)$. Then, by Lemma 4.17, the geodesic $\tilde{c}$ lies in a $(C_1 + 3D)$-neighborhood of $\mathrm{Cay}(\Gamma)$.

From property (O1), the image c of $\tilde{c}$ in X^σ is a $(2, 18\delta)$-quasi-geodesic. Thus, if $z = c(\infty)$, the point $\tilde{z}$ is a $(2, 18\delta)$-lift of z and $\phi_\sigma^{-1}(z) \subset U_{v_1}$ by Lemma 5.25. □

The next lemma shows that, *along parabolic sequences*, the strata satisfy condition (C2):

Lemma 5.34. *Let $q \in \partial(\Gamma^\sigma, \mathcal{P}_\infty^\sigma)$ be parabolic, let h_n be a sequence of pairwise distinct elements in $\Gamma_p^\sigma = \mathrm{Stab}_{\Gamma^\sigma}(p)$, and let K be a compact subset of C_q^σ. Then $\sigma(h_n)K$ eventually lies in any open neighborhood $\phi_\sigma^{-1}(p)$.*

Proof. By the σ-equivariance of ϕ_σ and the repelling strata C_z^σ, we only need to consider the case where $q = p \in \Pi^\sigma$. Let v be a parabolic vertex of $\mathcal{G}$ so that $q_v = p$. Since K is a compact subset of C_p^σ, it is covered by finitely many sets of the form $\sigma(h)B_w$, where $h \in \Gamma_p^\sigma$, and w is a vertex of $\mathcal{G}$ with $v \to w$ an edge in $\mathcal{G}$. But then, for any sequence $\xi_n \in K$, any accumulation point of $\sigma(h_n)\xi_n$ lies in $\phi_\sigma^{-1}(p)$, by definition of ϕ_σ. Thus $\sigma(h_n)K$ must eventually lie in any neighborhood of $\phi_\sigma^{-1}(p)$. □

Now we can show that condition (C2) holds in general:

Lemma 5.35. *Suppose that a sequence $\gamma_n \in \Gamma^\sigma$ satisfies $\gamma_n^{\pm 1} \to z_\pm$ for $z_\pm \in \partial(\Gamma, \mathcal{P}_\infty)$, and let $K \subset C_{z_-}^\sigma$ be compact. Then $\sigma(\gamma_n)K$ eventually lies in any neighborhood of $\phi_\sigma^{-1}(z_+)$.*

Proof. Consider the sequence of relative lengths $|\gamma_n|_{\hat{\Gamma}}^\sigma$. After taking a subsequence, these either tend to infinity, or else they are bounded.

Case 1: $|\gamma_n|_{\hat{\Gamma}}^\sigma \to \infty$: In this case, by applying Lemma 5.13 to the constant sequence σ, the sequence $\sigma(\gamma_n)$ is Q-divergent. Further, because of the previous lemma, Lemma 5.13 also implies that the Q-limit points of $\sigma(\gamma_n^\pm)$ all lie in Λ_σ. So, after extraction, the sequences $\sigma(\gamma_n^{\pm 1})$ respectively have unique Q-limit points $\xi_\pm \in \Lambda_\sigma$. Let $z'_\pm = \phi_\sigma(\xi_\pm)$. Since $C_z \subseteq$

$\mathrm{Opp}(\phi_\sigma^{-1}(z))$ for every $z \in \partial(\Gamma^\sigma, \mathcal{P}_\infty^\sigma)$, it follows from Lemma 3.6 that, if K is any compact subset of $C_{z'_-}^\sigma$, then $\sigma(\gamma_n)K$ eventually lies in any neighborhood of $\phi_\sigma^{-1}(z'_+)$. So we will be done if we can show $z_\pm = z'_\pm$.

However, since $(\Gamma^\sigma, \mathcal{P}_\infty^\sigma)$ is non-elementary, we can pick a compact set $K \subset C_{z'_-}$ so that $\phi_\sigma(K \cap \Lambda_\sigma)$ is a nonempty open subset Y of $\partial(\Gamma^\sigma, \mathcal{P}_\infty^\sigma)$. Since ϕ_σ is equivariant and continuous, $\gamma_n Y$ converges to z'_+, and since γ_n converges to z_+ in $\overline{X^\sigma}$, this is only possible if $z_+ = z'_+$. An identical argument applied to γ_n^{-1} shows that $z_- = z'_-$, and we can conclude this case.

Case 2: $|\gamma_n|_\Gamma^\sigma$ is bounded: For this case, we argue as in the proof of Lemma 5.15. Extract a subsequence so that

$$\gamma_n = g_1 h_1^{(n)} \cdots g_k h_k^{(n)} g_{k+1},$$

where each $g_i \in \Gamma^\sigma$ is fixed, $h_i^{(n)}$ is a sequence of pairwise distinct elements fixing some $p_i \in \Pi_\infty^\sigma$, and $g_i p_i \neq p_{i-1}$ for every $1 < i \leq k$. Let $z'_+ = g_1 p_1$ and let $z'_- = g_{k+1}^{-1} p_k$. By applying Lemma 5.34 and inducting on k, we can see that for any compact set $K \subset C_{z'_-}^\sigma$, the set $\sigma(\gamma_n)K$ eventually lies in any neighborhood of $\phi_\sigma^{-1}(z'_+)$. Once again, we can show that $z_+ = z'_+$ by choosing K so that its intersection with Λ_σ projects to a nonempty open subset of $\partial(\Gamma^\sigma, \mathcal{P}_\infty^\sigma)$, and again the same argument applied to γ_n^{-1} shows that $z'_- = z_-$. This completes the proof of the parabolic case.

□

Finally we can prove:

Proposition 5.36. *The representation σ is Q-extended geometrically finite, with boundary extension $\phi_\sigma : \Lambda_\sigma \to \partial(\Gamma^\sigma, \mathcal{P}_\infty^\sigma)$ and repelling strata C_z^σ defined above.*

Proof. We have already shown that the map ϕ_σ is continuous, equivariant, surjective, and antipodal, and we know from Lemma 5.32 and Lemma 5.35 that conditions (C1*) and (C2) in the definition of an extended convergence action are satisfied. From Lemma 5.13, σ is relatively Q-divergent, so Proposition 3.9 applies and σ must be Q-EGF.

If we only consider the statement of Proposition 3.9, it might seem possible that the Q-EGF boundary extension and repelling strata could be different from ϕ_σ and C_z^σ. However, examining the construction in the proof of Proposition 3.9 (see Appendix A) shows that ϕ_σ and C_z^σ will not be modified, so they serve as an actual boundary extension and repelling strata for σ. □

5.9. **Semicontinuity of the boundary sets.** We have shown the main part of Proposition 5.1. We complete the proof by observing:

Proposition 5.37. *The set Λ_σ is contained in the ε-neighborhood of Λ, and has nonempty intersection with $\mathcal{N}_\varepsilon(\phi^{-1}(Z))$.*

Proof. From property (G2), each vertex set of the automaton is contained in an ε-neighborhood of Λ, and by Lemma 5.25, each set $\phi_\sigma^{-1}(z)$ for $z \in \partial(\Gamma^\sigma, \mathcal{P}_\infty^\sigma)$ lies in some vertex set U_v, so this proves the first part of the proposition.

For the second part, we use property (G10) to see that there is at least one vertex set U_v contained in the ε-neighborhood of $\phi^{-1}(Z)$. From Lemma 5.33, Λ_σ has nonempty intersection with this set. □

Remark 5.38. In the special case where the subgroups $\sigma(P)$ are Q-divergent for all $P \in \mathcal{P}$, the construction of the limit set Λ_σ in the proof above ensures that, for each $p \in \Pi$, the set

$\phi_\sigma^{-1}(p)$ is precisely the Q-limit set of the subgroup $\sigma(\Gamma_p)$. Thus, in this case, the boundary set Λ_σ is precisely the Q-limit set of $\sigma(\Gamma)$.

6. Relative Anosov representations and continuity of the limit set

In this section we prove Theorem 1.8, which applies our main theorem to the special case of *relatively Q-Anosov representations* defined in work of Kapovich–Leeb [KL23] and Zhu [Zhu21]; see also [CZZ22]. There are several characterizations of relative Anosov representations (see e.g. [ZZ22]), but we will work with the one given by the proposition below. Here, and for the rest of the section, $(\Gamma, \mathcal{P})$ is again a relatively hyperbolic pair, and Q is a symmetric parabolic subgroup of a semisimple Lie group G.

Proposition 6.1 (See [Wei22, Sec. 4])**.** *Let $\rho : \Gamma \to G$ be a representation. The following are equivalent:*

(i) ρ is relatively Q-Anosov (in the sense of [ZZ22]).
(ii) ρ is Q-EGF, and has a boundary extension which is a homeomorphism.
(iii) ρ is Q-EGF, and has a boundary extension ϕ such that $\phi^{-1}(p)$ is a singleton for each $p \in \Pi$.

Moreover, if $\phi : \Lambda \to \partial(\Gamma, \mathcal{P})$ is the boundary extension in (ii) or (iii), then Λ is the Q-limit set of ρ.

Remark 6.2. When the set of peripheral subgroups $\mathcal{P}$ is empty, then all three of these conditions agree with the definition of a (non-relatively) *Q-Anosov representation* of the hyperbolic group Γ into G.

We prove the following, which is a stronger form of Theorem 1.8:

Proposition 6.3. *Let ρ a relatively Q-Anosov representation with Q-limit set Λ, and let $W \subseteq \mathrm{Hom}_{\mathrm{geom}}(\Gamma, G; \mathcal{P})$ be an extended Dehn filling space about ρ. Let $\rho_n \in W$ be a sequence of representations converging to ρ. Then:*

(1) The sequence ρ_n converges strongly to ρ, i.e. in $\mathrm{Hom}_{\mathrm{geom}}(\Gamma, G)$.
(2) For sufficiently large n, $\rho_n : \Gamma^{\rho_n} \to G$ is Q-EGF.
(3) There is a sequence of boundary sets Λ_n for ρ_n converging to Λ with respect to Hausdorff distance.

Moreover, if in addition each ρ_n is Q-divergent, the boundary sets in part (3) can all be taken to be the Q-limit set of ρ_n.

Proof. By Proposition 6.1, ρ is Q-EGF, with a homeomorphic boundary extension $\phi : \Lambda \to \partial(\Gamma, \mathcal{P})$. Parts (1) and (2) are immediate consequences of Proposition 5.1 and Proposition 5.2. Now, fix $\varepsilon > 0$. We also know from Proposition 5.1 that the boundary sets Λ_n for ρ_n eventually lie in the ε-neighborhood of Λ, so for part (3), we just need to show that Λ is eventually contained in the ε-neighborhood of Λ_n.

Since ϕ is a homeomorphism of compact sets, there is a finite family of compact subsets Z_i covering $\partial(\Gamma, \mathcal{P})$ such that the diameter of each $\phi^{-1}(Z_i)$ is at most $\varepsilon/2$. By Proposition 5.1 again, for sufficiently large n, Λ_n has nonempty intersection with the $\varepsilon/2$-neighborhood of each $\phi^{-1}(Z_i)$, which means an ε-neighborhood of Λ_n contains $\phi^{-1}(Z_i)$. Since the Z_i cover $\partial(\Gamma, \mathcal{P})$, this proves part (3). The final part of the proposition is an immediate consequence of Remark 5.38. ∎

6.1. **Filling to relative Anosov representations.** Usually, it is straightforward to detect whether the Dehn fillings in Theorem 1.4 and Theorem 1.8 give rise to (relatively) Anosov representations. The reason is the following relativization theorem of Wang, which (combined with our main theorem) immediately implies Corollary 1.6:

Theorem 6.4 ([Wan23], see also [Wei22, Sec. 4])**.** *Let $\rho : \Gamma \to G$ be a Q-EGF representation. Then ρ is (relatively) Q-Anosov if and only if ρ restricts to a (relatively) Q-Anosov representation on each $P \in \mathcal{P}$.*

7. The rank one case

In this section we prove Theorems 1.10 and 1.11 from the introduction. Below, let G be a **rank one** semisimple Lie group, and let $\mathbb{X}$ be the associated Riemannian symmetric space. We say that a representation $\rho : \Gamma \to G$ of a finitely generated group Γ is *geometrically finite* if it has finite kernel and geometrically finite image. We connect geometrically finite representations to EGF representations via Proposition 7.3 below, but first we introduce some terminology.

Definition 7.1. A *horospherical subgroup* of $H < G$ is any subgroup with noncompact closure in G which preserves a horosphere in the symmetric space $\mathbb{X}$. If $\rho : \Gamma \to G$ is a representation, then the *ρ-horospherical subgroups* are the maximal subgroups of Γ mapping to horospherical subgroups of G.

Remark 7.2. Horospherical subgroups are more usually called "parabolic subgroups" in the context of both rank-one Lie groups and convergence groups. However, this terminology conflicts with the definiton of a parabolic subgroup of a semisimple Lie group (which in this case is the *full* stabilizer in G of a point in $\partial\mathbb{X}$, including hyperbolic elements), so we avoid it.

Due to work of Bowditch [Bow12] [Bow95], whenever $\rho : \Gamma \to G$ is geometrically finite, then Γ is relatively hyperbolic, relative to any maximal collection $\mathcal{P}$ of pairwise nonconjugate ρ-horospherical subgroups. In this case, since ρ has finite kernel and discrete image, each group in $\mathcal{P}$ is a finite extension of a virtually nilpotent group. Then it follows from Gromov's theorem on groups with polynomial growth [Gro81] that each group in $\mathcal{P}$ is itself virtually nilpotent (see also [TZ24, Prop. 2.4] for a direct elementary proof of this fact).

It is also true that in this situation, the representation ρ is EGF with respect to any choice of $\mathcal{P}$ as above. More precisely, we have the following:

Proposition 7.3 (See [GW24, Prop 2.8 and Prop. 2.11])**.** *Let $(\Gamma, \mathcal{P})$ be a relatively hyperbolic pair, and let $\rho : \Gamma \to G$ be a representation. Suppose that every group in $\mathcal{P}$ is virtually nilpotent. Then the following are equivalent:*

(1) ρ is geometrically finite, and $\mathcal{P}$ contains a conjugate of each ρ-horospherical subgroup of Γ.
(2) ρ is EGF with respect to $\mathcal{P}$.

In this situation, if $\Lambda_\rho \subseteq \partial\mathbb{X}$ is the limit set of $\rho(\Gamma)$, there is a unique EGF boundary extension $\phi : \Lambda_\rho \to \partial(\Gamma, \mathcal{P})$. Moreover, if $\mathcal{P}$ only contains ρ-horospherical subgroups, this boundary extension is a homeomorphism.

Note that in the statement of the proposition, $\mathcal{P}$ could contain subgroups which are *not* ρ-horospherical; this matters because (especially in the context of Dehn filling) we want to be able to consider deformations ρ' of ρ where the ρ-horospherical subgroups are *not* in correspondence with the ρ'-horospherical subgroups.

Our main task in this section is to prove the proposition below, which immediately implies Theorem 1.10. Afterwards we will prove Theorem 1.11 as a corollary.

Proposition 7.4. *Let $\rho : \Gamma \to G$ be a geometrically finite representation and let $\mathcal{P}$ be a maximal collection of pairwise non-conjugate ρ-horospherical subgroups (so in particular $(\Gamma, \mathcal{P})$ is a relatively hyperbolic pair and ρ is EGF with respect to $\mathcal{P}$ by Proposition 7.3).*

Then, if $\rho_n : \Gamma \to G$ is a sequence of representations converging to ρ in $\mathrm{Hom}_{\mathrm{geom}}(\Gamma, G; \mathcal{P})$, the subset

$$\{\rho\} \cup \{\rho_n\}_{n\in\mathbb{N}}$$

is an extended Dehn filling space about ρ (with respect to the peripheral structure $\mathcal{P}$ and the boundary extension $\phi : \Lambda_\rho \to \partial(\Gamma, \mathcal{P})$ given in Proposition 7.3).

In the special case where each ρ_n in the proposition is faithful, then the result above is a direct consequence of Proposition 4.4 in [GW24]. In fact, very little work is needed to adapt the proof of the special case to the general statement, since the key lemma we need in both cases is exactly the same.

We state this key lemma below. To set this up, for any subset $V \subseteq \overline{\mathbb{X}}$, let $\mathcal{K}(V)$ denote the set of elements $g \in G$ which preserve some nonempty convex subset $C \subset V$. Then we have the following:

Lemma 7.5 (see [GW24, Lem. 4.6])**.** *Let $p \in \partial\mathbb{X}$, and let K, K' be compact subsets of $\partial\mathbb{X}$ not containing p. Then there exists an open neighborhood V of p in $\overline{\mathbb{X}}$, disjoint from K and K', so that the set of elements $g \in \mathcal{K}(V)$ satisfying $gK \cap gK' \neq \emptyset$ has compact closure in G.*

Remark 7.6. The cited lemma in [GW24] only states a special case of this result, but its proof applies verbatim to the general situation above.

Proof of Proposition 7.4. We proceed by contradiction, and suppose that the set $\{\rho\} \cup \{\rho_n\}_{n\in\mathbb{N}}$ is *not* an extended Dehn filling space. Since ρ_n converges to ρ in $\mathrm{Hom}_{\mathrm{geom}}(\Gamma, G; \mathcal{P})$, and each $P \in \mathcal{P}$ is ρ-horospherical, this can only occur if there exists:

- A ρ-horospherical subgroup $P \in \mathcal{P}$, with unique fixed point $p \in \partial(\Gamma, \mathcal{P})$,
- An open neighborhood U of $\phi^{-1}(p)$ in $\partial\mathbb{X}$,
- A compact subset $K \subset \partial\mathbb{X} \setminus U$,
- A finite set $F \subset P$, satisfying $\rho(h)K \subset U$ for every $h \in P \setminus F$, and
- A sequence of elements $h_n \in P$ with $\rho_n(h_n) \notin \rho_n(F)$,

such that $\rho_n(h_n)K \not\subset U$ for every n.

For any subgroup $H < G$, let $\Lambda(H)$ denote the limit set of H in $\partial\mathbb{X}$, let $\mathrm{Fix}(H)$ denote the fixed-point set of H in $\overline{\mathbb{X}}$, and let $\mathrm{Hull}(H)$ denote the convex hull in $\overline{\mathbb{X}}$ of $\Lambda(H) \cup \mathrm{Fix}(H)$. Since each $P \in \mathcal{P}$ is virtually nilpotent, and the restrictions $\rho_n|_P$ converge to $\rho|_P$ in the compact-open topology on $\mathrm{Hom}(P, G)$, it follows that $\mathrm{Hull}(\rho_n(P))$ converges to $\mathrm{Hull}(\rho(P))$, with respect to Hausdorff distance on any fixed metrization of $\overline{\mathbb{X}}$ (see [GW24, Lem. 4.5]). In particular, $\mathrm{Hull}(\rho_n(P))$ is nonempty for all large n, since $\mathrm{Hull}(\rho(P))$ is the singleton $\{p\}$.

Now, applying Lemma 7.5, there is an open neighborhood V of p in $\overline{\mathbb{X}}$ such that the set of elements $g \in \mathcal{K}(V)$ satisfying $gK \not\subset U$ is relatively compact in G. Then by the previous paragraph, $\mathrm{Hull}(\rho_n(P))$ eventually lies in V, and since each $\mathrm{Hull}(\rho_n(P))$ is a nonempty $\rho_n(h_n)$-invariant convex set, it follows that the set of elements appearing in the sequence $\rho_n(h_n)$ is relatively compact in G. Thus, a subsequence of $\rho_n(h_n)$ converges to some $g \in G$.

Since ρ_n converges to ρ in $\mathrm{Hom}_{\mathrm{geom}}(\Gamma, G; \mathcal{P})$, the restrictions $\rho_n|_P$ converge strongly to the restriction $\rho|_P$, and so actually $g = \rho(h)$ for some $h \in P$. Since $\rho_n(h_n)K$ is not contained in U, we must also have $\rho(h)K \not\subset U$ and thus h lies in the finite set F.

We also know that $\rho_n(h)$ converges to $\rho(h)$, so $\rho_n(h_n^{-1}h)$ converges to the identity. Now, the set of discrete subgroups of G is open in the space $\mathcal{C}(G)$ of closed subgroups of G with the Chabauty topology (see [BHK09, Prop 3.4]), which tells us that $\rho_n(P)$ is discrete for sufficiently large n. But in fact the proof of the cited proposition gives us a slightly stronger statement (see [GW24, Prop 4.1]), implying that the subgroups $\rho_n(P)$ are *uniformly* discrete—there is a neighborhood O of the identity in G so that $\rho_n(P) \cap O = \{\mathrm{id}\}$ for all sufficiently large n. This implies that $\rho_n(h_n) = \rho_n(h)$ for all sufficiently large n, which contradicts the fact that $\rho_n(h_n) \notin \rho_n(F)$ for all n. ☐

Proof of Theorem 1.11. The implication (2) $\implies$ (3) is the content of Proposition 7.4, and the implication (3) $\implies$ (1) follows from Theorem 1.8 since geometrically finite representations are always relatively Anosov. So we will be done if we show (1) $\implies$ (2).

For this, we can argue similarly to the proof of Proposition 4.4 in [GW24]. Assume that the sequence of subgroups $\rho_n(\Gamma)$ converges to the group $\rho(\Gamma)$ in the Chabauty topology. We wish to show that for any peripheral subgroup $P \in \mathcal{P}$, the subgroups $\rho_n(P)$ converge to $\rho(P)$ in the Chabauty topology. We employ Proposition 3.11. Algebraic convergence of representations passes to restrictions, which gives us (A) in the proposition. So, we just need to show condition (B), meaning we need to show that if h_n is a sequence in P, and $\rho_n(h_n)$ converges to $g \in G$, then $g \in \rho(P)$. So, suppose $\rho_n(h_n) \to g \in G$. Since we assume that $\rho_n \to \rho$ strongly, we know that $g \in \rho(\Gamma)$.

We know that P is virtually nilpotent, which means that $\rho_n(P)$ is as well. Arguing as in the proof of the previous proposition, this means that the ρ_n-invariant set $\mathrm{Hull}(\rho_n(P)) \subset \overline{\mathbb{X}}$ must converge to the singleton $\{z\}$, where $z \in \partial\mathbb{X}$ is the unique point fixed by $\rho(P)$. It follows that the limit g of $\rho_n(h_n)$ must fix z. However, since P is precisely the stabilizer of z for the ρ-action of Γ on $\partial\mathbb{X}$, we have $g \in \rho(P)$, as required. ☐

Appendix A. Relative Q-divergence and extended convergence actions

In this section of the paper, we prove Proposition 3.9. For convenience, we restate the proposition below:

Proposition A.1. *Let $(\Gamma, \mathcal{P})$ be a relatively hyperbolic pair, let Q be a symmetric parabolic subgroup of a semisimple Lie group G, and let $\rho : \Gamma \to G$ be a representation. Suppose that there exists a compact ρ-invariant set $\Lambda \subset G/Q$, an equivariant surjective antipodal map $\phi : \Lambda \to \partial(\Gamma, \mathcal{P}_\infty)$, and open sets $C_z \subset G/Q$ for each $z \in \partial(\Gamma, \mathcal{P}_\infty)$ satisfying conditions (C1*) and (C2) above.*

Then, if ρ is relatively Q-divergent, there exists a (possibly different) equivariant surjective antipodal map $\hat{\phi} : \hat{\Lambda} \to \partial(\Gamma, \mathcal{P}_\infty)$ and open sets $\{\hat{C}_z\}_{z \in \partial(\Gamma, \mathcal{P}_\infty)}$ satisfying conditions (C1) and (C2) in Definition 3.1; in other words, ρ is Q-EGF.

For the proof of the proposition, fix a representation $\rho : \Gamma \to G$, a map $\phi : \Lambda \to \partial(\Gamma, \mathcal{P}_\infty)$, and open sets $\{C_z\}_{z \in \partial(\Gamma, \mathcal{P}_\infty)}$ as in the statement, and assume that ρ is relatively Q-divergent. We have the following basic observation.

Lemma A.2. *Suppose that γ_n is a sequence in Γ converging to some $z \in \partial(\Gamma, \mathcal{P}_\infty)$. If $\rho(\gamma_n)$ is Q-divergent, then every Q-limit point of $\rho(\gamma_n)$ lies in $\phi^{-1}(z)$.*

Proof. Let ξ be a Q-limit point of $\rho(\gamma_n)$. Via Lemma 3.6, we can find a flag ξ^- and extract a subsequence so that $\rho(\gamma_n)$ converges to the constant map ξ uniformly on compact subsets of $\mathrm{Opp}(\xi^-)$. We can extract a further subsequence so that γ_n^{-1} converges to some $z_- \in \partial(\Gamma, \mathcal{P}_\infty)$.

Since $\mathrm{Opp}(\xi^-)$ is dense in G/Q, it has nonempty intersection with the open set C_{z_-}. Let K be a nonempty compact subset in this intersection. Then it follows from condition (C2) that $\rho(\gamma_n)K$ eventually lies in an arbitrary small neighborhood of $\phi^{-1}(z)$; since $\rho(\gamma_n)K$ must converge to $\{\xi\}$ this means that $\xi \in \phi^{-1}(z)$. □

Using this lemma, we show that one can shrink the set Λ into a certain nice form, while still preserving all of the assumptions. For each conical limit point $z \in \partial(\Gamma, \mathcal{P}_\infty)$, define $\Psi_z \subset G/Q$ to be the union of the set of Q-limit points of all sequences $\rho(\gamma_n)$, where $\gamma_n \to z$ in the Bowditch compactification of Γ. (Note that since z is conical, any sequence γ_n converging to z must satisfy $|\gamma_n|_{\hat{\Gamma}} \to \infty$ and therefore by assumption $\rho(\gamma_n)$ is Q-divergent and Ψ_z is nonempty.)

Then, define a subset $\hat{\Lambda} \subset G/Q$ via:

$$\hat{\Lambda} = \left(\bigcup_{\substack{z \in \partial(\Gamma, \mathcal{P}_\infty), \\ z \text{ parabolic}}} \phi^{-1}(z) \right) \cup \left(\bigcup_{\substack{z \in \partial(\Gamma, \mathcal{P}_\infty), \\ z \text{ conical}}} \Psi_z \right).$$

Lemma A.3. *The set $\hat{\Lambda}$ is a compact ρ-invariant subset of Λ.*

Proof. That $\hat{\Lambda} \subset \Lambda$ is an immediate consequence of Lemma A.2. The ρ-invariance of $\hat{\Lambda}$ follows from ρ-invariance of Λ, as well as the Γ-invariance of the set of conical limit points in $\partial(\Gamma, \mathcal{P}_\infty)$. So, we just need to show that $\hat{\Lambda}$ is compact.

Consider a sequence of flags $\xi_n \in \hat{\Lambda}$, and extract a subsequence so that $\xi_n \to \xi$. We wish to show $\xi \in \hat{\Lambda}$. Since $\hat{\Lambda} \subset \Lambda$, there is a sequence of points $z_n \in \partial(\Gamma, \mathcal{P}_\infty)$ such that $\xi_n \in \phi^{-1}(z_n)$. By compactness of Λ and continuity of ϕ, we know $\xi \in \Lambda$ and $z_n \to z = \phi(\xi)$. If z is parabolic, then $\phi^{-1}(z) \subset \hat{\Lambda}$, hence $\xi \in \hat{\Lambda}$, so assume z is conical.

Let γ_m be a sequence in Γ converging conically to z. By definition this means there are distinct points $u, v \in \partial(\Gamma, \mathcal{P}_\infty)$ such that $\gamma_m^{-1} z \to u$ and $\gamma_m^{-1} K \to \{v\}$ for all compact sets $K \subset \partial(\Gamma, \mathcal{P}_\infty) \setminus \{z\}$; note that this implies $\gamma_m^{-1} \to v$ in the Bowditch compactification $\overline{\Gamma}$. Since γ_m converges to a conical point, we must have $|\gamma_m|_{\hat{\Gamma}} \to \infty$, so $\rho(\gamma_m)$ is Q-divergent. After extracting a subsequence, we can apply Lemma A.2 and Lemma 3.6 to see that there is a flag $\xi^- \in \phi^{-1}(v)$ such that $\rho(\gamma_n)$ converges to the constant map ξ, uniformly on compacts in $\mathrm{Opp}(\xi^-)$.

Now, since z_n converges to z, for each n we can find an index m_n so that $m_n \to \infty$, but $\gamma_{m_n}^{-1} z_n$ does *not* accumulate at v. Then, since ϕ is antipodal and equivariant, this means that $\rho(\gamma_{m_n}^{-1})\xi_n \in \phi^{-1}(\gamma_{m_n}^{-1} z_n)$ lies in a compact subset of $\mathrm{Opp}(\xi^-)$. Therefore $\xi_n = \rho(\gamma_{m_n})\rho(\gamma_{m_n}^{-1})\xi_n$ must converge to ξ, and we are done. □

Remark A.4. It turns out that actually each set Ψ_z defined above must be a singleton (see [Wei22, Prop. 4.8]), but we will not show this at the moment.

Restrict ϕ to obtain a new map $\hat{\phi} : \hat{\Lambda} \to \partial(\Gamma, \mathcal{P}_\infty)$. This map is also equivariant, surjective, and antipodal. Next, we define a new family $\hat{C}_z$ of repelling strata. For each conical limit point $z \in \partial(\Gamma, \mathcal{P}_\infty)$, define

$$\hat{C}_z = \mathrm{Opp}(\hat{\phi}^{-1}(z)).$$

Then, for each parabolic point $p \in \Pi$, let P be the stabilizer of p in Γ, and define

$$\hat{C}_p = \mathrm{Opp}(\hat{\phi}^{-1}(p)) \cap \bigcup_{\gamma \in P} \rho(\gamma) C_p.$$

Finally, for an arbitrary parabolic point $z \in \partial(\Gamma, \mathcal{P}_\infty)$, write $z = gp$ for $p \in \Pi$, $g \in \Gamma$, and define $\hat{C}_z = \rho(g)\hat{C}_p$.

The sets $\hat{C}_z$ are nonempty open sets satisfying condition (C1*); this is a consequence of transversality of the map $\hat{\phi}$ and the fact that the original strata C_z satisfied (C1*). Next we show:

Proposition A.5. *The map $\hat{\phi} : \hat{\Lambda} \to \partial(\Gamma, \mathcal{P}_\infty)$ and sets $\hat{C}_z$ satisfy condition (C2) in Definition 3.1.*

Proof. Consider a sequence $\gamma_n \in \Gamma$ such that $\gamma_n^{\pm 1} \to z_\pm \in \partial(\Gamma, \mathcal{P}_\infty)$, and let $K \subset \hat{C}_{z_-}$ be compact. Fix an arbitrary subsequence of γ_n. We will show that there is a further subsequence so that $\rho(\gamma_n)K$ lies in an arbitrarily small neighborhood of $\hat{\phi}^{-1}(z_+)$. After extraction we can assume that either $|\gamma_n|_{\hat{\Gamma}} \to \infty$, or else $|\gamma_n|_{\hat{\Gamma}}$ is bounded.

In the first case, the sequences $\rho(\gamma_n^{\pm 1})$ are both Q-divergent by assumption, and by Lemma A.2 all of the Q-limit points of $\rho(\gamma_n^{\pm 1})$ respectively lie in $\hat{\phi}^{-1}(z_\pm)$. We have constructed the strata $\hat{C}_z$ so that $\hat{C}_z \subseteq \mathrm{Opp}(\hat{\phi}^{-1}(z))$ for every $z \in \partial(\Gamma, \mathcal{P}_\infty)$. In particular, this implies that $K \subset \hat{C}_{z_-}$ is opposite to every Q-limit point of the sequence $\rho(\gamma_n^{-1})$. Thus by Lemma 3.6, $\rho(\gamma_n)K$ eventually lies in an arbitrarily small neighborhood of the Q-limit points of $\rho(\gamma_n)$, hence in an arbitrarily small neighborhood of $\hat{\phi}^{-1}(z_+)$.

Otherwise, if $|\gamma_n|_{\hat{\Gamma}}$ is bounded, then both z_+ and z_- must be parabolic. Observe that, by definition, $\hat{C}_{z_-}$ is contained in a union of sets of the form $\rho(g)C_q$, where $q \in \partial(\Gamma, \mathcal{P}_\infty)$ is a parabolic point satisfying $gq = z_-$. The compact set K is contained in a finite union of these sets, and in fact K can be written as a finite union $K_1, \ldots, K_n$ of compact pieces, each of which lies in such a set $\rho(g_i)C_{q_i}$. Each g_i is fixed (independent of n), so $g_i^{-1}\gamma_n^{-1} \to g_i^{-1}z_-$, and $\gamma_n g_i \to z_+$. Then, as the original repelling strata C_{q_i} already satisfied condition (C2), the set

$$\rho(\gamma_n)K_i = \rho(\gamma_n g_i)\rho(g_i^{-1})K_i$$

eventually lies in an arbitrarily small neighborhood of $\phi^{-1}(z_+) = \hat{\phi}^{-1}(z_+)$, and therefore the same is true for K. ☐

The proposition below completes the proof of Proposition A.1.

Proposition A.6. *The map $\hat{\phi} : \hat{\Lambda} \to \partial(\Gamma, \mathcal{P}_\infty)$ and sets $\hat{C}_z$ satisfy condition (C1) in Definition 3.1.*

Proof. Fix $K \subset \partial(\Gamma, \mathcal{P}_\infty)$ compact. It suffices to show that for every $w \in \partial(\Gamma, \mathcal{P}_\infty) \setminus K$, the set $\bigcap_{z \in K} \hat{C}_z$ contains a neighborhood of $\hat{\phi}^{-1}(w)$. Suppose for a contradiction that this is not the case. We have already seen that the strata $\hat{C}_z$ satisfy condition (C1*), so the only possibility is that there is a sequence of points $z_n \in K$ so that for every neighborhood U of $\hat{\phi}^{-1}(w)$, the set $\hat{C}_n = \hat{C}_{z_n}$ eventually fails to contain U.

Up to a subsequence, every z_n is either conical or parabolic. Suppose first that every z_n is conical, in which case $\hat{C}_n = \mathrm{Opp}(\hat{\phi}^{-1}(z_n))$ for every n. Thus, there is a sequence of flags $\xi_n \in \hat{\phi}^{-1}(z_n)$, and a sequence of flags $\eta_n \in G/Q$, so that η_n accumulates in $\hat{\phi}^{-1}(w)$, and ξ_n, η_n are not transverse. After extraction, ξ_n converges to $\xi \in \hat{\phi}^{-1}(K)$ and η_n converges to $\eta \in \hat{\phi}^{-1}(w)$. The flags ξ and η are not transverse, which contradicts transversality of $\hat{\phi}$.

So, now consider the case where each z_n is parabolic. Let $X = X(\Gamma, \mathcal{P})$ be the cusped space, and fix a bi-infinite geodesic $c_n : (-\infty, \infty) \to X$ with its forward endpoint at z_n and backward endpoint at w. This geodesic must eventually enter a combinatorial horoball in X centered at z_n, so let $\gamma_n \in \Gamma$ be the earliest point where this occurs. Then, $\gamma_n^{-1} z_n$ is a parabolic point in $\partial(\Gamma, \mathcal{P}_\infty)$, in the boundary of a combinatorial horoball in X passing through the identity, and so up to subsequence $\gamma_n^{-1} z_n = p$ for a fixed parabolic point $p \in \Pi$.

We may assume that z_n converges to some $z \in K$. Now, if the sequence γ_n is bounded in $\mathrm{Cay}(\Gamma)$, then eventually $z_n = z$ for all n, and therefore an open neighborhood of $\hat{\phi}^{-1}(w)$ is contained in every $\hat{C}_n = \hat{C}_z$ by condition (C1*). So, assume that $|\gamma_n|_\Gamma \to \infty$. We claim that in this case, γ_n converges to z. To see this, observe that since z_n does not accumulate at w, the bi-infinite geodesic c_n passes within a uniform distance of the identity in X. This means that the point γ_n must actually lie uniformly close (in X) to a point $r_n(t_n)$, where r_n is a geodesic ray in X from the identity to z_n, and $t_n \to \infty$. Since $z_n \to z$ it follows that $r_n(t_n)$, hence γ_n, tends to z.

Now, since our chosen strata satisfy condition (C1*), and $w \neq z$, there is a neighborhood U of $\hat{\phi}^{-1}(w)$ whose closure lies in $\hat{C}_z$. Extract a subsequence so that $\gamma_n^{-1} \to y$ for some $y \in \partial(\Gamma, \mathcal{P}_\infty)$. From Proposition A.5, the repelling strata satisfy condition (C2), so $\rho(\gamma_n^{-1})U$ eventually lies in an arbitrarily small neighborhood of $\hat{\phi}^{-1}(y)$. However, we also know that $y \neq p$: there is a bi-infinite geodesic $\gamma_n^{-1} c_n$ in X from p to $\gamma_n^{-1} w$ passing through the identity, and since $w \neq z$, we have $\gamma_n^{-1} w \to y$ and therefore there is also a geodesic in X joining p to y.

By (C1*) again, the interior of $\hat{C}_p$ contains $\hat{\phi}^{-1}(y)$, and therefore eventually $\hat{C}_p$ contains $\rho(\gamma_n^{-1})U$. Then by equivariance, $\hat{C}_n = \rho(\gamma_n)\hat{C}_p$ contains U for all sufficiently large n. This contradicts our original assumption. □

References

[Aco16] Miguel Acosta. "Spherical CR Dehn surgeries". In: *Pacific J. Math.* 284.2 (2016), pp. 257–282. ISSN: 0030-8730,1945-5844.

[Aco19] Miguel Acosta. "Spherical CR uniformization of Dehn surgeries of the Whitehead link complement". In: *Geom. Topol.* 23.5 (2019), pp. 2593–2664. ISSN: 1465-3060,1364-0380.

[AGM09] Ian Agol, Daniel Groves, and Jason Fox Manning. "Residual finiteness, QCERF and fillings of hyperbolic groups". In: *Geom. Topol.* 13.2 (2009), pp. 1043–1073. ISSN: 1465-3060,1364-0380.

[BDLM] Samuel Ballas, Jeffrey Danciger, Gye-Seon Lee, and Ludovic Marquis. *Exotic real projective Dehn filling spaces.* To appear.

[BH20] Brendan Burns Healy and G. Christopher Hruska. "Cusped spaces and quasi-isometries of relatively hyperbolic groups". In: *arXiv e-prints*, arXiv:2010.09876 (Oct. 2020), arXiv:2010.09876. arXiv: 2010.09876 [math.GR].

[BH99] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature.* Vol. 319. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 1999, pp. xxii+643. ISBN: 3-540-64324-9.

[BHK09] Martin R. Bridson, Pierre de la Harpe, and Victor Kleptsyn. "The Chabauty space of closed subgroups of the three-dimensional Heisenberg group". English. In: *Pac. J. Math.* 240.1 (2009), pp. 1–48. ISSN: 1945-5844.

[Bog97] O. V. Bogopol'skiĭ. "Infinite commensurable hyperbolic groups are bi-Lipschitz equivalent". In: *Algebra i Logika* 36.3 (1997), pp. 259–272, 357. ISSN: 0373-9252.

[Bow12] B. H. Bowditch. "Relatively Hyperbolic Groups". In: *International Journal of Algebra and Computation* 22.03 (May 2012), p. 1250016.

[Bow95] B. H. Bowditch. "Geometrical finiteness with variable negative curvature". In: *Duke Math. J.* 77.1 (1995), pp. 229–274. ISSN: 0012-7094,1547-7398.

[Bow99] B. H. Bowditch. "Convergence groups and configuration spaces". In: *Geometric group theory down under (Canberra, 1996)*. de Gruyter, Berlin, 1999, pp. 23–54.

[BPS19] Jairo Bochi, Rafael Potrie, and Andrés Sambarino. "Anosov representations and dominated splittings". In: *J. Eur. Math. Soc. (JEMS)* 21.11 (2019), pp. 3343–3414. ISSN: 1435-9855.

[CLM20] Suhyoung Choi, Gye-Seon Lee, and Ludovic Marquis. "Convex projective generalized Dehn filling". In: *Ann. Sci. Éc. Norm. Supér. (4)* 53.1 (2020), pp. 217–266. ISSN: 0012-9593,1873-2151.

[CZZ22] Richard Canary, Tengren Zhang, and Andrew Zimmer. "Cusped Hitchin representations and Anosov representations of geometrically finite Fuchsian groups". In: *Adv. Math.* 404 (2022), Paper No. 108439, 67. ISSN: 0001-8708,1090-2082.

[Ger09] Victor Gerasimov. "Expansive convergence groups are relatively hyperbolic". In: *Geom. Funct. Anal.* 19.1 (2009), pp. 137–169. ISSN: 1016-443X,1420-8970.

[GGKW17] François Guéritaud, Olivier Guichard, Fanny Kassel, and Anna Wienhard. "Anosov Representations and Proper Actions". In: *Geometry & Topology* 21.1 (2017), pp. 485–584.

[GM08] Daniel Groves and Jason Fox Manning. "Dehn Filling in Relatively Hyperbolic Groups". In: *Israel Journal of Mathematics* 168 (2008), pp. 317–429.

[GM21] Daniel Groves and Jason Fox Manning. "Quasiconvexity and Dehn filling". In: *American Journal of Mathematics* 143.1 (2021), pp. 95–124.

[GMS19] Daniel Groves, Jason Fox Manning, and Alessandro Sisto. "Boundaries of Dehn fillings". In: *Geom. Topol.* 23.6 (2019), pp. 2929–3002. ISSN: 1465-3060,1364-0380.

[GR70] H. Garland and M. S. Raghunathan. "Fundamental domains for lattices in (R-)rank 1 semisimple Lie groups". In: *Ann. of Math. (2)* 92 (1970), pp. 279–326. ISSN: 0003-486X.

[Gro81] Michael Gromov. "Groups of polynomial growth and expanding maps (with an appendix by Jacques Tits)". In: *Publications Mathématiques de l'IHÉS* 53 (1981), pp. 53–78.

[GW12] Olivier Guichard and Anna Wienhard. "Anosov representations: domains of discontinuity and applications". In: *Invent. Math.* 190.2 (2012), pp. 357–438. ISSN: 0020-9910.

[GW24] Antonin Guilloux and Theodore Weisman. "Limits of limit sets in rank-one symmetric spaces". In: *arXiv e-prints*, arXiv:2407.04301 (July 2024), arXiv:2407.04301. arXiv: 2407.04301 [math.GT].

[KL23] Michael Kapovich and Bernhard Leeb. "Relativizing characterizations of Anosov subgroups, I". In: *Groups Geom. Dyn.* 17.3 (2023). With an appendix by Gregory A. Soifer, pp. 1005–1071. ISSN: 1661-7207,1661-7215.

[KLP17] Michael Kapovich, Bernhard Leeb, and Joan Porti. "Anosov Subgroups: Dynamical and Geometric Characterizations". In: *European Journal of Mathematics* 3.4 (2017), pp. 808–898.

[Lab06] François Labourie. "Anosov Flows, Surface Groups and Curves in Projective Space". In: *Inventiones Mathematicae* 165.1 (2006), pp. 51–114.

[Mar84] G. A. Margulis. "Arithmeticity of the irreducible lattices in the semisimple groups of rank greater than 1". In: *Invent. Math.* 76.1 (1984), pp. 93–120. ISSN: 0020-9910,1432-1297.

[McM99] Curtis T. McMullen. "Hausdorff dimension and conformal dynamics. I: Strong convergence of Kleinian groups". English. In: *J. Differ. Geom.* 51.3 (1999), pp. 471–515. ISSN: 0022-040X.

[MMW22] Kathrynn Mann, Jason Fox Manning, and Theodore Weisman. *Stability of hyperbolic groups acting on their boundaries*. 2022.

[MMW24] Kathryn Mann, Jason Fox Manning, and Theodore Weisman. "Topological stability of relatively hyperbolic groups acting on their boundaries". In: *arXiv e-prints*, arXiv:2402.06144 (Feb. 2024), arXiv:2402.06144. arXiv: `2402.06144 [math.GR]`.

[MR18] Bruno Martelli and Stefano Riolo. "Hyperbolic Dehn filling in dimension four". In: *Geom. Topol.* 22.3 (2018), pp. 1647–1716. ISSN: 1465-3060,1364-0380.

[Osi07] Denis V. Osin. "Peripheral Fillings of Relatively Hyperbolic Groups". In: *Inventiones Mathematicae* 167.2 (2007), pp. 295–326.

[Sch07] Richard Evan Schwartz. *Spherical CR Geometry and Dehn Surgery (AM-165)*. Princeton University Press, 2007. ISBN: 9780691128108.

[Sch89] Viktor Schroeder. "A cusp closing theorem". In: *Proc. Amer. Math. Soc.* 106.3 (1989), pp. 797–802. ISSN: 0002-9939,1088-6826.

[Tuk94] Pekka Tukia. "Convergence groups and Gromov's metric hyperbolic spaces". In: *New Zealand J. Math.* 23.2 (1994), pp. 157–187. ISSN: 1171-6096.

[TZ24] Konstantinos Tsouvalas and Feng Zhu. "Topological restrictions on relatively Anosov representations". In: *arXiv e-prints*, arXiv:2401.03050 (Jan. 2024), arXiv:2401.03050. arXiv: `2401.03050 [math.GR]`.

[Wan23] Tianqi Wang. *Notions of Anosov representation of relatively hyperbolic groups*. 2023. arXiv: `2309.15636 [math.GT]`.

[Wei22] Theodore Weisman. *An Extended Definition of Anosov Representation for Relatively Hyperbolic Groups*. Sept. 1, 2022. arXiv: `arXiv:2205.07183`. (Visited on 11/13/2022). preprint.

[Wei23] Theodore Weisman. "Examples of extended geometrically finite representations". In: *arXiv e-prints*, arXiv:2311.18653 (Nov. 2023), arXiv:2311.18653. arXiv: `2311.18653 [math.GT]`.

[Yam04] Asli Yaman. "A topological characterisation of relatively hyperbolic groups". In: *J. Reine Angew. Math.* 566 (2004), pp. 41–89. ISSN: 0075-4102.

[Zhu21] Feng Zhu. "Relatively dominated representations". In: *Ann. Inst. Fourier (Grenoble)* 71.5 (2021), pp. 2169–2235. ISSN: 0373-0956,1777-5310.

[ZZ22] Feng Zhu and Andrew Zimmer. "Relatively Anosov representations via flows I: theory". In: *arXiv e-prints*, arXiv:2207.14737 (July 2022), arXiv:2207.14737. arXiv: `2207.14737 [math.GT]`.

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