AN EXTENDED DEFINITION OF ANOSOV REPRESENTATION FOR RELATIVELY HYPERBOLIC GROUPS

THEODORE WEISMAN

ABSTRACT. We define a new family of discrete representations of relatively hyperbolic groups which unifies many existing definitions and examples of geometrically finite behavior in higher rank. The definition includes the relative Anosov representations defined by Kapovich-Leeb and Zhu, and Zhu-Zimmer, as well as holonomy representations of various different types of "geometrically finite" convex projective manifolds. We prove that these representations are all stable under deformations whose restriction to the peripheral subgroups satisfies a dynamical condition, in particular allowing for deformations which do not preserve the conjugacy class of the peripheral subgroups.

CONTENTS

1. INTRODUCTION

1.1. **Overview.** Classically, the best understood discrete subgroups of rank one Lie groups have been those that are *convex cocompact*, or equivalently quasi-isometrically embedded. Only slightly less well-behaved, however, are the geometrically finite subgroups. While geometrically finite subgroups are not always quasi-isometrically embedded, all of their distortion is confined to isolated "cuspidal" regions of their orbits. Thus, these subgroups can still be understood using hyperbolic geometry, by piecing together the behavior of their peripheral subgroups.

Since the practice of isolating non-hyperbolic behavior has been succesful in rank one, it is reasonable to try and extend the technique when investigating the still mysterious world of discrete subgroups of higher-rank Lie groups. The natural generalization of convex cocompact groups in higher rank is given by *Anosov subgroups*. Originally defined by Labourie [Lab06]

Date: July 4, 2024.

and Guichard-Wienhard [GW12], Anosov subgroups are a class of Gromov-hyperbolic discrete subgroups of semisimple Lie groups, which have many dynamical and geometric properties in common with convex cocompact groups. They have allowed many tools and ideas from rank-one geometry to be applied fruitfully in higher rank.

The main goal in this paper is therefore to provide a higher-rank version of geometrical finiteness by defining a new class of subgroups which have isolated "non-Anosov" behavior. As in rank one, in this situation we can hope to understand the entire subgroup by stitching together what happens on the isolated pieces.

Formally, we introduce the theory of *extended geometrically finite* (or *EGF*) representations as a unifying framework for studying "relativized Anosov" groups. An EGF representation is always a discrete finite-kernel representation of a relatively hyperbolic group Γ into some semisimple Lie group G. An important feature of the definition is that it places essentially no restrictions on the induced representations of the peripheral subgroups of Γ. This strongly contrasts with previous (less general) notions of relative Anosov representation due to Kapovich-Leeb [KL23] and Zhu [Zhu21] (see also [ZZ22]). These definitions unavoidably impose strict requirements on peripherals, and thus cannot capture many interesting examples of intrinsically higher-rank behavior.

The starting point for our framework is a definition of Anosov representation in terms of topological dynamics: if Γ is a hyperbolic group, G is a semisimple Lie group, and $P \subset G$ is a symmetric parabolic subgroup, a representation $\rho : \Gamma \to G$ is P-Anosov if there is a ρ-equivariant embedding $\xi : \partial\Gamma \to G/P$ of the Gromov boundary $\partial\Gamma$ of Γ satisfying certain dynamical properties.

Existing notions of relative Anosov representation simply replace ξ with an embedding of the Bowditch boundary of a relatively hyperbolic group. Our idea is to instead *reverse the direction* of the boundary map: we characterize geometrical finiteness via the existence of an equivariant map *from* a closed subset of a flag manifold *to* the Bowditch boundary of a relatively hyperbolic group, rather than the other way around.

This "backwards" boundary map does not need to be a homeomorphism, which makes our definition much more flexible. For instance, the relative Anosov subgroups of Kapovich-Leeb and Zhu can have no subgroup which is isomorphic to a lattice in a higher-rank Lie group—a limitation which is *not* shared by EGF representations. The definition also makes EGF representations better suited for studying the rich world of *convex projective structures* on manifolds, which provide a number of examples of discrete subgroups of Lie groups displaying an intriguing mix of "higher rank" and "rank one" phenomena. In particular, EGF representations interact well with the theory of *convex cocompact* projective actions developed by Danciger-Guéritaud-Kassel [DGK17]; see Section 1.4 below.

Relative stability. Another significant advantage of the flexibility inherent in our definition is that it allows for a strong *stability* property (see Theorem 1.4 below), which generalizes the stability of Anosov representations originally demonstrated by Labourie [Lab06]. It is not true that an arbitrary (sufficiently small) deformation of an EGF representation is still EGF; indeed, it is possible to find small deformations of geometrically finite representations in rank one which are not even discrete. However, we prove that any EGF representation $\rho : \Gamma \to G$ is *relatively* stable: any small deformation of ρ in $\mathrm{Hom}(\Gamma, G)$ which satisfies a condition on the peripheral subgroups is also EGF.

This peripheral condition—which we call *peripheral stability*—is very general, and can hold even in the absence of a topological conjugacy between the actions of the original peripheral subgroups and their deformations. As a result, EGF representations provide a unifying framework for understanding the transitions between discrete relatively hyperbolic groups with qualitatively different cuspidal behavior (see e.g. [Wei23b, Sec. 5]). In particular, the notion of peripheral stability can even describe transitions between "Anosov" and "non-Anosov" behavior of cusp groups. This means that our framework can be used to understand situations where non-Anosov subgroups occur as limits of Anosov subgroups.

In fact, even in the rank-one case (where Anosov subgroups are precisely same as convex cocompact groups), such transitions are still not completely understood, and in upcoming work [GW24] we plan to explore applications of the theory of EGF representations in this setting specifically.

We note also that the main stability theorem for EGF representations can be used to deduce new results about less dramatic transitions between representations. For instance, since EGF representations generalize the relative Anosov representations of Kapovich-Leeb and Zhu, one can apply the theorem to obtain a relative stability result for relative Anosov representations (see Section 1.5 below for a comparison of this result with independent work of Zhu-Zimmer). In another direction, see [Wei23b, Section 6] for an application of the theorem towards an *absolute* (i.e. non-relative) stability property for a family of discrete non-hyperbolic groups; these results provide new examples of stably quasi-isometrically embedded subgroups of Lie groups which are neither Anosov nor rigid.

Relative automata. To prove the general stability theorem, we introduce a general tool: a *relative automaton* giving a "relative coding" for points in the Bowditch boundary of an arbitrary relatively hyperbolic group. The construction is loosely based on Sullivan's symbolic coding of points in the limit set of a convex cocompact group in rank one (recently adapted and generalized by Kapovich-Kim-Lee [KKL19]), as well as an "automatic" description of Anosov representations due to Bochi-Potrie-Sambarino [BPS19]. We expand on the basic idea in two different ways simultaneously.

First, rather than coding *points* in the "limit set" of some discrete faithful representation, we essentially code *fibers* in the invariant set Λ surjecting onto the Bowditch boundary of our relatively hyperbolic group. Second, we provide a way to code parabolic points (or more accurately, "parabolic fibers") in a way which is compatible with the coding for generic points in the Bowditch boundary.

We remark again that even though both of these innovations were originally developed to better understand the behavior of discrete subgroups of higher-rank Lie groups, they are also directly useful outside of this context. For example, in [MMW22], [MMW24], we show how to apply the ideas appearing in this paper towards the theory abstract (relatively) hyperbolic groups, and in future work [GW24] we will explore applications to geometrically finite subgroups in rank-one. It seems possible that there are additional applications beyond even these, for instance towards subgroups of mapping class groups.

1.2. **Main definition.** In the rest of the introduction we give some more detail regarding the central definition of this paper and the main results surrounding it.

If Γ is a relatively hyperbolic group, relative to a collection $\mathcal{H}$ of peripheral subgroups, then Γ acts as a *convergence group* on the Bowditch boundary $\partial(\Gamma, \mathcal{H})$. We recall the definition here.

Definition 1.1. Let Γ act on a topological space M. The group Γ is said to act as a *convergence group* if for every infinite sequence of distinct elements $\gamma_n \in \Gamma$, there exist points $a, b \in M$ and a subsequence $\gamma_m \in \Gamma$ such that γ_m converges uniformly on compacts in $M - \{a\}$ to the constant map b.

When γ_n is a sequence of distinct elements in a relatively hyperbolic group Γ, then γ_n converges to b uniformly on compacts in $\partial(\Gamma, \mathcal{H}) - \{a\}$ if and only if γ_n converges to b in the compactification $\overline{\Gamma} = \Gamma \sqcup \partial(\Gamma, \mathcal{H})$ and the inverse sequence γ_n^{-1} converges to a.

Recall that if a group Γ acts by homeomorphisms on a Hausdorff space X, the pair (Γ, X) is called a *topological dynamical system*. We say that an *extension* of (Γ, X) is a topological dynamical system (Γ, Y) together with a Γ-equivariant surjective map $\phi : Y \to X$.

In this paper, when $(\Gamma, \mathcal{H})$ is a relatively hyperbolic pair (i.e. Γ is hyperbolic relative to a collection $\mathcal{H}$ of peripheral subgroups), we will consider *embedded extensions* of the topological dynamical system $(\Gamma, \partial(\Gamma, \mathcal{H}))$. We want these embedded extensions to respect the convergence group action of $(\Gamma, \partial(\Gamma, \mathcal{H}))$ in some sense, so we introduce the following definition:

Definition 1.2. Let $(\Gamma, \mathcal{H})$ be a relatively hyperbolic pair, with Γ acting on a connected compact metrizable space M by homeomorphisms. Let $\Lambda \subset M$ be a closed Γ-invariant set.

We say that a continuous equivariant surjective map $\phi : \Lambda \to \partial(\Gamma, \mathcal{H})$ *extends the convergence action of* Γ if for each $z \in \partial(\Gamma, \mathcal{H})$, there exists an open set $C_z \subset M$ containing $\Lambda - \phi^{-1}(z)$, satisfying the following:

If γ_n is a sequence in Γ with $\gamma_n^{\pm 1} \to z_\pm$ for $z_\pm \in \partial(\Gamma, \mathcal{H})$, then for any compact set $K \subset C_{z_-}$ and any open set U containing $\phi^{-1}(z_+)$, for sufficiently large n, $\gamma_n \cdot K$ lies in U.

Now let G denote a semisimple Lie group with no compact factor. The central definition of the paper is the following:

Definition 1.3. Let Γ be a relatively hyperbolic group, let $\rho : \Gamma \to G$ be a representation, and let $P \subset G$ be a symmetric parabolic subgroup. We say that ρ is *extended geometrically finite* (EGF) with respect to P if there exists a closed $\rho(\Gamma)$-invariant set $\Lambda \subset G/P$ and a continuous ρ-equivariant surjective antipodal map $\phi : \Lambda \to \partial(\Gamma, \mathcal{H})$ extending the convergence action of Γ.

The map ϕ is called a *boundary extension* of the representation ρ, and the closed invariant set Λ is called the *boundary set*.

We refer to Section 4 for the definition of "antipodal map" in this context.

1.3. **Main results.** Like (relative) Anosov representations, extended geometrically finite representations are always discrete with finite kernel (see 4.1). The central result of this paper says that EGF representations have a *relative stability* property: if ρ is an EGF representation, then certain small relative deformations of ρ must also be EGF.

To state the theorem, we define a notion of a *peripherally stable* subspace of $\mathrm{Hom}(\Gamma, G)$. The precise definition is given in Section 9, but roughly speaking, a subspace $\mathcal{W} \subseteq \mathrm{Hom}(\Gamma, G)$ is *peripherally stable* if the large-scale dynamical behavior of the peripheral subgroups of Γ is in some sense preserved by small deformations inside of $\mathcal{W}$. We emphasize again that the action of a deformed peripheral subgroup does *not* need to be even topologically conjugate to the action of the original peripheral subgroup.

We prove the following:

Theorem 1.4. *Let $\rho : \Gamma \to G$ be EGF with respect to P, let $\phi : \Lambda \to \partial(\Gamma, \mathcal{H})$ be a boundary extension, and let $\mathcal{W} \subseteq \mathrm{Hom}(\Gamma, G)$ be peripherally stable at (ρ, ϕ). For any compact subset Z of $\partial(\Gamma, \mathcal{H})$ and any open set $V \subset G/P$ containing $\phi^{-1}(Z)$, there is an open subset $\mathcal{W}' \subset \mathcal{W}$ containing ρ such that each $\rho' \in \mathcal{W}'$ is EGF with respect to P, and has an EGF boundary extension ϕ' satisfying $\phi'^{-1}(Z) \subset V$.*

Remark 1.5. When $\rho : \Gamma \to G$ is a P-Anosov representation of a hyperbolic group Γ, then the associated boundary embedding $\partial\Gamma \to G/P$ also varies continuously with ρ in the compact-open topology on maps $\partial\Gamma \to G/P$. Since EGF representations come with boundary *extensions* (rather than embeddings), Theorem 1.4 only gives us a *semicontinuity* result.

We expect that it is possible to extend the methods of this paper to prove stronger continuity results when the original EGF representation $\rho : \Gamma \to G$ satisfies additional assumptions; see the discussion following Corollary 1.12.

While the peripheral stability condition in Theorem 1.4 is mildly technical, we can also apply it to yield more concrete results.

Definition 1.6. Let $(\Gamma, \mathcal{H})$ be a relatively hyperbolic pair, and let $\rho : \Gamma \to G$ be a representation. The space of *cusp-preserving* representations $\mathrm{Hom}_{\mathrm{cp}}(\Gamma, G, \mathcal{H}, \rho)$ is the set of representations $\rho' : \Gamma \to G$ such that for each peripheral subgroup $H \in \mathcal{H}$, we have $\rho'|_H = g \cdot \rho|_H \cdot g^{-1}$ for some $g \in G$ (which may depend on H).

Corollary 1.7. *Let $\rho : \Gamma \to G$ be an EGF representation. Then there is a neighborhood of ρ in $\mathrm{Hom}_{\mathrm{cp}}(\Gamma, G, \mathcal{H}, \rho)$ consisting of EGF representations.*

Corollary 1.7 gives a very restrictive example of a peripherally stable subspace of $\mathrm{Hom}(\Gamma, G)$. But, in general the peripheral stability condition is flexible enough to allow peripheral subgroups to deform in nontrivial ways.

In particular, it is possible to find peripherally stable deformations of an EGF representation $\rho : \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ which change the Jordan block decomposition of elements in the peripheral subgroups. For instance, one can deform an EGF representation in $\mathrm{PGL}(d, \mathbb{R})$ with unipotent peripheral subgroups into an EGF representation with diagonalizable peripheral subgroups—see Example 9.3.

1.4. **Examples.** The related paper [Wei23b] is focused on examples of EGF representations. For illustrative purposes, however, we briefly describe some examples here.

1.4.1. *Convex projective structures.* A host of examples of Anosov representations arise from the theory of *convex projective structures*; see e.g. [Ben04], [Ben06b], [Kap07], [DGK18], [DGK$^+$21]. In fact, work of Danciger-Guéritaud-Kassel [DGK17] and Zimmer [Zim21] implies that Anosov representations can be essentially characterized as holonomy representations of *convex cocompact* projective orbifolds with hyperbolic fundamental group. However, convex projective structures also yield a number of interesting examples of discrete non-Anosov subgroups of $\mathrm{PGL}(d, \mathbb{R})$. In many cases, the groups in question are relatively hyperbolic, and appear to have "geometrically finite" properties.

The theory of EGF representations is well-suited to these examples. For instance, in [Wei23b], we apply some of our previous work [Wei23a] together with work of Islam-Zimmer [IZ22] to see that whenever a subgroup $\Gamma \subset \mathrm{PGL}(d, \mathbb{R})$ is relatively hyperbolic and *projectively convex cocompact* in the sense of [DGK17], then the inclusion $\Gamma \hookrightarrow \mathrm{PGL}(d, \mathbb{R})$ is EGF with respect to the parabolic subgroup stabilizing a flag of type $(1, d-1)$ in $\mathbb{R}^d$. If Γ is not hyperbolic, then these examples are *not* covered by other definitions of relative Anosov representations (see [Wei23a, Remark 1.14]); such non-hyperbolic examples have been constructed in e.g. [Ben06a], [BDL15], [CLM20], [CLM22], [DGK$^+$21], [BV23].

In [CM14], Crampon-Marquis introduced several definitions of "geometrical finiteness" for *strictly* convex projective manifolds. Zhu [Zhu21] proved that the manifolds satisfying one of their definitions[1] have "relative Anosov" holonomy in the sense of [KL23], [Zhu21], [ZZ22] (see Section 1.5), which means they also have EGF holonomy by Theorem 1.9 below. Examples can be found by deforming geometrically finite groups in $\mathrm{PO}(d,1)$ into $\mathrm{PGL}(d+1,\mathbb{R})$ while keeping the conjugacy classes of cusp groups fixed (see [Bal14], [BM20], [CLT18]), or via Coxeter reflection groups [CLM22].

There are, however, more general notions of "geometrically finite" convex projective structures. In [CLT18], Cooper-Long-Tillmann considered the situation of a convex projective manifold M (with strictly convex boundary) which is a union of a compact piece and finitely many ends homeomorphic to $N \times [0,\infty)$, where N is a compact manifold with virtually nilpotent fundamental group. The ends of such a manifold are called "generalized cusps," and the possible "types" of generalized cusps were later classified by Ballas-Cooper-Leitner [BCL20]. Examples of projective manifolds with generalized cusps have been produced by Ballas [Bal21], Ballas-Marquis [BM20], and Bobb [Bob19]. In general the holonomy representations of these manifolds are *not* "relative Anosov" in the sense of [ZZ22], but in [Wei23b] we prove that they do provide additional examples of EGF representations. The proof is an application of Theorem 1.4: it turns out that peripheral stability is actually flexible enough to allow for deformation between the different Ballas-Cooper-Leitner generalized cusp types.

Remark 1.8. We do not (yet) have a general result asserting that all strictly convex compact projective manifolds with generalized cusps have EGF holonomy, but there are indications that this should be true; see for example [Cho10], [Wol20] and the general setup in [IZ22], [BV23].

After a version of this paper originally appeared as a preprint, Blayac-Viaggi [BV23] also produced still more general examples of convex projective n-manifolds which decompose into a compact piece and several projective "cusps." In these examples (which can arise as limits of convex cocompact representations), each cusp is finitely covered by a product $N \times S^1 \times [0,\infty)$, where N is a closed hyperbolic manifold of dimension $n-2$. Consequently, these manifolds do not have "generalized cusps" in the sense of Cooper-Long-Tillmann, and their fundamental groups cannot even admit relative Anosov representations. Nevertheless, Blayac-Viaggi showed that the holonomy representations of their examples are always EGF.

1.4.2. *Other examples.* In [Wei23b], we construct additional examples of EGF representations by considering *compositions* of projectively convex cocompact representations $\rho : \Gamma \to \mathrm{PGL}(V)$ with the symmetric representation $\tau_k : \mathrm{PGL}(V) \to \mathrm{PGL}(\mathrm{Sym}^k V)$. We show that, assuming the peripheral subgroups in Γ are all virtually abelian, then the composition $\tau_k \circ \rho$ is still EGF; this holds even though the compositions are *not* believed to be convex cocompact.

We are also able to prove that the entire space $\mathrm{Hom}(\Gamma, \mathrm{PGL}(\mathrm{Sym}^k V))$ is peripherally stable about $\tau_k \circ \rho$. Via Theorem 1.4, this gives a new source of examples of stable discrete subgroups of higher-rank Lie groups.

1.5. **Comparison with relative Anosov representations.** Previously, Kapovich-Leeb [KL23] and Zhu [Zhu21] independently introduced several notions of a *relative Anosov* representation. Later work of Zhu-Zimmer [ZZ22] showed that Zhu's definition (that of a *relatively dominated* representation) is equivalent to one of the Kapovich-Leeb definitions (specifically, the definition of a *relatively asymptotically embedded* representation). In the special case where the domain group is isomorphic to a Fuchsian group, these definitions

[1] Crampon-Marquis originally claimed that all of their definitions of "geometrically finite" were equivalent; this appears to have been an error.

also agree with a notion of relative Anosov representation for Fuchsian groups introduced by Canary-Zhang-Zimmer [CZZ21].

For the rest of this paper we refer to representations satisfying any of these equivalent definitions as *relative Anosov* representations. Note that we will never simply say "Anosov representation" when we mean to refer to a *relative* Anosov representation (this differs from the convention in [CZZ21]).

The theorem below says that extended geometrically finite representations are a strict generalization of relative Anosov representations. Additionally, it provides a precise characterization of when an EGF representation is also relative Anosov.

Theorem 1.9. *Let $(\Gamma, \mathcal{H})$ be a relatively hyperbolic pair, and let $P \subset G$ be a symmetric parabolic subgroup. A representation $\rho : \Gamma \to G$ is relatively P-Anosov if and only if ρ is EGF with respect to P, and has an injective boundary extension $\phi : \Lambda \to \partial(\Gamma, \mathcal{H})$.*

We emphasize again that almost all of the EGF examples mentioned in Section 1.4 are *not* relative Anosov, so the theorem tells us that in many cases it is actually *not* possible to construct an injective boundary extension for a given EGF representation.

Remark 1.10. By Proposition 4.8, any EGF representation has a boundary extension which is injective on preimages of conical limit points. So, in the case where the peripheral structure $\mathcal{H}$ is trivial (meaning that Γ is a hyperbolic group and $\partial(\Gamma, \mathcal{H})$ is identified with the Gromov boundary $\partial\Gamma$ of Γ), Theorem 1.9 implies that EGF representations are precisely the same as (non-relative) Anosov representations.

This actually gives a new characterization of Anosov representations, since a priori the EGF boundary extension ϕ surjecting onto the Gromov boundary of a hyperbolic group does not need to be a homeomorphism; the theorem tells us that if such a boundary extension exists, then it is possible to replace ϕ with an injective boundary extension, whose inverse is the Anosov boundary map.

1.5.1. *Stability for relative Anosov representations.* In [KL23], Kapovich-Leeb suggested that a relative stability result should hold for relative Anosov representations, but not did not give a precise statement. By applying Theorem 1.4, Proposition 4.8, and Theorem 1.9, we obtain the following stability theorem:

Theorem 1.11. *Let $(\Gamma, \mathcal{H})$ be a relatively hyperbolic pair, let $\rho : \Gamma \to G$ be a relative P-Anosov representation, and let $\mathcal{W} \subset \mathrm{Hom}(\Gamma, G)$ be a peripherally stable subspace, such that for each $H \in \mathcal{H}$ and each $\rho' \in \mathcal{W}$, the restriction $\rho'|_H$ is P-divergent with P-limit set a singleton. Then an open neighborhood of ρ in $\mathcal{W}$ consists of relative P-Anosov representations of Γ.*

If we restrict the allowable peripheral deformations to conjugacies, this result reduces to:

Corollary 1.12. *Let $(\Gamma, \mathcal{H})$ be a relatively hyperbolic pair, and let $\rho : \Gamma \to G$ be a relative P-Anosov representation. There is an open neighborhood of ρ in $\mathrm{Hom}_{\mathrm{cp}}(\Gamma, G, \mathcal{H}, \rho)$ consisting of relative P-Anosov representations.*

Remark 1.13. In the special case where Γ is isomorphic to a Fuchsian group, Corollary 1.12 follows from previous work of Canary-Zhang-Zimmer [CZZ21]. Corollary 1.12 itself was also proved independently by Zhu-Zimmer [ZZ22], who showed further that the associated relative boundary maps vary continuously (in fact, analytically). The methods used in [CZZ21] and [ZZ22] are considerably different from those in this paper, and they do *not* imply Theorem 1.4 or Theorem 1.11 (and they do not apply to most of the examples of deformations considered in the companion paper [Wei23b]).

As Zhu-Zimmer observe, it seems unlikely that their techniques can be easily adapted for the general study of EGF representations, or for the general class of relative deformations considered in this article. On the other hand, while we expect that the methods in this paper could be used to generalize the Zhu-Zimmer result regarding *continuously* varying boundary embeddings for relative Anosov representations, it does not seem simple to use our approach to study analytic variation of the boundary maps.

Remark 1.14. After this paper was originally posted, Wang [Wan23b] showed that the situation of an EGF representation ρ with P-divergent image can be interpreted in terms of *restricted Anosov* representations, i.e. representations which are Anosov "along a subflow" of a certain flow space associated to the representation (see [Wan23a]). Using these ideas, Wang proves a version of Corollary 1.7 for this special class of EGF representations.

1.6. **Further applications, and potential future applications.**

1.6.1. *Anosov relativization.* When Γ is a relatively hyperbolic group, the Bowditch boundary of Γ (and thus, the definition of an EGF representation) depends on the choice of peripheral structure $\mathcal{H}$ for Γ. In general, there might be more than one possible choice: for instance, if a group Γ is hyperbolic relative to a collection $\mathcal{H}$ of hyperbolic subgroups, then Γ is itself hyperbolic, relative to an *empty* collection of peripheral subgroups (see [DS05, Corollary 1.14]).

In this paper, we prove the following *Anosov relativization theorem*:

Theorem 1.15. *Let $(\Gamma, \mathcal{H})$ be a relatively hyperbolic pair, and suppose that each $H \in \mathcal{H}$ is hyperbolic. If $\rho : \Gamma \to G$ is an EGF representation with respect to P for the peripheral structure $\mathcal{H}$, and ρ restricts to a P-Anosov representation on each $H \in \mathcal{H}$, then ρ is a P-Anosov representation of Γ.*

A potential application of Theorem 1.15 is the construction of new examples of Anosov representations: one could start with an EGF representation $\rho : \Gamma \to G$ which is *not* Anosov, and then attempt to find a peripherally stable deformation of ρ which restricts to an Anosov representation on peripheral subgroups. Theorem 1.4 and Theorem 1.15 would then imply that the original representation ρ can be realized as a non-Anosov limit of Anosov representations in the peripherally stable deformation space.

1.6.2. *Limits of Anosov representations.* In [LLS21], Lee-Lee-Stecker considered the deformation space of Anosov representations $\rho : \Gamma_{p,q,r} \to \mathrm{SL}(3, \mathbb{R})$, where $\Gamma_{p,q,r}$ is a triangle reflection group, and showed that certain components of this space have representations in their boundary which are *not* Anosov. Interestingly, these limiting representations still have equivariant injective boundary maps from $\partial\Gamma_{p,q,r}$ into the space of full flags in $\mathbb{R}^3$, but they fail to be Anosov because the boundary maps fail to be transverse.

The limiting representations constructed by Lee-Lee-Stecker cannot be relatively Anosov, but they do appear to be EGF. Together with the Anosov relativization theorem mentioned above, this provides evidence that EGF representations could serve as a useful tool in the study of boundaries of spaces of Anosov representations. In addition, it gives a potential source of examples of EGF representations which do not directly derive from convex projective structures.

1.6.3. *Deformations in rank one.* Even in rank one, the deformation theory of geometrically finite representations is not completely understood. In [Bow98], Bowditch described circumstances which guarantee that a small deformation of a geometrically finite group $\Gamma \subset \mathrm{PO}(d, 1)$ is still geometrically finite, but his criteria do not have an obvious analog in other rank one Lie groups. Moreover, the conditions Bowditch gives are too strict to allow for deformations which change the homeomorphism type of the limit set $\Lambda(\Gamma)$. Such deformations exist and are often peripherally stable, meaning that the EGF framework could be used to understand them further. It even seems possible that a version of the theory could be applied in circumstances where the isomorphism type of Γ is allowed to change.

1.7. **Outline of the paper.** We begin by providing some background in Sections 2 and 3, and then give the full formal definition of EGF representations in Section 4. In that section we also prove Theorem 1.9 (giving the connection between EGF representations and relative Anosov representations) and Theorem 1.15 (the Anosov relativization theorem). Some of these proofs assume the results of later sections, but they are not relied upon anywhere else in the paper.

The rest of the paper is devoted to the proof of our main stablity theorem for EGF representations (Theorem 1.4). In Section 5 and Section 6, we develop the main technical tool needed for the proof, which involves using the notion of an extended convergence group action to construct the *relative quasigeodesic automaton* alluded to previously. Then, in Section 7, we prove a key result (Proposition 7.11) regarding a metric on certain open subsets of flag manifolds G/P, which we use to relate the results of the previous sections to relatively hyperbolic group actions on G/P. Then, we use all of these tools to develop an alternative characterization of EGF representations in Section 8, and finally prove our main theorem in Section 9.

1.8. **Acknowledgements.** The author thanks his PhD advisor, Jeff Danciger, for encouragement and many helpful conversations—without which this paper could not have been written. The author also thanks Katie Mann and Jason Manning for assistance simplifying some of the arguments in Sections 5 and 6. Further thanks are owed to Daniel Allcock, Dick Canary, Fanny Kassel, Max Riestenberg, Anna Wienhard, Feng Zhu, and Andy Zimmer for providing feedback on various versions of this project.

This work was supported in part by NSF grants DMS-1937215 and DMS-2202770.

2. Relative hyperbolicity

In this section we discuss some of the basic theory of relatively hyperbolic groups, mostly to establish the notation and conventions we will use throughout the paper. We refer to [BH99], [Bow12], [DS05] for background on hyperbolic groups and relatively hyperbolic groups. See also section 3 of [KL23] for an overview (which we follow in part here).

Notation 2.1. Throughout this paper, if X is a metric space, A is a subset of X, and $r \geq 0$, we let $N_X(A, r)$ denote the open r-neighborhood in X about A. For a point $x \in X$, we let $B_X(x, r)$ denote the open r-ball about x.

When the metric space X is implied from context, we will often just write $N(A, r)$ or $B(x, r)$.

2.1. **Geometrically finite actions.** Recall that a finitely generated group Γ is *hyperbolic* (or *word-hyperbolic* or *δ-hyperbolic* or *Gromov-hyperbolic*) if and only if it acts properly discontinuously and cocompactly on a δ-hyperbolic proper geodesic metric space Y.

A *relatively hyperbolic group* is also a group with an action by isometries on a δ-hyperbolic proper geodesic metric space Y, but instead of asking for the action to cocompact, we ask for the action to be in some sense "geometrically finite."

To be precise, this means that Y has a Γ-invariant decomposition into a *thick part* Y_{th} and a countable collection $\mathcal{B}$ of *horoballs*. For a horoball B, we let $\mathrm{ctr}(B)$ denote the center of B in ∂Y, and we let Γ_p denote the stabilizer of any $p \in \partial Y$.

Definition 2.2. Let Γ be a finitely generated group acting on a hyperbolic metric space Y, and let $\mathcal{B}$ be a countable collection of horoballs in Y, invariant under the action of Γ on Y. If:

(1) The action of Γ on the closure of $Y_{\mathrm{th}} = Y - \bigcup_{B \in \mathcal{B}} B$ is cocompact, and
(2) for each $B \in \mathcal{B}$, the stabilizer of $\mathrm{ctr}(B)$ in Γ is infinite,

then we say that Γ is a *relatively hyperbolic group*, relative to the collection $\mathcal{H} = \{\mathrm{Stab}_\Gamma(p) : p = \mathrm{ctr}(B) \text{ for } B \in \mathcal{B}\}$.

Definition 2.3. Let Γ be a relatively hyperbolic group, relative to a collection of subgroups $\mathcal{H}$.

- The centers of the horoballs in $\mathcal{B}$ are called *parabolic points* for the Γ-action on ∂Y. The set of parabolic points in ∂Y is denoted $\partial_{\mathrm{par}} Y$.
- The parabolic point stablizers $\mathcal{H} = \{\mathrm{Stab}_\Gamma(p) : p \in \partial_{\mathrm{par}} Y\}$ are called *peripheral subgroups*. We often write Γ_p for $\mathrm{Stab}_\Gamma(p)$.

A group Γ might be hyperbolic relative to different collections $\mathcal{H}$, $\mathcal{H}'$ of peripheral subgroups. The collection $\mathcal{H}$ of peripheral subgroups is sometimes called a *peripheral structure* for Γ.

Definition 2.4. Let Γ be a finitely generated group, and let $\mathcal{H}$ be a collection of subgroups. We say that $(\Gamma, \mathcal{H})$ is a *relatively hyperbolic pair* if Γ is hyperbolic relative to $\mathcal{H}$.

2.2. **The Bowditch boundary.**

Definition 2.5. Let $(\Gamma, \mathcal{H})$ be a relatively hyperbolic pair, so that $\mathcal{H}$ is the set of stabilizers of parabolic points for an action of Γ on a metric space Y as in Definition 2.2. We say that Y is a *Gromov model* for the pair $(\Gamma, \mathcal{H})$.

In general there is *not* a unique choice of Gromov model for a given relatively hyperbolic pair $(\Gamma, \mathcal{H})$, even up to quasi-isometry. There are various "canonical" constructions for a preferred quasi-isometry class of Gromov model, with certain desirable metric properties (see e.g. [Bow12], [GM08]).

Given *any* two Gromov models Y, Y' for $(\Gamma, \mathcal{H})$, there is always a Γ-equivariant homeomorphism $\partial Y \to \partial Y'$ [Bow12]. The Γ-space ∂Y is the *Bowditch boundary* of $(\Gamma, \mathcal{H})$. We will denote it by $\partial(\Gamma, \mathcal{H})$, or sometimes just $\partial \Gamma$ when the collection of peripheral subgroups is understood from context. Since a Gromov model Y is a proper hyperbolic metric space, $\partial(\Gamma, \mathcal{H})$ is always compact and metrizable.

Definition 2.6. We say a relatively hyperbolic pair $(\Gamma, \mathcal{H})$ is *elementary* if Γ is finite or virtually cyclic, or if $\mathcal{H} = \{\Gamma\}$.

Whenever $(\Gamma, \mathcal{H})$ is nonelementary, its Bowditch boundary contains at least three points. The convergence properties of the action of Γ on $\partial(\Gamma, \mathcal{H})$ (see below) imply that in this case, $\partial(\Gamma, \mathcal{H})$ is *perfect* (i.e. contains no isolated points).

2.2.1. *Cocompactness on pairs.* Let Y be a Gromov model for a relatively hyperbolic pair $(\Gamma, \mathcal{H})$. Since Y is hyperbolic, proper, and geodesic, for any compact subset $K \subset Y$, the space of bi-infinite geodesics passing through K is compact.

Given any distinct pair of points $u, v \in \partial Y$, there is a bi-infinite geodesic c in Y joining u to v. Since a horoball in a hyperbolic metric space has just one point in its ideal boundary, this geodesic must pass through the thick part Y_{th} of Y, so up to the action of Γ it passes through a fixed compact subset $K \subset Y_{\mathrm{th}}$.

This implies:

Proposition 2.7. *The action of* Γ *on the space of distinct pairs in* $\partial(\Gamma, \mathcal{H})$ *is cocompact.*

2.3. **Convergence group actions.** If a group Γ acts on a proper geodesic hyperbolic metric space Y, we can characterize the geometrical finiteness of the action entirely in terms of the topological dynamics of the action on ∂Y. In particular, we can understand geometrical finiteness by studying properties of *convergence group actions*. See [Tuk94], [Tuk98], [Bow99] for further detail on such actions, and justifications for the results stated in this section.

Definition 2.8. Let Γ act as a convergence group (see Definition 1.1) on a topological space Z.

(1) A point $z \in Z$ is a *conical limit point* if there exists a sequence $\gamma_n \in \Gamma$ and distinct points $a, b \in Z$ such that $\gamma_n z \to a$ and $\gamma_n y \to b$ for any $y \neq z$.
(2) An infinite subgroup H is a *parabolic subgroup* if it fixes a point $p \in Z$, and every infinite-order element of H fixes exactly one point in Z.
(3) A point $p \in Z$ is a *parabolic point* if it is the fixed point of a parabolic subgroup.
(4) A parabolic point p is *bounded* if its stabilizer Γ_p acts cocompactly on $Z - \{p\}$.

The name "conical limit point" makes more sense in the context of convergence group actions on boundaries of hyperbolic metric spaces.

Definition 2.9. Let Y be a hyperbolic metric space, and let $z \in \partial Y$. We say that a sequence $y_n \in Y$ *limits conically to* z if there is a geodesic ray $c : \mathbb{R}^+ \to Y$ limiting to z and a constant $D > 0$ such that

$$d_Y(y_n, c(t_n)) < D$$

for some sequence $t_n \to \infty$.

A bounded neighborhood of a geodesic in a hyperbolic metric space looks like a "cone," hence "conical limit."

Proposition 2.10 ([Tuk94], [Tuk98])**.** *Let* Γ *be a group acting properly discontinuously by isometries on a proper geodesic hyperbolic metric space* Y*, and fix a basepoint* $y_0 \in Y$*.*

Then Γ *acts on* ∂Y *as a convergence group. Moreover, a point* $z \in \partial Y$ *is a conical limit point (in the dynamical sense of Definition 2.8) if and only if there is a sequence* $\gamma_n \cdot y_0$ *limiting conically to* z *(in the geometric sense of Definition 2.9). In this case, there are distinct points* $a, b \in \partial Y$ *such that* $\gamma_n^{-1} \cdot z \to a$ *and* $\gamma_n^{-1} z' \to b$ *for any* $z' \neq z$ *in* ∂Y*.*

If $\gamma_n \cdot y_0$ limits conically to a point $z \in \partial Y$ for some (hence any) basepoint $y_0 \in Y$, we just say that γ_n limits conically to z.

Theorem 2.11 ([Bow12])**.** *Let* Γ *be a finitely generated group acting by isometries on a hyperbolic metric space* Y*. Then* Γ *is a relatively hyperbolic group, acting on* Y *as in Definition 2.2, if and only if:*

(1) The induced action of Γ *on* ∂Y *is a convergence group action.*
(2) Every point $z \in \partial Y$ *is either a conical limit point or a bounded parabolic point.*

Whenever a group Γ acts as a convergence group on a perfect compact metrizable space Z, every point in Z is either a conical limit point or a bounded parabolic point, and the stabilizer of each parabolic point is finitely generated, we say the Γ-action on Z is *geometrically finite*. This is justified by a theorem of Yaman [Yam04], which says that any such group action is induced by the action of a relatively hyperbolic group on a Gromov model Y whose boundary is equivariantly homeomorphic to Z. We can then identify the space Z with the Bowditch boundary $\partial(\Gamma, \mathcal{H})$. The set of parabolic points in Z coincides exactly with the set of fixed points of peripheral subgroups.

Definition 2.12. Let $(\Gamma, \mathcal{H})$ be a relatively hyperbolic pair. We write

$$\partial(\Gamma, \mathcal{H}) = \partial_{\mathrm{con}}(\Gamma, \mathcal{H}) \sqcup \partial_{\mathrm{par}}(\Gamma, \mathcal{H}),$$

where $\partial_{\mathrm{con}}(\Gamma, \mathcal{H})$ and $\partial_{\mathrm{par}}(\Gamma, \mathcal{H})$ are respectively the conical limit points and parabolic points in $\partial(\Gamma, \mathcal{H})$.

2.3.1. *Compactification of* Γ *and divergent sequences.* When $(\Gamma, \mathcal{H})$ is a relatively hyperbolic pair, there is a natural topology on the set

$$\overline{\Gamma} = \Gamma \sqcup \partial(\Gamma, \mathcal{H})$$

making it into a *compactification* of Γ (i.e. $\overline{\Gamma}$ is compact, $\partial(\Gamma, \mathcal{H})$ and Γ are both embedded in $\overline{\Gamma}$, and Γ is an open dense subset of $\overline{\Gamma}$). Specifically, we view Γ as a subset of (any) Gromov model Y, via an orbit map $\gamma \mapsto \gamma \cdot y_0$ for some basepoint $y_0 \in Y$. Since Γ acts properly on Y, this is a proper embedding, so if we compactify Y by adjoining its visual boundary $\partial(\Gamma, \mathcal{H})$, we compactify Γ as well; this does not depend on the choice of basepoint y_0 or even the choice of space Y.

Definition 2.13. A sequence $\gamma_n \in \Gamma$ is *divergent* if it leaves every bounded subset of Γ (equivalently, if a subsequence of it consists of pairwise distinct elements).

Up to subsequence, a divergent sequence $\gamma_n \in \Gamma$ converges to a point $z \in \partial(\Gamma, \mathcal{H})$. When $(\Gamma, \mathcal{H})$ is non-elementary, the point z is determined solely by the action of Γ on $\partial(\Gamma, \mathcal{H})$: we have $\gamma_n \to z$ if and only if $\gamma_n \cdot x \to z$ for all but a single $x \in \partial(\Gamma, \mathcal{H})$.

2.4. **The coned-off Cayley graph.** Whenever $(\Gamma, \mathcal{H})$ is a relatively hyperbolic pair, there are only finitely many conjugacy classes of groups in $\mathcal{H}$. We can fix a finite set $\mathcal{P}$ of conjugacy representatives for the groups in $\mathcal{H}$. The set $\mathcal{P}$ corresponds to a finite set $\Pi \subset \partial_{\mathrm{par}}\Gamma$ of parabolic points, such that

$$\mathcal{P} = \{\Gamma_p : p \in \Pi\}.$$

Then Π contains exactly one point in each Γ-orbit in $\partial_{\mathrm{par}}\Gamma$.

Definition 2.14. Let $(\Gamma, \mathcal{H})$ be a relatively hyperbolic pair, and fix a finite generating set S for Γ and finite collection of conjugacy representatives $\mathcal{P}$ for $\mathcal{H}$.

The *coned-off Cayley graph* $\mathrm{Cay}(\Gamma, S, \mathcal{P})$ is a metric space obtained from the Cayley graph $\mathrm{Cay}(\Gamma, S)$ as follows: for each coset gP_i for $P_i \in \mathcal{P}$, we add a vertex $v(gP_i)$. Then, we add an edge of length 1 from each $h \in gP_i$ to $v(gP_i)$.

The quasi-isometry class of $\mathrm{Cay}(\Gamma, S, \mathcal{P})$ is independent of the choice of generating set S. When $(\Gamma, \mathcal{H})$ is a relatively hyperbolic pair, $\mathrm{Cay}(\Gamma, S, \mathcal{P})$ is a hyperbolic metric space. It is *not* a proper metric space if $\mathcal{H}$ is nonempty. The Gromov boundary of $\mathrm{Cay}(\Gamma, S, \mathcal{P})$ is equivariantly homeomorphic to the set $\partial_{\mathrm{con}}\Gamma$ of conical limit points in $\partial(\Gamma, \mathcal{H})$.

3. Lie theory notation and background

For the rest of the paper, we let G be a connected semisimple Lie group with no compact factor and finite center. We will be concerned with representations $\rho : \Gamma \to G$, where Γ is a relatively hyperbolic group. We want to consider the action of $\rho(\Gamma)$ on the flag manifold G/P, where P is a parabolic subgroup of G.

In this section, we give an overview of the definitions and notation we will use to describe the dynamical behavior of the Γ-action on G/P. We mostly follow the notation of [GGKW17], but we will also identify the connection to the language of [KLP17].

The exposition here is fairly brief, since most of this paper does not use much of the technical theory of semisimple Lie groups and their associated Riemannian symmetric spaces. In fact, in nearly every case, our approach will be to use a representation of G to reduce to the case $G = \mathrm{PGL}(n, \mathbb{R})$. The most important part of this section is 3.5, which identifies the connection between *P-divergence* (or equivalently *τ_{mod}-regularity*) and *contracting dynamics* in G.

Standard references for the general theory are [Ebe96], [Hel01], and [Kna02]. See also section 3 of [Rie21] for a careful discussion of the theory as it relates to Anosov representations and the work of Kapovich-Leeb-Porti.

3.1. Parabolic subgroups.

Let K be a maximal compact subgroup of the semisimple Lie group G, and let X be the Riemannian symmetric space G/K. A subgroup $P \subset G$ is a *parabolic subgroup* if it is the stabilizer of a point in the visual boundary $\partial_\infty X$ of X. Two parabolic subgroups P, Q are *opposite* if there is a bi-infinite geodesic c in X so that P is the stabilizer of $c(\infty)$ and Q is the stabilizer of $c(-\infty)$.

The compact homogeneous G-space G/P is called a *flag manifold*. If P and Q are parabolic subgroups, then we say that two flags $\xi^+ \in G/P$ and $\xi^- \in G/Q$ are *opposite* if the stabilizers of ξ^+, ξ^- are opposite parabolic subgroups. (In particular a conjugate of Q must be opposite to P).

3.2. Root space decomposition.

Let $\mathfrak{g}$ be the Lie algebra of G, and let $\mathfrak{k}$ be the Lie algebra of the maximal compact K. We can decompose $\mathfrak{g}$ as $\mathfrak{k} \oplus \mathfrak{p}$, and fix a maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{p}$. The restriction of the Killing form B to $\mathfrak{p}$ is positive definite, so any maximal abelian $\mathfrak{a} \subset \mathfrak{p}$ is naturally endowed with a Euclidean structure.

Each element of the abelian subalgebra $\mathfrak{a}$ acts semisimply on $\mathfrak{g}$, with real eigenvalues. So we let $\Sigma \subset \mathfrak{a}^*$ denote the set of *roots* for this choice of $\mathfrak{a}$, i.e. the set of nonzero linear functionals $\alpha \in \mathfrak{a}^*$ such that the linear map $\mathfrak{g} \to \mathfrak{g}$ given by $X - \alpha(X)I$ has nonzero kernel for every $X \in \mathfrak{a}$. We have a *restricted root space decomposition*

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha,$$

where $X \in \mathfrak{a}$ acts on $\mathfrak{g}_\alpha$ by multiplication by $\alpha(X)$.

We choose a set of *simple roots* $\Delta \subset \Sigma$ so that each $\alpha \in \Sigma$ can be uniquely written as a linear combination of elements of Δ with coefficients either all nonnegative or all nonpositive. We let Σ_+ denote the *positive roots*, i.e. roots which are nonnegative linear combinations of elements of Δ.

The simple roots Δ determine a *Euclidean Weyl chamber*

$$\mathfrak{a}^+ = \{x \in \mathfrak{a} : \alpha(x) \geq 0, \text{ for all } \alpha \in \Delta\}.$$

The kernels of the roots $\alpha \in \Delta$ are the *walls* of the Euclidean Weyl chamber.

Choosing a maximal compact K, a maximal abelian $\mathfrak{a} \subset \mathfrak{p}$, and a Euclidean Weyl chamber $\mathfrak{a}^+$ determines a *Cartan projection*

$$\mu : G \to \mathfrak{a}^+,$$

uniquely determined by the equation $g = k \exp(\mu(g))k'$, where $k, k' \in K$ and $\mu(g) \in \mathfrak{a}^+$.

3.3. **P-divergence.** Fix a subset θ of the simple roots Δ. We define a *standard parabolic subgroup* P_θ^+ to be the normalizer of the Lie algebra

$$\bigoplus_{\alpha \in \Sigma_\theta^+} \mathfrak{g}_\alpha,$$

where Σ_θ^+ is the set of positive roots which are *not* in the span of $\Delta - \theta$. The *opposite subgroup* P^- is the normalizer of

$$\bigoplus_{\alpha \in \Sigma_\theta^+} \mathfrak{g}_{-\alpha}.$$

Every parabolic subgroup $P \subset G$ is conjugate to a unique standard parabolic subgroup P_θ^+, and every pair of opposite parabolics (P^+, P^-) is simultaneously conjugate to a unique pair (P_θ^+, P_θ^-).

For a fixed $\theta \subset \Delta$, the group P_θ^+ is the stabilizer of the endpoint of a geodesic ray $\exp(tZ) \cdot p$, where $p \in X$ is the image of the identity in G/K, and for any $\alpha \in \Delta$, the element $Z \in \mathfrak{a}^+$ satisfies

$$\alpha(Z) = 0 \iff \alpha \in \Delta - \theta.$$

Definition 3.1. Let g_n be a sequence in G. The sequence g_n is P_θ^+-*divergent* if for every $\alpha \in \theta$, we have

$$\alpha(\mu(g_n)) \to \infty.$$

That is, the Cartan projections of the sequence g_n *drift away from the walls* of $\mathfrak{a}$ determined by the subset $\theta \subset \Delta$.

For a general parabolic subgroup $P \subset G$, we say that g_n is P-*divergent* if g_n is P_θ^+-divergent for P_θ^+ conjugate to P.

3.4. **Affine charts.**

Definition 3.2. Let P^+, P^- be opposite parabolic subgroups in G. Given a flag $\xi \in G/P^-$, we define

$$\mathrm{Opp}(\xi) = \{\eta \in G/P^+ : \xi \text{ is opposite to } \eta\}.$$

We call a set of the form $\mathrm{Opp}(\xi)$ for some $\xi \in G/P^-$ an *affine chart* in G/P^+.

An affine chart is the unique open dense orbit of $\mathrm{Stab}_G(\xi)$ in G/P^+. When $G = \mathrm{PGL}(d, \mathbb{R})$ and P^+ is the stabilizer of a line $\ell \subset \mathbb{R}^d$, G/P^+ is identified with $\mathbb{P}(\mathbb{R}^d)$ and this notion of affine chart agrees with the usual one in $\mathbb{P}(\mathbb{R}^d)$.

3.5. **Dynamics in flag manifolds.** There is a close connection between P-divergence in the group G and the *topological dynamics* of the action of G on the associated flag manifold G/P. Kapovich-Leeb-Porti frame this connection in terms of a *contraction property* for P-divergent sequences.

Definition 3.3 ([KLP17], Definition 4.1)**.** Let g_n be a sequence of group elements in G. We say that g_n is P^+-*contracting* if there exist $\xi \in G/P^+$, $\xi_- \in G/P^-$ such that g_n converges uniformly to ξ on compact subsets of $\mathrm{Opp}(\xi_-)$.

The flag ξ is the uniquely determined *limit* of the sequence g_n.

Definition 3.4. For an arbitrary sequence $g_n \in G$, a P^+*-limit point* of g_n in G/P^+ is the limit point of some P^+-contracting subsequence of g_n.

The P^+*-limit set* of a group $\Gamma \subset G$ is the set of P^+-limit points of sequences in Γ.

The importance of contracting sequences is captured by the following:

Proposition 3.5 ([KLP17], Proposition 4.15)**.** *A sequence $g_n \in G$ is P^+-divergent if and only if every subsequence of g_n has a P^+-contracting subsequence.*

Proposition 3.5 implies in particular that if $g_n \in G$ is P^+-divergent, then up to subsequence there is an open subset $U \subset G/P^+$ such that $g_n \cdot U$ converges to a singleton in G/P^+. It turns out that this "weak contraction property" is enough to characterize P^+-divergence.

Proposition 3.6. *Let g_n be a sequence in G, and suppose that for some nonempty open subset $U \subset G/P^+$, we have $g_n \cdot U \to \{\xi\}$ for $\xi \in G/P^+$. Then g_n is P^+-divergent, and has a unique P^+-limit point $\xi \in G/P^+$.*

We provide a proof of this fact in Appendix A.

3.5.1. *Dynamics of inverses of P^+-divergent sequences.* When g_n is a P^+-divergent sequence, the inverse sequence is P^--divergent. Kapovich-Leeb-Porti show that this can be framed in terms of the dynamical behavior of the inverse sequence.

Lemma 3.7 ([KLP17], Lemma 4.19)**.** *For $g_n \in G$ and flags $\xi_- \in G/P^-, \xi_+ \in G/P^+$, the following are equivalent:*

(1) g_n is P^+-contracting and $g_n|_{\mathrm{Opp}(\xi_-)} \to \xi_+$ uniformly on compacts.
(2) g_n is P^+-divergent, g_n has unique P^+-limit point ξ_+, and g_n^{-1} has unique P^--limit point ξ_-.

3.6. τ_{mod}**-regularity.** P-divergent sequences are equivalent to the τ_{mod}*-regular* sequences discussed in the work of Kapovich-Leeb-Porti, where τ_{mod} is the unique face corresponding to P in a *spherical model Weyl chamber.* We explain the connection here.

Remark 3.8. The language of τ_{mod}-regularity is not used anywhere else in this paper, so this part of the background is provided for convenience only and may be safely skipped.

For any point $p \in X$, we let $\mathfrak{p}$ be the uniquely determined subspace of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ is the Lie algebra of the stabilizer of p in G.

Let $z \in \partial_\infty X$. There is a point $p \in X$, a maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{p}$, a Euclidean Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$, and a unit-length $Z \in \mathfrak{a}^+$ such that z is the endpoint of the geodesic ray $c(t) = \exp(tZ) \cdot p$.

Up to the action of the stabilizer of z, the point p, the maximal abelian subalgebra $\mathfrak{a}$, the Euclidean Weyl chamber $\mathfrak{a}^+$, and the unit vector $Z \in \mathfrak{a}^+$ are uniquely determined. In addition, the stabilizer in G of the triple $(p, \mathfrak{a}, \mathfrak{a}^+)$ acts trivially on $\mathfrak{a}^+$.

This means that we can identify the space $\partial_\infty X/G$ with the set of unit vectors in any Euclidean Weyl chamber $\mathfrak{a}^+$. This set has the structure of a *spherical simplex.* We let σ_{mod} denote the *model spherical Weyl chamber $\partial_\infty X/G$.*

We let $\pi : \partial_\infty X \to \sigma_{\mathrm{mod}}$ be the *type map* to the model spherical Weyl chamber. For fixed $z \in \partial_\infty X$, we let P_z denote the parabolic subgroup stabilizing z.

After choosing a maximal compact K, a maximal abelian $\mathfrak{a} \subset \mathfrak{p}$, and a Euclidean Weyl chamber $\mathfrak{a}^+$, the data of a *face* τ_{mod} of the spherical simplex σ_{mod} is the same as the data of a *subset* of the simple roots of G: the set of roots identifies a collection of walls of the Euclidean Weyl chamber $\mathfrak{a}^+$. The intersection of those walls with the unit sphere in $\mathfrak{a}$ is uniquely identified with a face of σ_{mod}.

Definition 3.9. Let τ_{mod} be a face of the model spherical Weyl chamber σ_{mod}. We say that a sequence $g_n \in G$ is τ_{mod}*-regular* if g_n is P_z-divergent for some $z \in \partial_\infty X$ such that $\pi(z) \in \tau_{\mathrm{mod}}$.

For a fixed model face $\tau_{\mathrm{mod}} \subset \sigma_{\mathrm{mod}}$, we let $P_{\tau_{\mathrm{mod}}}$ denote any parabolic subgroup which is the stabilizer of a point $z \in \pi^{-1}(\tau_{\mathrm{mod}})$. All such parabolic subgroups are conjugate, so as a G-space the flag manifold $G/P_{\tau_{\mathrm{mod}}}$ depends only on the model face τ_{mod}.

4. EGF representations and relative Anosov representations

In this section we cover basic properties of the central objects of this paper: *extended geometrically finite representations* from a relatively hyperbolic group Γ to a semisimple Lie group G with no compact factor and trivial center. We also show that they generalize a definition of relative Anosov representation (Theorem 1.9), and prove our Anosov relativization theorem (Theorem 1.15).

We refer also to Section 2 of the related paper [Wei23b] for an overview of the definition in the special case where $G = \mathrm{PGL}(d, \mathbb{R})$ or $\mathrm{SL}(d, \mathbb{R})$ and the parabolic subgroup P is the stabilizer of a flag of type $(1, d-1)$ in $\mathbb{R}^d$.

Definition 4.1. Let P be a parabolic subgroup of G. We say that P is *symmetric* if $P = P^+$ is conjugate to a subgroup P^- opposite to P.

When $P = P^+$ is symmetric, we can identify G/P^+ with G/P^-, so that it makes sense to say that two flags $\xi_1, \xi_2 \in G/P$ are *opposite*.

Definition 4.2. Let P be symmetric, and let A, B be two subsets of G/P. We say that A and B are *opposite* if every $\xi \in A$ is opposite to every $\nu \in B$.

Definition 4.3. Let $(\Gamma, \mathcal{H})$ be a relatively hyperbolic pair, and let $\Lambda \subset G/P$ for a symmetric parabolic P. We say that a continuous surjective map $\phi : \Lambda \to \partial(\Gamma, \mathcal{H})$ is *antipodal* if for every pair of distinct points $z_1, z_2 \in \partial(\Gamma, \mathcal{H})$, $\phi^{-1}(z_1)$ is opposite to $\phi^{-1}(z_2)$.

We recall the main definition of the paper here:

Definition 1.3. Let Γ be a relatively hyperbolic group, let $\rho : \Gamma \to G$ be a representation, and let $P \subset G$ be a symmetric parabolic subgroup. We say that ρ is *extended geometrically finite* (EGF) with respect to P if there exists a closed $\rho(\Gamma)$-invariant set $\Lambda \subset G/P$ and a continuous ρ-equivariant surjective antipodal map $\phi : \Lambda \to \partial(\Gamma, \mathcal{H})$ extending the convergence action of Γ.

The map ϕ is called a *boundary extension* of the representation ρ, and the closed invariant set Λ is called the *boundary set*.

Remark 4.4. Unfortunately, the boundary set $\Lambda \subset G/P$ is *not* necessarily uniquely determined by the representation ρ. In many contexts, we will be able to make a natural choice, but we do not give a procedure for doing so in general.

4.1. Discreteness and finite kernel.

When $\rho : \Gamma \to G$ is EGF, the action of $\rho(\Gamma)$ on the boundary set Λ is by definition an extension of the topological dynamical system $(\Gamma, \partial(\Gamma, \mathcal{H}))$. When Γ is non-elementary, convergence dynamics imply that the homomorphism $\Gamma \to \mathrm{Homeo}(\partial(\Gamma, \mathcal{H}))$ has finite kernel and discrete image. So the map $\Gamma \to \mathrm{Homeo}(\Lambda)$ must also have discrete image and finite kernel, and therefore so does the representation $\rho : \Gamma \to G$. The case where Γ is elementary can be verified directly.

4.2. **Shrinking the sets C_z.** Let $\rho : \Gamma \to G$ be an EGF representation with boundary map $\phi : \Lambda \to \partial(\Gamma, \mathcal{H})$. By assumption, we know there exists an open subset $C_z \subset G/P$ for each $z \in \partial(\Gamma, \mathcal{H})$, satisfying the *extended convergence dynamics* conditions (Definition 1.2). In general, there is not a canonical choice for the set C_z. We are able to make some assumptions about the properties of the C_z, however.

Proposition 4.5. *Let $\rho : \Gamma \to G$ be an EGF representation with boundary extension ϕ. For any $z \in \partial(\Gamma, \mathcal{H})$, we can choose the set C_z to be a subset of*

$$\mathrm{Opp}(\phi^{-1}(z)) := \{\xi \in G/P : \xi \text{ is opposite to } \nu \text{ for every } \nu \in \phi^{-1}(z)\}.$$

Proof. Since $\phi^{-1}(z)$ is closed, $\mathrm{Opp}(\phi^{-1}(z))$ is an open subset of G/P. And, transversality of ϕ implies that $\mathrm{Opp}(\phi^{-1}(z))$ contains $\Lambda - \phi^{-1}(z)$. So the intersection $C_z \cap \mathrm{Opp}(\phi^{-1}(z))$ is open and nonempty, meaning we can replace C_z with this intersection. □

4.3. **An equivalent characterization of EGF representations.** It is often possible to prove properties of relatively hyperbolic groups by first showing that the property holds for conical subsequences in the group, and then showing that the property holds inside of peripheral subgroups. There is a characterization of the EGF property along these lines, which is frequently useful for constructing examples of EGF representations (see [Wei23b]).

Proposition 4.6. *Let $\rho : \Gamma \to G$ be a representation of a relatively hyperbolic group, and let $\Lambda \subset G/P$ be a closed $\rho(\Gamma)$-invariant set, where $P \subset G$ is a symmetric parabolic subgroup. Suppose that $\phi : \Lambda \to \partial(\Gamma, \mathcal{H})$ is a continuous surjective ρ-equivariant antipodal map.*

Then ρ is an EGF representation with EGF boundary extension ϕ if and only if both of the following conditions hold:

(a) For any sequence $\gamma_n \in \Gamma$ limiting conically to some point in $\partial(\Gamma, \mathcal{H})$, $\rho(\gamma_n^{\pm 1})$ is P-divergent and every P-limit point of $\rho(\gamma_n^{\pm 1})$ lies in Λ.

(b) For every parabolic point $p \in \partial_{\mathrm{par}}(\Gamma, \mathcal{H})$, there exists an open set $C_p \subset G/P$, with $\Lambda - \phi^{-1}(p) \subset C_p$, such that for any compact $K \subset C_p$ and any open set U containing $\phi^{-1}(p)$, for all but finitely many $\gamma \in \Gamma_p$, we have $\rho(\gamma) \cdot K \subset U$.

The proof of Proposition 4.6 requires the technical machinery of *relative quasigeodesic automata*, so we defer it to Section 8. At the end of Section 8, we also provide another (weaker) characterization of EGF representations which may be of interest.

4.4. **Properties of Λ.**

Proposition 4.7. *Let $(\Gamma, \mathcal{H})$ be a relatively hyperbolic pair, and let $\rho : \Gamma \to G$ be a representation which is EGF with respect to a symmetric parabolic P, with boundary extension $\phi : \Lambda \to \partial(\Gamma, \mathcal{H})$. Then Λ contains the P-limit set of $\rho(\Gamma)$.*

Proof. Let $\xi \in G/P$ be a flag in the P-limit set of $\rho(\Gamma)$. Then there is a P-contracting sequence $\rho(\gamma_n)$ for $\gamma_n \in \Gamma$ and a flag $\xi_- \in G/P$ such that $\rho(\gamma_n)\eta$ converges to ξ for any η in $\mathrm{Opp}(\xi_-)$. Up to subsequence $\gamma_n^{\pm 1}$ converges to $z_\pm \in \partial(\Gamma, \mathcal{H})$, so for any flag $\eta \in C_{z_-}$, the sequence $\rho(\gamma_n)\eta$ subconverges to a point in $\phi^{-1}(z_+)$. But since $\mathrm{Opp}(\xi_-)$ is open and dense, for some $\eta \in C_{z_-}$ we have $\rho(\gamma_n)\eta \to \xi$ and hence $\xi \in \phi^{-1}(z_+)$. □

In particular, Proposition 4.7 implies that the EGF boundary set $\Lambda \subset G/P$ of an EGF representation $\rho : \Gamma \to G$ must always contain the *P-proximal limit set* of $\rho(\Gamma)$. (Recall that $g \in G$ is *P-proximal* if it has a unique attracting fixed point in G/P; the *P-proximal limit set* of a subgroup of G is the closure of the set of attracting fixed points of P-proximal elements).

We will see that most of the power of EGF representations lies in the fact that their associated boundary extensions $\phi : \Lambda \to \partial(\Gamma, \mathcal{H})$ do not have to be homeomorphisms (so the Bowditch boundary of Γ does not need to be equivariantly embedded in any flag manifold). However, it turns out that it is always possible to choose the boundary extension ϕ so that it has a well-defined inverse on *conical limit points* in $\partial(\Gamma, \mathcal{H})$. In fact, we can even get a somewhat precise description of all the fibers of ϕ. Concretely, we have the following:

Proposition 4.8. *Let $\rho : \Gamma \to G$ be an EGF representation, with boundary extension $\phi : \Lambda \to \partial(\Gamma, \mathcal{H})$. There is a $\rho(\Gamma)$-invariant closed subset $\Lambda' \subset G/P$ and a ρ-equivariant map $\phi' : \Lambda' \to \partial(\Gamma, \mathcal{H})$ such that:*

(1) $\phi' : \Lambda' \to \partial(\Gamma, \mathcal{H})$ is also a boundary extension for ρ,
(2) for every $z \in \partial_{\mathrm{con}}(\Gamma, \mathcal{H})$, $\phi'^{-1}(z)$ is a singleton, and
(3) for every $p \in \partial_{\mathrm{par}}(\Gamma, \mathcal{H})$, $\phi'^{-1}(p)$ is the closure of the set of all accumulation points of orbits $\gamma_n \cdot x$ for γ_n a sequence of distinct elements in Γ_p and $x \in C_p$.

We will prove Proposition 4.8 at the end of Section 9, where it will follow as a consequence of the proof of the relative stability theorem for EGF representations (Theorem 1.4)—see Remark 9.18.

We will rely on both Proposition 4.6 and Proposition 4.8 to prove the rest of the results in this section (which are not needed anywhere else in this paper).

4.5. **Relatively Anosov representations.** EGF representations give a strict generalization of the relative Anosov representations mentioned in the introduction. We give a precise definition here.

Definition 4.9 ([KL23, Definition 7.1] or [ZZ22, Definition 1.1]; see also [ZZ22, Proposition 4.4])**.** Let Γ be a subgroup of G and suppose $(\Gamma, \mathcal{H})$ is a relatively hyperbolic pair. Let $P \subset G$ be a symmetric parabolic subgroup.

The subgroup Γ is *relatively P-Anosov* if it is P-divergent, and there is a Γ-equivariant antipodal embedding $\partial(\Gamma, \mathcal{H}) \to G/P$ whose image Λ is the P-limit set of Γ.

Here, we say an embedding $\psi : \partial(\Gamma, \mathcal{H}) \to G/P$ is *antipodal* if for every distinct ξ_1, ξ_2 in $\partial(\Gamma, \mathcal{H})$, $\psi(\xi_1)$ and $\psi(\xi_2)$ are opposite flags.

Remark 4.10. Several remarks on the definition are in order:

(a) In [KL23], Kapovich-Leeb provide several possible ways to relativize the definition of an Anosov representation; Definition 4.9 agrees with essentially their most general definition, that of a *relatively asymptotically embedded* representation.
(b) When Γ is a *hyperbolic* group (and the collection of peripheral subgroups $\mathcal{H}$ is empty), then the Bowditch boundary $\partial(\Gamma, \mathcal{H})$ is identified with the Gromov boundary $\partial\Gamma$. In this case, Definition 4.9 coincides with the usual definition of an Anosov representation.
(c) In general, it is possible to define (relatively) P-Anosov representations for a *non-symmetric* parabolic subgroup P. However, there is no loss of generality in assuming that P is symmetric: a representation $\rho : \Gamma \to G$ is P-Anosov if and only if it is P'-Anosov for a symmetric parabolic subgroup $P' \subset G$ depending only on P.

Proposition 4.11. *Let $\rho : \Gamma \to G$ be an EGF representation with respect to P, and suppose that the boundary extension $\phi : \Lambda \to \partial(\Gamma, \mathcal{H})$ is a homeomorphism. Then:*

(1) $\rho(\Gamma)$ is P-divergent, and Λ is the P-limit set of $\rho(\Gamma)$.
(2) The sets C_z for $z \in \partial(\Gamma, \mathcal{H})$ can be taken to be

$$\mathrm{Opp}(\phi^{-1}(z)) = \{\nu \in G/P : \nu \text{ is opposite to } \phi^{-1}(z)\}.$$

Proof. (1). Let γ_n be any infinite sequence of elements in Γ. After extracting a subsequence, we have $\gamma_n^{\pm 1} \to z_\pm$, and since ϕ is a homeomorphism, $\rho(\gamma_n)$ converges to the point $\phi^{-1}(z_+)$ uniformly on compacts in the open set C_{z_-}. Then Proposition 3.6 implies that $\rho(\gamma_n)$ is P-divergent, with unique P-limit point $\phi^{-1}(z_+) \in \Lambda$.

(2). The fact that ϕ is antipodal is exactly the statement that the sets $\mathrm{Opp}(\phi^{-1}(z))$ contain $\Lambda - \phi^{-1}(z)$ for every $z \in \partial(\Gamma, \mathcal{H})$, so we just need to see that the appropriate dynamics hold for these sets. Let γ_n be an infinite sequence in Γ with $\gamma_n^{\pm 1} \to z_\pm$ for $z_\pm \in \partial(\Gamma, \mathcal{H})$.

We know that for open subsets $U_\pm \subset G/P$, we have $\rho(\gamma_n)\cdot U_+ \to \phi^{-1}(z_+)$ and $\rho(\gamma_n^{-1})U_- \to \phi^{-1}(z_-)$, uniformly on compacts. Proposition 3.6 implies that $\rho(\gamma_n)$ and $\rho(\gamma_n^{-1})$ are both P-divergent with unique P-limit points $\phi^{-1}(z_+)$, $\phi^{-1}(z_-)$. So in fact by Lemma 3.7 $\rho(\gamma_n)$ converges to $\phi^{-1}(z_+)$ uniformly on compacts in $\mathrm{Opp}(\phi^{-1}(z_-))$. ☐

Using the previous proposition, we can see the relationship between relatively Anosov representations and EGF representations (Theorem 1.9). We restate this theorem as the following:

Proposition 4.12. *Let $(\Gamma, \mathcal{H})$ be a relatively hyperbolic pair, and let $P \subset G$ be a symmetric parabolic subgroup. A representation $\rho : \Gamma \to G$ is relatively P-Anosov in the sense of Definition 4.9 if and only if ρ is EGF with respect to P, and has an injective boundary extension $\phi : \Lambda \to \partial(\Gamma, \mathcal{H})$.*

Proof. Proposition 4.11 ensures that if ρ is an EGF representation, and the boundary extension ϕ is a homeomorphism, then ρ is P-divergent and ϕ^{-1} is an antipodal embedding whose image is the P-limit set.

On the other hand, if ρ is relatively P-Anosov, with boundary embedding $\psi : \partial(\Gamma, \mathcal{H}) \to \Lambda$, for each $z \in \partial(\Gamma, \mathcal{H})$, we can take

$$C_z = \mathrm{Opp}(\psi(z)).$$

Antipodality means that C_z contains $\Lambda - \psi(z)$, and P-divergence and Lemma 3.7 imply that $\rho(\Gamma)$ has the appropriate convergence dynamics. ☐

4.6. **Relativization.** We now turn to the situation where we have an EGF representation of a *hyperbolic* group Γ with a *nonempty* collection of peripheral subgroups. That is, for some invariant set $\Lambda \subset G/P$, we have an EGF boundary extension $\phi : \Lambda \to \partial(\Gamma, \mathcal{H})$, where $\partial(\Gamma, \mathcal{H})$ is the Bowditch boundary of Γ with peripheral structure $\mathcal{H}$.

We want to prove Theorem 1.15, which says that in this situation, if ρ restricts to a P-Anosov representation on each $H \in \mathcal{H}$, then ρ is a P-Anosov representation of Γ. For the rest of this section, we assume that Γ is a hyperbolic group, and $\mathcal{H}$ is a collection of subgroups of Γ so that the pair $(\Gamma, \mathcal{H})$ is relatively hyperbolic. We let $\rho : \Gamma \to G$ be an EGF representation for the pair $(\Gamma, \mathcal{H})$ with respect to a symmetric parabolic subgroup $P \subset G$, and we assume that for each $H \in \mathcal{H}$, $\rho|_H : H \to G$ is P-Anosov, with Anosov limit map $\psi_H : \partial H \to G/P$.

The main step in the proof is to observe that it is always possible to choose the boundary extension $\phi : \Lambda \to \partial(\Gamma, \mathcal{H})$ so that Λ is equivariantly homeomorphic to the Gromov boundary of Γ (which we here denote $\partial\Gamma$).

Whenever Γ is a hyperbolic group and $\mathcal{H}$ is a collection of subgroups so that $(\Gamma, \mathcal{H})$ is a relatively hyperbolic pair, there is an explicit description of the Bowditch boundary $\partial(\Gamma, \mathcal{H})$ in terms of the Gromov boundary $\partial\Gamma$ of Γ—see [Ger12], [GP13], or [Tra13]. Specifically, we can say:

Proposition 4.13. *There is an equivariant surjective continuous map $\phi_\Gamma : \partial\Gamma \to \partial(\Gamma, \mathcal{H})$ such that for each conical limit point z in $\partial(\Gamma, \mathcal{H})$, $\phi_\Gamma^{-1}(z)$ is a singleton, and for each parabolic point $p \in \partial(\Gamma, \mathcal{H})$ with $H = \mathrm{Stab}_\Gamma(p)$, $\phi_\Gamma^{-1}(p)$ is an embedded copy of ∂H in $\partial\Gamma$.*

In our situation, we can see that the boundary extension $\phi : \Lambda \to \partial(\Gamma, \mathcal{H})$ satisfies similar properties.

Lemma 4.14. *There is a closed $\rho(\Gamma)$-invariant subset $\Lambda' \subset G/P$ and an EGF boundary extension $\phi' : \Lambda' \to \partial(\Gamma, \mathcal{H})$ such that:*

1. *For each conical limit point $z \in \partial(\Gamma, \mathcal{H})$, $\phi'^{-1}(z)$ is a singleton.*
2. *For each parabolic point $p \in \partial(\Gamma, \mathcal{H})$, with $H = \mathrm{Stab}_\Gamma(p)$, we have $\phi'^{-1}(p) = \psi_H(\partial H)$.*

Proof. We choose Λ' as in Proposition 4.8. The only thing we need to check is that for $H = \mathrm{Stab}_\Gamma(p)$, the set $\psi_H(\partial H)$ is exactly the closure of the set of accumulation points of $\rho(H)$-orbits in C_p. But since we may assume C_p is contained in $\mathrm{Opp}(\psi_H(\partial H))$, this follows immediately from the fact that $\rho(H)$ is P-divergent and the closed set $\psi_H(\partial H)$ is the P-limit set of $\rho(H)$.

□

Next we need a lemma which will allow us to characterize the Gromov boundary of Γ as an extension of the Bowditch boundary $\partial(\Gamma, \mathcal{H})$. First recall that if Γ acts as a convergence group on a space Z, the *limit set* of Γ is the set of points $z \in Z$ such that for some $y \in Z$ and some sequence $\gamma_n \in \Gamma$, we have

$$\gamma_n|_{Z-\{y\}} \to z$$

uniformly on compacts.

Lemma 4.15. *Let Γ act on compact metrizable spaces X and Y, and let $\phi_X : X \to \partial(\Gamma, \mathcal{H})$, $\phi_Y : Y \to \partial(\Gamma, \mathcal{H})$ be continuous equivariant surjective maps such that for every conical limit point $z \in \partial(\Gamma, \mathcal{H})$, $\phi_X^{-1}(z)$ and $\phi_Y^{-1}(z)$ are both singletons, and for every parabolic point $p \in \partial(\Gamma, \mathcal{H})$, $H = \mathrm{Stab}_\Gamma(p)$ acts as a convergence group on X and Y, with limit sets $\phi_X^{-1}(p)$, $\phi_Y^{-1}(p)$ equivariantly homeomorphic to ∂H.*

Then for any sequences $z_n, z_n' \in \partial_{\mathrm{con}}(\Gamma, \mathcal{H})$, we have

$$\lim_{n\to\infty} \phi_X^{-1}(z_n) = \lim_{n\to\infty} \phi_X^{-1}(z_n')$$

if and only if

$$\lim_{n\to\infty} \phi_Y^{-1}(z_n) = \lim_{n\to\infty} \phi_Y^{-1}(z_n').$$

Proof. We proceed by contradiction, and suppose that for a pair of sequences $z_n, z_n' \in \partial_{\mathrm{con}}(\Gamma, \mathcal{H})$, we have

$$\lim_{n\to\infty} \phi_X^{-1}(z_n) = \lim_{n\to\infty} \phi_X^{-1}(z_n') = x,$$

but

$$\lim_{n\to\infty} \phi_Y^{-1}(z_n) \neq \lim_{n\to\infty} \phi_Y^{-1}(z_n').$$

After taking a subsequence we may assume z_n converges to $z \in \partial(\Gamma, \mathcal{H})$, and that $y_n = \phi_Y^{-1}(z_n)$ converges to y and $y_n' = \phi_Y^{-1}(z_n')$ converges to y' for $y \neq y'$. By continuity, we have

$$\phi_Y(y) = \phi_Y(y') = \phi_X(x) = z.$$

Since ϕ_X and ϕ_Y are bijective on $\phi_X^{-1}(\partial_{\mathrm{con}}(\Gamma, \mathcal{H}))$ and $\phi_Y^{-1}(\partial_{\mathrm{con}}(\Gamma, \mathcal{H}))$ respectively, we must have $z = p$ for a parabolic point $p \in \partial_{\mathrm{par}}(\Gamma, \mathcal{H})$. Let $H = \mathrm{Stab}_\Gamma(p)$.

Since p is a bounded parabolic point, we can find sequences of group elements $h_n, h_n' \in H$ so that for a fixed compact subset $K \subset \partial(\Gamma, \mathcal{H}) - \{p\}$, we have

$$h_n z_n \in K, \quad h_n' z_n' \in K. \tag{1}$$

This implies that no subsequence of $h_n y_n$ or $h_n' y_n'$ converges to a point in $\phi_Y^{-1}(p)$.

Then, since H acts as a convergence group on Y with limit set $\phi_Y^{-1}(p)$, up to subsequence there are points $u, u' \in \phi_Y^{-1}(p)$ so that h_n converges to a point in $\phi_Y^{-1}(p)$ uniformly on compacts in $Y - \{u\}$, and h_n' converges to a point in $\phi_Y^{-1}(p)$ uniformly on compacts in $Y - \{u'\}$. So, we must have $u = y$ and $u' = y'$.

This means that the sequences h_n^{-1} and $h_n'^{-1}$ have distinct limits in the compactification $\overline{H} = H \sqcup \partial H$. So, there are distinct points $v, v' \in \phi_X^{-1}(p)$ so that (again up to subsequence) h_n converges to a point in $\phi_X^{-1}(p)$ uniformly on compacts in $X - \{v\}$, and h_n' converges to a point in $\phi_X^{-1}(p)$ uniformly on compacts in $X - \{v'\}$. Without loss of generality, we can assume $x \neq v$.

But then $\phi_X^{-1}(z_n)$ lies in a compact subset of $X - \{v\}$, so $h_n \phi_X^{-1}(z_n)$ converges to a point in $\phi_X^{-1}(p)$ and $h_n z_n$ converges to p. But this contradicts (1) above. □

Proposition 4.16. *If the set Λ satisfies the conclusions of Lemma 4.14, then Λ is equivariantly homeomorphic to the Gromov boundary of Γ.*

Proof. Let $\phi_\Gamma : \partial\Gamma \to \partial(\Gamma, \mathcal{H})$ denote the quotient map identifying the limit set of each $H \in \mathcal{H}$ to the parabolic point p with $H = \mathrm{Stab}_\Gamma(p)$. For each conical limit point $z \in \partial(\Gamma, \mathcal{H})$, the fiber $\phi_\Gamma^{-1}(z)$ is a singleton. So, there is an equivariant bijection f from $\phi_\Gamma^{-1}(\partial_{\mathrm{con}}(\Gamma, \mathcal{H}))$ to $\phi^{-1}(\partial_{\mathrm{con}}(\Gamma, \mathcal{H}))$.

Moreover, since $\phi_\Gamma^{-1}(\partial_{\mathrm{con}}(\Gamma, \mathcal{H}))$ is Γ-invariant, and the action of Γ on its Gromov boundary $\partial\Gamma$ is minimal, $\phi_\Gamma^{-1}(\partial_{\mathrm{con}}(\Gamma, \mathcal{H}))$ is dense in $\partial\Gamma$. We claim that f extends to a continuous injective map $\partial\Gamma \to \Lambda$ by defining $f(x) = \lim f(x_n)$ for any sequence $x_n \to x$.

To see this, we can apply Lemma 4.15, taking $\partial\Gamma = X$ and $\Lambda = Y$. We know that Γ always acts on its own Gromov boundary as a convergence group (so in particular each $H \in \mathcal{H}$ acts on $\partial\Gamma$ as a convergence group with limit set ∂H). And, since ρ restricts to a P-Anosov representation on each $H \in \mathcal{H}$, for any infinite sequence $h_n \in H$, up to subsequence there are $u, u_- \in \psi_H(\partial H)$ so that $\rho(h_n)$ converges to u uniformly on compacts in $\mathrm{Opp}(u_-)$. Antipodality of ϕ implies that $\rho(h_n)$ converges to u uniformly on compacts in $\Lambda - \psi_H(\partial H)$. The other hypotheses of Lemma 4.15 follow from Proposition 4.13 and Lemma 4.14.

We still need to check that f is actually surjective. We know that f restricts to a bijection on $\phi_\Gamma^{-1}(\partial_{\mathrm{con}}(\Gamma, \mathcal{H}))$, and that f takes $\phi_\Gamma^{-1}(p)$ to $\phi^{-1}(p)$ for each parabolic point p in $\partial(\Gamma, \mathcal{H})$. So we just need to check that for every $H \in \mathcal{H}$, f restricts to a surjective map $\partial H \to \psi_H(\partial H)$. If H is non-elementary, this must be the case because the action of H on ∂H is minimal and f maps ∂H into $\psi_H(\partial H)$ as an invariant closed subset. Otherwise, H is virtually cyclic and ∂H, $\psi_H(\partial H)$ both contain exactly two points. Then injectivity of f implies surjectivity.

So we conclude that there is a continuous bijection $f : \partial\Gamma \to \Lambda$, and since $\partial\Gamma$ is compact and Λ is metrizable, f is a homeomorphism. □

We let $f : \Lambda \to \partial\Gamma$ denote the equivariant homeomorphism from Proposition 4.16. The final step in the proof of Theorem 1.15 is the following:

Proposition 4.17. *The equivariant homeomorphism $f : \Lambda \to \partial\Gamma$ extends the convergence action of Γ on its Gromov boundary $\partial\Gamma$.*

Proof. By Proposition 4.6, we just need to show that if $\gamma_n \in \Gamma$ is a conical limit sequence with $\gamma_n^{\pm 1} \to z_\pm$ for $z_\pm \in \partial\Gamma$, then every P-limit point of $\rho(\gamma_n^{\pm 1})$ lies in Λ.

We consider two cases:

Case 1: $\phi \circ f(z_+)$ is a parabolic point p in $\partial(\Gamma, \mathcal{H})$. In this case, γ_n lies along a quasigeodesic ray in Γ limiting to some $z_+ \in \partial H$, with $H = \mathrm{Stab}_\Gamma(p)$. This means that for a bounded sequence $b_n \in \Gamma$, we have $\gamma_n b_n \in H$. Since ρ restricts to a P-Anosov representation on H, this means that $\rho(\gamma_n b_n)$ is P-divergent and every P-limit point of $\rho(\gamma_n b_n)$ lies in $\psi_H(z_+)$. For the same reason, every P-limit point of $\rho(b_n^{-1}\gamma_n^{-1})$ lies in $\psi_H(z_+)$.

Up to subsequence b_n is a constant b. We can use Proposition 3.5 to see that $\rho(\gamma_n)$ has the same P-limit set as $\rho(\gamma_n b)$. This P-limit set lies in Λ. And, every P-limit point of $\rho(\gamma_n^{-1})$ is a b-translate of a P-limit point of $\rho(b^{-1}\gamma_n^{-1})$. This P-limit set also lies in Λ.

Case 2: $\phi \circ f(z_+)$ is a conical limit point in $\partial(\Gamma, \mathcal{H})$. In this case, a subsequence of γ_n is a conical limit sequence for the action of Γ on $\partial(\Gamma, \mathcal{H})$, and the desired result follows from the "only if" part of Proposition 4.6.

□

Proof of Theorem 1.15. Let Γ be hyperbolic, let $\mathcal{H}$ be a collection of subgroups such that $(\Gamma, \mathcal{H})$ is a relatively hyperbolic pair, and let $\rho : \Gamma \to G$ be an EGF representation with respect to P, for the peripheral structure $\mathcal{H}$.

Suppose that ρ restricts to a P-Anosov representation on each $H \in \mathcal{H}$. Proposition 4.17 implies that ρ is *also* an EGF representation of Γ for its *empty* peripheral structure, whose boundary extension can be chosen to be a homeomorphism. Then Theorem 1.9 says that ρ is relatively P-Anosov (again for the empty peripheral structure on Γ). This ensures that ρ is actually (non-relatively) P-Anosov; see e.g. [KLP17, Theorem 1.1]. □

5. Relative quasigeodesic automata

In the next three sections, we develop the technical tools needed to prove the main results of the paper: namely, a *relative quasigeodesic automaton* for a relatively hyperbolic group Γ acting on a flag manifold G/P, and a *system of open sets* in G/P which is in some sense *compatible* with both the relative quasigeodesic automaton and the action of Γ on G/P.

The basic idea is motivated by the computational theory of hyperbolic groups. Given a hyperbolic group Γ with finite generating set S, it is always possible to find a finite directed graph $\mathcal{G}$, with edges labeled by elements of S, so that directed paths on $\mathcal{G}$ starting at a fixed vertex $v_{\mathrm{id}} \in \mathcal{G}$ are in one-to-one correspondence with geodesic words in Γ. The graph $\mathcal{G}$ is called a *geodesic automaton* for Γ.

Geodesic automata are really a manifestation of the local-to-global principle for geodesics in hyperbolic metric spaces: the fact that the automaton exists means that it is possible to recognize a geodesic path in a hyperbolic group just by looking at bounded-length subpaths.

In this section of the paper, we consider a *relative* version of a geodesic automaton. This is a finite directed graph $\mathcal{G}$ which encodes the behavior of quasigeodesics in the coned-off Cayley graph of a relatively hyperbolic group Γ. Eventually, our goal is to build such an automaton by looking at the dynamics of the action of Γ on its Bowditch boundary $\partial(\Gamma, \mathcal{H})$. The main result of this section is Proposition 5.13, which says that we can construct such a *relative quasigeodesic automaton* for a relatively hyperbolic pair $(\Gamma, \mathcal{H})$ using an open covering of the Bowditch boundary $\partial(\Gamma, \mathcal{H})$ which satisfies certain technical conditions.

In this section of the paper and the next, we will work in the general context of a relatively hyperbolic group Γ acting by homeomorphisms on a connected compact metrizable space M, before returning to the case where M is a flag manifold G/P for the rest of the paper.

Throughout the rest of this section, we fix a *non-elementary* relatively hyperbolic pair $(\Gamma, \mathcal{H})$, and let $\Pi \subset \partial_{\mathrm{par}}(\Gamma, \mathcal{H})$ be a finite set, containing exactly one point from each Γ-orbit in $\partial_{\text{par}}(\Gamma, \mathcal{H})$. We also fix a finite generating set S for Γ, which allows us to refer to the coned-off Cayley graph $\text{Cay}(\Gamma, S, \mathcal{P})$ (Definition 2.14).

Definition 5.1. A Γ-*graph* is a finite directed graph $\mathcal{G}$ where each vertex v is labelled with a subset $T_v \subset \Gamma$, which is either:

- A singleton $\{\gamma\}$, with $\gamma \neq \text{id}$, or
- A cofinite subset of a coset $g\Gamma_p$ for some $p \in \Pi$, $g \in \Gamma$.

A sequence $\{\alpha_n\} \subset \Gamma$ is a $\mathcal{G}$-*path* if $\alpha_n \in T_{v_n}$ for a vertex path $\{v_n\}$ in $\mathcal{G}$.

Remark 5.2. We will often refer to "the" vertex path $\{v_n\}$ corresponding to a $\mathcal{G}$-path $\{\alpha_n\}$, although we will never actually verify that such a vertex path is uniquely determined by the sequence of group elements $\{\alpha_n\}$ in Γ.

A vertex of a Γ-graph which is labeled by a cofinite subset of a (necessarily unique) coset $g\Gamma_p$ is a *parabolic vertex*. If v is a parabolic vertex, we let $p_v = g \cdot p$ denote the corresponding parabolic point in $\partial_{\text{par}}(\Gamma, \mathcal{H})$.

Remark 5.3. It will be convenient to allow parabolic vertices to be labeled by *cofinite* subsets of peripheral cosets (instead of just the entire coset) when we construct Γ-graphs using the convergence dynamics of the Γ-action on $\partial(\Gamma, \mathcal{H})$.

Definition 5.4. Let $z \in \partial(\Gamma, \mathcal{H})$. We say that a $\mathcal{G}$-path $\{\alpha_n\}$ *limits to* z if either:

- $z \in \partial_{\text{con}}(\Gamma, \mathcal{H})$, $\{\alpha_n\}$ is infinite, and the sequence
$$\{\gamma_n = \alpha_1 \cdots \alpha_n\}_{n=1}^{\infty}$$
limits to z in the compactification $\overline{\Gamma} = \Gamma \sqcup \partial(\Gamma, \mathcal{H})$, or
- $z \in \partial_{\text{par}}(\Gamma, \mathcal{H})$, $\{\alpha_n\}$ is a finite $\mathcal{G}$-path whose corresponding vertex path $\{v_n\}$ ends at a parabolic vertex v_N, and
$$z = \alpha_1 \cdots \alpha_{N-1} p_{v_N}.$$

Definition 5.5. Let $\mathcal{G}$ be a Γ-graph. The *endpoint* of a finite $\mathcal{G}$-path $\{\alpha_n\}_{n=1}^{N}$ is
$$\alpha_1 \cdots \alpha_N.$$

Definition 5.6. A Γ-graph $\mathcal{G}$ is a *relative quasigeodesic automaton* if:

(1) There is a constant $D > 0$ so that for any infinite $\mathcal{G}$-path α_n, the sequence
$$\{\gamma_n = \alpha_1 \cdots \alpha_n\} \subset \Gamma$$
lies Hausdorff distance at most D from a geodesic ray in $\text{Cay}(\Gamma, S, \mathcal{P})$, based at the identity.

(2) For every $z \in \partial(\Gamma, \mathcal{H})$, there exists a $\mathcal{G}$-path limiting to z.

One way to think of a relative quasigeodesic automaton is that it gives us a system for finding quasigeodesic representatives of every element in the group. More concretely, we have the following:

Lemma 5.7. *Let $\mathcal{G}$ be a relative quasigeodesic automaton. There is a constant $R > 0$ so that set of endpoints of $\mathcal{G}$-paths is R-dense in Γ.*

Proof. If Γ is hyperbolic and $\mathcal{H}$ is empty, then this is a consequence of the Morse lemma and the fact that the union of the images of all infinite geodesic rays based at the identity in Γ is coarsely dense in Γ (see [Bog97]).

If $\mathcal{H}$ is nonempty, there is some $R > 0$ so that the union of all of the cosets $g \cdot \Gamma_p$ for $p \in \Pi$ is R-dense in Γ. So it suffices to show that for each $p \in \Pi$, there is some $R > 0$ so that all but R elements in any coset $g \cdot \Gamma_p$ are the endpoints of a $\mathcal{G}$-path.

For any such coset $g \cdot \Gamma_p$, we can find a finite $\mathcal{G}$-path $\{\alpha_n\}_{n=1}^{N-1}$ limiting to the vertex $g \cdot p$. That is,

$$g \cdot p = \alpha_1 \cdots \alpha_{N-1} p_{v_N}.$$

By definition $p_{v_N} = g' \cdot p$ with T_{v_N} a cofinite subset of the coset $g'\Gamma_p$ That is,

$$g \cdot \Gamma_p = \alpha_1 \cdots \alpha_{N-1} g'\Gamma_p,$$

so for all but finitely many $\gamma \in g \cdot \Gamma_p$ (depending only on the size of the complement of T_{v_N} in $g' \cdot \Gamma_p$), we can find $\alpha_N \in g'\Gamma_p$ with

$$\alpha_1 \cdots \alpha_N = \gamma.$$

□

Remark 5.8. In general, we do *not* require the set of elements in Γ labelling the vertices of a relative quasigeodesic automaton $\mathcal{G}$ to generate the group Γ (although the proposition above implies that they at least generate a finite-index subgroup).

5.1. Compatible systems of open sets.

A relative quasigeodesic automaton always exists for any relatively hyperbolic group (although we will not prove this fact in full generality). We will give a way to construct a relative quasigeodesic automaton using the convergence group action of a group acting on its Bowditch boundary.

Definition 5.9. Suppose that Γ acts on a metrizable space M by homeomorphisms, and let $\mathcal{G}$ be a Γ-graph. A *$\mathcal{G}$-compatible system of open sets* for the action of Γ on M is an assignment of an open subset $U_v \subset M$ to each vertex v of $\mathcal{G}$ such that for each edge $e = (v, w)$ in $\mathcal{G}$, for some $\varepsilon > 0$, we have

$$\alpha \cdot N_M(U_w, \varepsilon) \subset U_v \tag{2}$$

for all $\alpha \in T_v$.

Remark 5.10. If $\mathcal{G}$ has no parabolic vertices (so each set T_v contains a single group element $\alpha_v \in \Gamma$), then (2) is equivalent to requiring $\alpha_v \cdot \overline{U_w} \subset U_v$ for every edge (v, w) in $\mathcal{G}$. When $\mathcal{G}$ has parabolic vertices (so T_v may be infinite), (2) may be a stronger condition.

Proposition 5.11. *Let $\mathcal{G}$ be a Γ-graph, and let $\{U_v : v$ vertex of $\mathcal{G}\}$ be a $\mathcal{G}$-compatible system of subsets of $\partial(\Gamma, \mathcal{H})$ for the action of Γ on $\partial(\Gamma, \mathcal{H})$.*

There is a constant $D > 0$ satisfying the following: let $\{\alpha_n\}$ be an infinite $\mathcal{G}$-path, corresponding to a vertex path $\{v_n\}$, and suppose the sequence $\{\gamma_n = \alpha_1 \cdots \alpha_n\}$ is divergent in Γ. Then for any point z in the intersection

$$U_\infty = \bigcap_{n=1}^{\infty} \alpha_1 \cdots \alpha_n U_{v_{n+1}},$$

the sequence γ_n lies within Hausdorff distance D of a geodesic ray in $\mathrm{Cay}(\Gamma, S, \mathcal{P})$ *tending towards z.*

Proof. Fix a point $z \in U_\infty$, and write $z = z_+$ and $U_n = U_{v_n}$. We first claim that there is a uniform $\varepsilon > 0$ and a point $z_- \in \partial(\Gamma, \mathcal{H})$ such that

$$d(\gamma_n^{-1} z_+, \gamma_n^{-1} z_-) > \varepsilon \tag{3}$$

for all $n \geq 0$.

To prove the claim, choose a uniform $\varepsilon > 0$ so that for every vertex v in $\mathcal{G}$, we have $N(U_v, \varepsilon) \neq \partial(\Gamma, \mathcal{H})$, and for every edge (v, w) in $\mathcal{G}$ and every $\alpha \in T_v$, we have $\alpha \cdot N(U_w, \varepsilon) \subset U_v$. Then we choose some $z_- \in \partial(\Gamma, \mathcal{H}) - \overline{N}(U_1, \varepsilon)$.

By the $\mathcal{G}$-compatibility condition, we know that for any n, $\gamma_n U_{n+1} \subset \ldots \subset \gamma_1 U_2 \subset U_1$, so we know that $d(z_+, z_-) > \varepsilon$.

Then, for any $n \geq 1$, we have

$$\gamma_n^{-1} z_+ \in U_{n+1}.$$

Moreover since $\gamma_n N(U_{n+1}, \varepsilon) \subset U_1$, we also have

$$\gamma_n^{-1} z_- \in \partial(\Gamma, \mathcal{H}) - N(U_{n+1}, \varepsilon).$$

So for all n we have $d(\gamma_n^{-1} z_+, \gamma_n^{-1} z_-) > \varepsilon$, which establishes that (3) holds for all n.

Now, consider a bi-infinite geodesic c in a cusped space Y for Γ joining z_+ and z_-. The sequence of geodesics $\gamma_n^{-1} \cdot c$ has endpoints in $\partial Y = \partial(\Gamma, \mathcal{H})$ lying distance at least ε apart, so each geodesic in the sequence passes within a uniformly bounded neighborhood of a fixed basepoint $y_0 \in Y$. Therefore $\gamma_n \cdot y_0$ lies in a uniformly bounded neighbood of the geodesic c.

Since γ_n is divergent, $\gamma_n y_0$ can only accumulate at either z_+ or z_-. But in fact $\gamma_n y_0$ can only accumulate at z_+—for in the construction of c above, we could have chosen any z_- in the nonempty open set $\partial(\Gamma, \mathcal{H}) - \overline{N}(U_1, \varepsilon)$, and since $\partial(\Gamma, \mathcal{H})$ is perfect there is at least one such $z_-' \neq z_-$.

This implies that γ_n is a conical limit sequence in Γ, limiting to z_+. Since the distance between γ_n and γ_{n+1} is bounded in $\mathrm{Cay}(\Gamma, S, \mathcal{P})$, the desired conclusion follows. ☐

Definition 5.12. Let $\mathcal{G}$ be a Γ-graph. An infinite $\mathcal{G}$-path $\{\alpha_n\}$ is *divergent* if the sequence $\{\gamma_n = \alpha_1 \cdots \alpha_n\}$ leaves every bounded subset of Γ.

We say that a Γ-graph $\mathcal{G}$ is divergent if *every* infinite $\mathcal{G}$-path is divergent.

Whenever $\{U_v\}$ is a $\mathcal{G}$-compatible system of open sets for a Γ-graph $\mathcal{G}$, one can think of a $\mathcal{G}$-path $\{\alpha_n\}$ as giving a *symbolic coding* of a point in the intersection

$$\alpha_1 \cdots \alpha_n U_{n+1}.$$

The following proposition gives a way to construct such a coding for a given point $z \in \partial(\Gamma, \mathcal{H})$, given an appropriate pair of open coverings of the Bowditch boundary $\partial(\Gamma, \mathcal{H})$ compatible with a Γ-graph $\mathcal{G}$.

Proposition 5.13. *Let $\mathcal{G}$ be a* divergent *Γ-graph. Suppose that for each vertex $a \in \mathcal{G}$, there exist open subsets V_a, W_a of $\partial(\Gamma, \mathcal{H})$ such that the following conditions hold:*

(1) The sets $\{W_a\}$ give a $\mathcal{G}$-compatible system of sets for the action of Γ on $\partial(\Gamma, \mathcal{H})$.
(2) For all vertices a, we have $V_a \subset W_a$ and $\overline{W_a} \neq \partial(\Gamma, \mathcal{H})$.
(3) The sets V_a give an open covering of $\partial(\Gamma, \mathcal{H})$.
(4) For every $z \in \partial(\Gamma, \mathcal{H})$ and every non-parabolic vertex a such that $z \in V_a$, there is an edge (a, b) in $\mathcal{G}$ such that $\alpha_a^{-1} \cdot z \in V_b$ for $\{\alpha_a\} = T_a$.
(5) For every $z \in \partial(\Gamma, \mathcal{H})$ and every parabolic *vertex a such that $z \in V_a - \{p_a\}$, there is an edge (a, b) in $\mathcal{G}$ and $\alpha \in T_a$ such that $\alpha^{-1} \cdot z \in V_b$.*

Then $\mathcal{G}$ is a relative quasigeodesic automaton for Γ.

Proof. Proposition 5.11 implies that any infinite $\mathcal{G}$-path lies finite Hausdorff distance from a geodesic ray in $\mathrm{Cay}(\Gamma, S, \mathcal{P})$. So, we just need to show that every $z \in \partial(\Gamma, \mathcal{H})$ is the limit of a $\mathcal{G}$-path.

The idea behind the proof is to use the fact that the sets V_a cover $\partial(\Gamma, \mathcal{H})$ to show that we can keep "expanding" a neighborhood of z in $\partial(\Gamma, \mathcal{H})$ to construct a path in $\mathcal{G}$ limiting

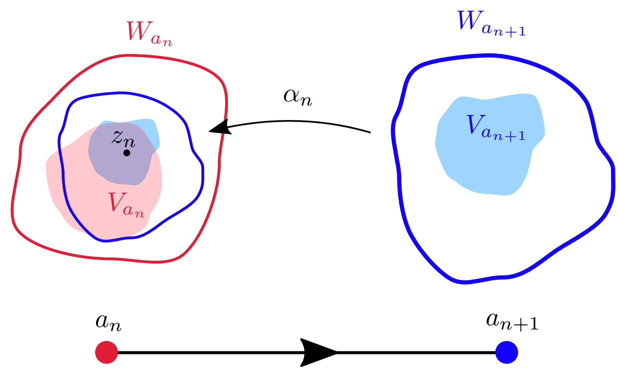

Figure 1. Illustration for the proof of Proposition 5.13. The group element α_n nests an ε-neighborhood of $W_{a_{n+1}}$ inside of W_{a_n} whenever $\alpha_n \cdot V_{a_{n+1}}$ intersects V_{a_n}.

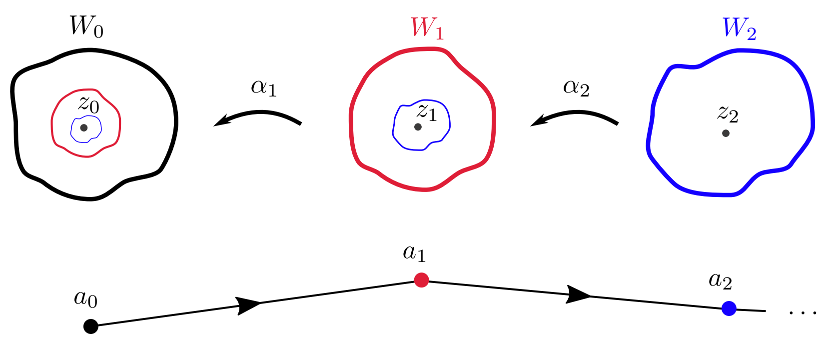

Figure 2. By iterating the nesting procedure backwards, we produce an infinite $\mathcal{G}$-path and a sequence of subsets intersecting in the initial point $z = z_0$.

to z. The $\{V_a\}$ covering tells us how to find the next edge in the path, and the $\{W_a\}$ cover gives us the $\mathcal{G}$-compatible system we need to show that the path is a geodesic.

We let A denote the vertex set of $\mathcal{G}$. When $a \in A$ is not a parabolic vertex, we write $T_a = \{\gamma_a\}$.

Case 1: z is a conical limit point. Fix $a \in A$ so that $z \in V_a$. We take $z_0 = z$, $a_0 = a$, and define sequences $\{z_n\}_{n=0}^\infty \subset \partial_{\text{con}}\Gamma$, $\{a_n\}_{n=0}^\infty \subset A$, and $\{\alpha_n\}_{n=1}^\infty \subset \Gamma$ as follows:

- If a_n is not a parabolic vertex, then we choose $\alpha_{n+1} = \gamma_{a_n}$. Let $z_{n+1} = \alpha_{n+1}^{-1} \cdot z_n$. Since conical limit points are invariant under the action of Γ, z_{n+1} is a conical limit point. By condition 4, there is a vertex a_{n+1} satisfying $z_{n+1} \in V_{a_{n+1}}$ with (a_n, a_{n+1}) an edge in $\mathcal{G}$.
- If a_n is a parabolic vertex, then since z_n is a conical limit point, $z_n \neq p$ for $p = p_{a_n}$. Then condition 5 implies that there exists some $\alpha_{n+1} \in T_{a_n}$ so that $\alpha_{n+1}^{-1} \cdot z_n \in V_{a_{n+1}}$ for an edge (a_n, a_{n+1}) in $\mathcal{G}$. Again, $z_{n+1} = \alpha_{n+1}^{-1} \cdot z_n$ must be a conical limit point since $\partial_{\text{con}}\Gamma$ is Γ-invariant.

The sequence $\{\alpha_n\}$ necessarily gives a $\mathcal{G}$-path. By assumption the sequence

$$\gamma_n = \alpha_1 \cdots \alpha_n$$

is divergent. And by construction $z = \gamma_n z_n$ lies in $\gamma_n W_{a_n}$ for all n. So, Proposition 5.11 implies that γ_n is a conical limit sequence, limiting to z. See Figure 2.

Case 2: z is a parabolic point. As before fix $a \in A$ so that $z \in V_a$, and take $z_0 = z$, $a_0 = a$. We inductively define sequences z_n, a_n, α_n as before, but we claim that for some finite N, a_N is a parabolic vertex with $z_N = p_{a_N}$. For if not, we can build an infinite $\mathcal{G}$-path (as in the previous case) limiting to z. But then, Proposition 5.11 would imply that z is actually a conical limit point. So, we must have

$$z = \gamma_N a_N = \alpha_1 \cdots \alpha_N a_N$$

as required.

□

Remark 5.14. In a typical application of Proposition 5.13, it will not be possible to construct the open coverings $\{V_a\}$ and $\{W_a\}$ so that $V_a = W_a$ for all vertices a. In particular we expect this to be impossible whenever $\partial(\Gamma, \mathcal{H})$ is connected.

To conclude this section, we make one more observation about systems of $\mathcal{G}$-compatible sets as in Proposition 5.13.

Lemma 5.15. *Let Γ be a relatively hyperbolic group, let $\mathcal{G}$ be a Γ-graph, and let $\{V_a\}$, $\{W_a\}$ be an assignment of open subsets of $\partial(\Gamma, \mathcal{H})$ to vertices of $\mathcal{G}$ satisfying the hypotheses of Proposition 5.13.*

Fix $z \in \partial_{\mathrm{con}}\Gamma$ and $N \in \mathbb{N}$. There exists $\delta > 0$ so that if $d(z, z') < \delta$, then there are $\mathcal{G}$-paths $\{\alpha_n\}, \{\beta_n\}$ limiting to z, z' respectively, with $\alpha_i = \beta_i$ for all $i < N$.

Proof. Let $\{\alpha_n\}$ be a $\mathcal{G}$-path limiting to z coming from the construction in Proposition 5.13, passing through vertices v_n. We choose $\delta > 0$ small enough so that if $d(z, z') < \delta$, then z' lies in the set

$$\alpha_1 \cdots \alpha_N V_{v_{n+1}}.$$

Then for every $i < N$, we have

$$\alpha_i^{-1}\alpha_{i-1}^{-1} \cdots \alpha_1^{-1} z' \in V_{v_{i+1}}.$$

As in Proposition 5.13, we can then extend $\{\alpha_n\}_{n=1}^{N-1}$ to a $\mathcal{G}$-path limiting to z'. □

6. Extended convergence dynamics

Let Γ be a relatively hyperbolic group acting on a connected compact metrizable space M. In this section, we will show that if the action of Γ on M extends the convergence dynamics of Γ (Definition 1.2), then we can construct a relative quasigeodesic automaton $\mathcal{G}$ and a $\mathcal{G}$-compatible system of open subsets of M which are in some sense reasonably well-behaved with respect to the group action.

To give the precise statement, we let $\Lambda \subset M$ be a closed Γ-invariant subset, and let $\phi : \Lambda \to \partial(\Gamma, \mathcal{H})$ be an equivariant, surjective, and continuous map satisfying the following: for each $z \in \partial(\Gamma, \mathcal{H})$, there is an open set $C_z \subset M$ containing $\Lambda - \phi^{-1}(z)$ such that:

(1) For any sequence $\gamma_n \in \Gamma$ limiting conically to z, with $\gamma_n^{-1} \to z_-$, any open set U containing $\phi^{-1}(z)$, and any compact $K \subset C_{z_-}$, we have $\gamma_n \cdot K \subset U$ for all sufficiently large n.

(2) For any parabolic point p, any compact $K \subset C_p$, and any open set U containing $\phi^{-1}(p)$, for all but finitely many $\gamma \in \Gamma_p$, we have $\gamma \cdot K \subset U$.

Note that in particular, any map extending convergence dynamics satisfies these conditions. For the rest of this section, however, we *only* assume that (1) and (2) both hold for our map ϕ. In this context, we will show:

Proposition 6.1. *For any $\varepsilon > 0$, there is a relative quasigeodesic automaton $\mathcal{G}$ for Γ, a $\mathcal{G}$-compatible system of open sets $\{U_v\}$ for the action of Γ on M, and a $\mathcal{G}$-compatible system of open sets $\{W_v\}$ for the action of Γ on $\partial(\Gamma, \mathcal{H})$ such that:*

(1) For every vertex v, there is some $z \in W_v$ so that
$$\phi^{-1}(W_v) \subset U_v \subset N_M(\phi^{-1}(z), \varepsilon).$$
(2) For every $p \in \Pi$, there is a parabolic vertex a with $p_a = p$. Moreover, for every parabolic vertex w with $p_w = g \cdot p$, (a, b) is an edge of $\mathcal{G}$ if and only if (w, b) is an edge of $\mathcal{G}$.
(3) If $q = g \cdot p$ for $p \in \Pi$, a is a parabolic vertex with $p_a = q$, and (a, b) is an edge of $\mathcal{G}$, then $q \in W_a$ and $U_b \subset C_p$.

Remark 6.2. By equivariance of ϕ, for each $p \in \partial_{\text{par}}\Gamma$, we can replace C_p with $\Gamma_p \cdot C_p$ and assume that C_p is Γ_p-invariant (and that if $q = g \cdot p$, then $C_q = g \cdot C_p$).

The proof of Proposition 6.1 involves some technicalities, so we first outline the general approach:

(1) For each $z \in \partial(\Gamma, \mathcal{H})$, we construct a pair V_z, W_z of small open neighborhoods of z and a subset $T_z \subset \Gamma$ so that for each $\alpha \in T_z$, α^{-1} is "expanding" about some point in V_z. When z is a conical limit point, then we can choose a single element $\alpha_z \in \Gamma$ which expands about every point in V_z. When z is a parabolic point, we may use a different element of Γ to "expand" about each $u \in V_z - \{z\}$.

We choose V_z, W_z, and T_z so that if α^{-1} is "expanding" about $u \in V_z$, and $\alpha^{-1}u \in V_y$, then $\alpha^{-1}W_z \supset W_y$. See Figure 3.

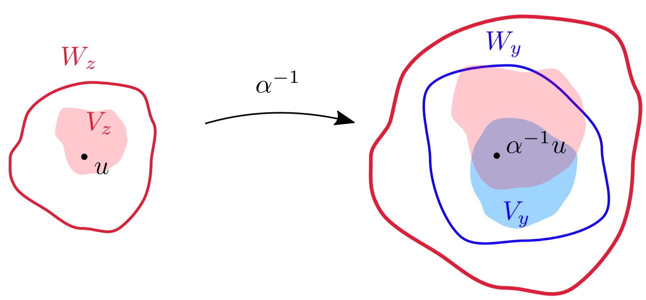

FIGURE 3. The group element α^{-1} is "expanding" about $u \in V_z$. We will construct V_z, W_z and V_y, W_y so that if $\alpha^{-1}u$ lies in V_y, then $\alpha^{-1}W_z$ contains W_y. Equivalently, we get the containment $\alpha W_y \subset W_z$ illustrated earlier in Figure 1.

(2) Using compactness of $\partial(\Gamma, \mathcal{H})$, we pick a finite set of points $a \in \partial(\Gamma, \mathcal{H})$ so that the sets $\{V_a\}$ give an open covering of $\partial(\Gamma, \mathcal{H})$. These points in $\partial(\Gamma, \mathcal{H})$ are identified with the vertices of a Γ-graph $\mathcal{G}$. We define the edges of $\mathcal{G}$ in such a way so that if, for some $\alpha \in T_a$, α^{-1} expands about $u \in V_a$ and $\alpha^{-1}u \in V_b$, then there is an edge from a to b. This ensures that $\{W_a\}$ is a $\mathcal{G}$-compatible system of open subsets of $\partial(\Gamma, \mathcal{H})$.

(3) Simultaneously, we construct a $\mathcal{G}$-compatible system $\{U_a\}$ of open sets in M by taking U_a to be a small neighborhood of $\phi^{-1}(a)$. The idea is to use the extended convergence dynamics to ensure that if, for some $\alpha \in T_z$, α^{-1} "expands" about some $u \in V_z$ and the point $\alpha^{-1}u$ lies in V_y, then $\alpha^{-1}U_z$ contains U_y. See Figure 6 below.
(4) Finally, we use Proposition 5.13 to prove that $\mathcal{G}$ is actually a relative quasigeodesic automaton. The open sets V_a, W_a are constructed exactly to satisfy the conditions of the proposition, so the main thing to check in this step is that the graph $\mathcal{G}$ is actually divergent (using the action of Γ on M).

Throughout the rest of the section, we will work with fixed metrics on both $\partial(\Gamma, \mathcal{H})$ and M. Critically, none of our "expansion" arguments will depend sensitively on the precise choice of metric. That is, in the sketch above, when we say that some group element $\alpha \in \Gamma$ "expands" on a small open subset U of a metric space X, we just mean that αU is quantifiably "bigger" than U, and *not* that for any $x, y \in U$, we have $d(\alpha \cdot x, \alpha \cdot y) \geq C \cdot d(x, y)$ for some expansion constant C. Lemma 6.5 and Lemma 6.7 below describe precisely what we mean by "bigger." The general idea is captured by the following example.

Example 6.3. We consider the group $\mathrm{PGL}(2, \mathbb{Z})$. While $\mathrm{PGL}(2, \mathbb{Z})$ is virtually a free group (and therefore word-hyperbolic), it is also relatively hyperbolic, relative to the collection $\mathcal{H}$ of conjugates of the parabolic subgroup $\left\{\begin{pmatrix} \pm 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{Z}\right\}$.

Since $\mathrm{PGL}(2, \mathbb{Z})$ acts with finite covolume on the hyperbolic plane $\mathbb{H}^2$, the Bowditch boundary of the pair $(\mathrm{PGL}(2, \mathbb{Z}), \mathcal{H})$ is equivariantly identified with $\partial\mathbb{H}^2$, the visual boundary of $\mathbb{H}^2$. Given a non-parabolic point $w \in \partial\mathbb{H}^2$, we can find an element of $\mathrm{PGL}(2, \mathbb{Z})$ which "expands" a neighborhood of w. There are two distinct possibilities:

(1) Suppose w is in a small neighborhood V_z of a conical limit point $z \in \partial\mathbb{H}^2$. Then choose some loxodromic element $\gamma \in \mathrm{PGL}(2, \mathbb{Z})$ whose attracting fixed point is close to z. Then, if W_z is a slightly larger neighborhood of z, $\gamma^{-1} \cdot W_z$ is large enough to contain a uniformly large neighborhood of $\gamma^{-1} \cdot w$. See Figure 4.
(2) On the other hand, suppose w is in a small neighborhood V_q of a parabolic fixed point $q \in \partial\mathbb{H}^2$, but $w \neq q$. We can find some element $\gamma \in \Gamma_q = \mathrm{Stab}_\Gamma(q)$ so that γ^{-1} takes w into a fundamental domain for the action of Γ_q on $\partial\mathbb{H}^2 - \{q\}$. Then, if W_q is a slightly larger neighborhood of q, $\gamma^{-1} \cdot W_q$ is again large enough to contain a uniformly large neighborhood of $\gamma^{-1} \cdot w$. See Figure 5.

There is a slight issue with this approach: in the second case above (when w is close to a parabolic point q), it is actually not quite good enough to "expand" a neighborhood of w by using Γ_q to push w into a fundamental domain for Γ_q on $\partial\mathbb{H}^2 - \{q\}$. The reason is that there might be no such fundamental domain which is actually far away from $\partial\mathbb{H}^2 - \{q\}$. We resolve this issue by instead choosing γ to lie in a *coset* $g\Gamma_p$, where $q = gp$ for some $p \in \Pi$. Then $\gamma^{-1} \cdot w$ lies in a fundamental domain for Γ_p on $\partial\mathbb{H}^2 - \{p\}$, which allows us to get *uniform* control on the size of the expanded neighborhood $\gamma^{-1}W_q$.

The two technical lemmas below (Lemma 6.5 and Lemma 6.7) essentially say that one can set up this kind of expansion *simultaneously* on the Bowditch boundary of our relatively hyperbolic group Γ *and* in a neighborhood of the Γ-invariant set $\Lambda \subset M$. The precise formulation of the expansion condition found in these two lemmas is best motivated by the proof of Proposition 6.10 below, which shows that the "expanding" open sets we construct give rise to a $\mathcal{G}$-compatible system of open sets on a Γ-graph $\mathcal{G}$.

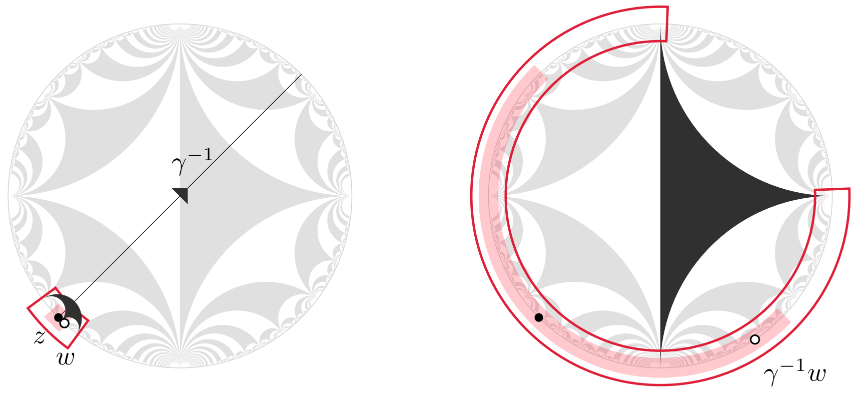

Figure 4. For any point w in a sufficiently small neighborhood V_z (pink) of z, the expanded neighborhood $\gamma^{-1}W_z$ (red) contains a uniform neighborhood of $\gamma^{-1}w$.

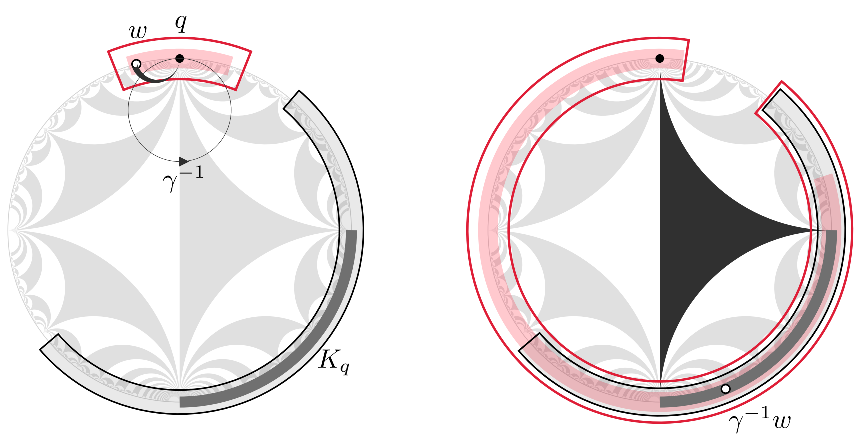

Figure 5. For any point $w \neq q$ in a neighborhood V_q (pink) of the parabolic point q, we find some $\gamma \in \Gamma_q$ so that $\gamma^{-1}w$ lies in K_q (dark gray), a fundamental domain for the action of Γ_q on $\partial\mathbb{H}^2 - \{q\}$. The expanded neighborhood $\gamma^{-1}W_q$ (red) contains a uniform neighborhood of K_q, so $\gamma^{-1}W_q$ contains a uniform neighborhood of $\gamma^{-1}w$.

Lemma 6.4. *There exists $\varepsilon > 0$ (depending on ϕ and D) so that for any $a, b \in \partial(\Gamma, \mathcal{H})$ with $d(a,b) > D$, the ε-neighborhood of $\phi^{-1}(a)$ in M is contained in C_b.*

Proof. Since $\phi^{-1}(z)$ is closed in M, such an $\varepsilon > 0$ exists for any fixed pair of distinct $(a,b) \in \partial(\Gamma, \mathcal{H})^2$. Then the result follows, since the space of pairs $(a,b) \in (\partial(\Gamma, \mathcal{H}))^2$ satisfying $d(a,b) > D$ is compact. ☐

Lemma 6.5. *There exists $\varepsilon_{\text{con}} > 0, \delta_{\text{con}} > 0$ satisfying the following: for any $\varepsilon > 0$, $\delta > 0$ with $\varepsilon < \varepsilon_{\text{con}}$, $\delta < \delta_{\text{con}}$, and every conical limit point z, we can find:*

- *A group element* $\gamma_z \in \Gamma$
- *Open subsets* $W_z, V_z \subset \partial(\Gamma, \mathcal{H})$ *with* $z \in V_z \subset W_z$

such that:

(1) $\mathrm{diam}(W_z) < \delta$,
(2) In $\partial(\Gamma, \mathcal{H})$*, we have*

$$N_{\partial\Gamma}(\gamma_z^{-1} V_z, \delta) \subset \gamma_z^{-1} W_z.$$

(3) In M *we have*

$$N_M(\gamma_z^{-1}\phi^{-1}(W_z), 2\varepsilon) \subset \gamma_z^{-1} N_M(\phi^{-1}(z), \varepsilon).$$

Remark 6.6. Conditions (1) and (2) together imply that for any $y, z \in \partial_{\mathrm{con}}\Gamma$, if $\gamma_z^{-1} V_z$ intersects V_y, then $\gamma_z W_y \subset W_z$. Later, we will see that condition (3) implies that if $\gamma_z^{-1} V_z$ intersects V_y, then also $\gamma_z N_M(\phi^{-1}(y), 2\varepsilon) \subset N_M(\phi^{-1}(z), \varepsilon)$ (giving us the inclusion indicated by Figure 3).

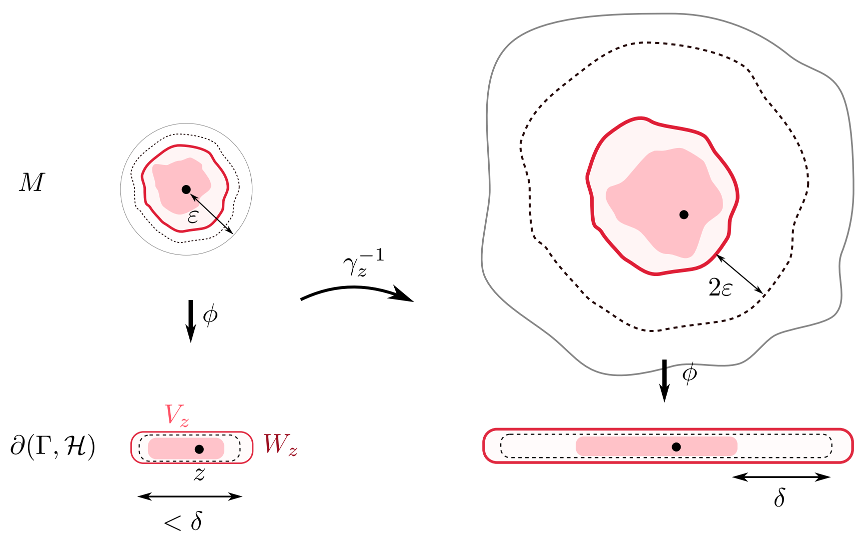

FIGURE 6. The group element γ_z^{-1} is "expanding" about $V_z \subset \partial(\Gamma, \mathcal{H})$: while W_z has diameter $< \delta$, $\gamma_z^{-1} W_z$ contains a δ-neighborhood of $\gamma_z^{-1} V_z$. At the same time, γ_z^{-1} enlarges an ε-neighborhood of $\phi^{-1}(z)$ in M, so that it contains a 2ε-neighborhood of $\gamma_z^{-1}\phi^{-1}(W_z)$.

Proof. For a conical limit point z, we choose a sequence γ_n so that for distinct $a, b \in \partial(\Gamma, \mathcal{H})$, we have $\gamma_n^{-1} z \to a$ and $\gamma_n^{-1} w \to b$ for any $w \neq z$. That is, γ_n limits conically to z in $\overline{\Gamma}$, and γ_n^{-1} converges (not necessarily conically) to b. Since the Γ-action on distinct pairs in $\partial(\Gamma, \mathcal{H})$ is cocompact (Proposition 2.7), we may assume that $d(a, b) > D$ for a uniform constant $D > 0$.

We choose $\varepsilon_{\mathrm{con}} > 0$ from Lemma 6.4 so that if $a, b \in \partial(\Gamma, \mathcal{H})$ satisfy $d(a, b) > D/2$, then a $2\varepsilon_{\mathrm{con}}$-neighborhood of $\phi^{-1}(a)$ is contained in C_b. Let $\varepsilon > 0$ satisfy $\varepsilon < \varepsilon_{\mathrm{con}}$, and let δ satisfy $\delta < \delta_{\mathrm{con}} := D/4$.

By the triangle inequality, we have $d(c, b) > D/2$ for all $c \in B_{\partial\Gamma}(a, 2\delta)$, so the closed 2ε-neighborhood of $\phi^{-1}(B_{\partial\Gamma}(a, 2\delta))$ is contained in C_b. This means that we can choose n

large enough so that
$$\gamma_n \cdot N_M(\phi^{-1}(B(a,2\delta)), 2\varepsilon)$$
is contained in $N_M(\phi^{-1}(z), \varepsilon)$ and
$$\gamma_n \cdot B_{\partial\Gamma}(a, 2\delta)$$
is contained in $B_{\partial\Gamma}(z, \delta/2)$. We let $\gamma_z = \gamma_n$ for this large n, and take
$$W_z = \gamma_z \cdot B_{\partial\Gamma}(a, 2\delta)$$
and
$$V_z = \gamma_z \cdot B_{\partial\Gamma}(a, \delta).$$
□

The next lemma is a version of Lemma 6.5 for parabolic points. As before, we want to show that for a point q in the Bowditch boundary, we can find a neighborhood W_q of q in $\partial(\Gamma, \mathcal{H})$ with uniformly bounded diameter δ, and group elements $\gamma \in \Gamma$ so that γ^{-1} enlarges W_q enough to contain a 2δ-neighborhood of $\gamma^{-1}z$, for some z close to q. Simultaneously we want to choose γ so that γ^{-1} enlarges an ε-neighborhood of $\phi^{-1}(q)$ in a similar manner. This case is more complicated, because we need to allow γ to depend on the point $z \in W_q$: if q is a parabolic point in $\partial(\Gamma, \mathcal{H})$, then in general there is *not* a single group element in Γ which expands distances in a neighborhood of q.

Lemma 6.7. *For each point $p \in \Pi$, there exist constants $\varepsilon_p > 0$, $\delta_p > 0$ such that for any $q = g \cdot p \in \Gamma \cdot p$, any $\varepsilon < \varepsilon_p$, and any $\delta < \delta_p$, we can find:*

- *A cofinite subset T_q of the coset $g\Gamma_p$,*
- *Open neighborhoods V_q, W_q of $\partial(\Gamma, \mathcal{H})$, with $q \in V_q \subset W_q$,*
- *Open neighborhoods $\hat{V}_q, \hat{W}_q$ of $\partial(\Gamma, \mathcal{H})$ with $\hat{V}_q \subset \hat{W}_q$*

such that:

(1) $\mathrm{diam}(W_q) < \delta$, *and* $\phi^{-1}(W_q) \subset N(\phi^{-1}(q), \varepsilon)$.
(2) in $\partial(\Gamma, \mathcal{H})$, we have
$$N_{\partial\Gamma}(\hat{V}_q, \delta) \subset \hat{W}_q.$$
(3) For every $z \in V_q - \{q\}$, there exists $\gamma \in T_q$ with $\gamma^{-1} \cdot z \in \hat{V}_q$.
(4) For every $\gamma \in T_q$, we have
$$N_M(\phi^{-1}(\hat{W}_q), 2\varepsilon) \subset \gamma^{-1} N_M(\phi^{-1}(q), \varepsilon)$$
and
$$\hat{W}_q \subset \gamma^{-1} W_q.$$
(5) $N_M(\phi^{-1}(\hat{W}_q), 2\varepsilon)$ is contained in C_p and $g\Gamma_p \cdot \hat{V}_q$ contains $\partial(\Gamma, \mathcal{H}) - \{q\}$.

Remark 6.8. If $z \in V_q - \{q\}$ and $\gamma^{-1}z \in \hat{V}_q$ for some $\gamma \in T_q$, we think of γ^{-1} as "expanding" about z. Conditions (1) and (2) imply that if $\gamma^{-1}z \in V_y$ for some $\gamma \in T_q$, then $\hat{W}_q$ contains W_y, and by condition (4), $\gamma^{-1}W_q$ contains W_y. Here V_y, W_y are the sets from either Lemma 6.5 or Lemma 6.7.

Proof. Pick a compact set $K \subset \partial(\Gamma, \mathcal{H}) - \{p\}$ so that $\Gamma_p \cdot K$ covers $\partial(\Gamma, \mathcal{H}) - \{p\}$. Choose δ_p small enough so that the closure of $N_{\partial\Gamma}(K, 2\delta_p)$ does not contain p. Then, for any $\delta < \delta_p$, we can pick
$$\hat{V}_q = N_{\partial\Gamma}(K, \delta), \quad \hat{W}_q = N_{\partial\Gamma}(K, 2\delta).$$

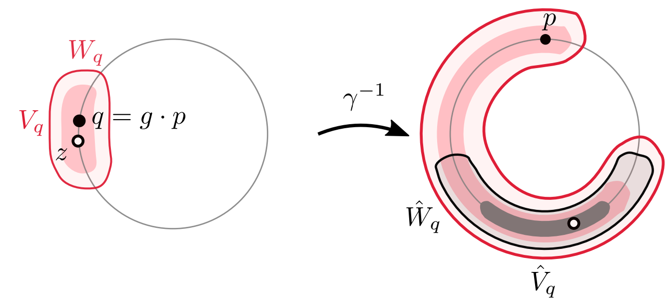

FIGURE 7. The behavior of sets in $\partial(\Gamma, \mathcal{H})$ described by Lemma 6.7. Given $z \in V_q$, we pick an element $\gamma \in g\Gamma_p$ so that a uniformly large neighborhood of $\gamma^{-1}z$ is contained in $\gamma^{-1}W_q$. We cannot pick γ^{-1} to expand the metric everywhere close to q—some points in V_q get contracted close to p.

We can choose ε_p sufficiently small so that a $2\varepsilon_p$-neighborhood of $\phi^{-1}(N_{\partial\Gamma}(K, 2\delta_p))$ is contained in C_p. Now, fix $\varepsilon < \varepsilon_p$. We claim that for a cofinite subset $T_q \subset g \cdot \Gamma_p$, for any $\gamma \in T_q$, we have

$$\gamma \cdot \hat{W}_q \subset B_{\partial\Gamma}(q, \delta/2) \tag{4}$$

$$\gamma \cdot N_M(\phi^{-1}(\hat{W}_q), 2\varepsilon) \subset N_M(\phi^{-1}(q), \varepsilon) \tag{5}$$

To see that this claim holds, it suffices to verify that for any infinite sequence γ_n of distinct group elements in $g\Gamma_p$, (4) and (5) both hold for all sufficiently large n.

We write $\gamma_n = g \cdot \gamma_n'$ for $\gamma_n' \in \Gamma_p$. Then γ_n' converges uniformly to p on compact subsets of $\partial(\Gamma, \mathcal{H}) - \{p\}$, so γ_n converges uniformly to q on compact subsets of $\partial(\Gamma, \mathcal{H}) - \{p\}$, implying that (4) eventually holds. And by our assumptions, we know that

$$\gamma_n' \cdot N_M(\phi^{-1}(\hat{W}_q), 2\varepsilon) \subset g^{-1} \cdot N_M(\phi^{-1}(q), \varepsilon)$$

for sufficiently large n, implying that (5) also eventually holds.

So we can take W_q to be the set

$$\{q\} \cup \bigcup_{\gamma \in T_q} \gamma \cdot \hat{W}_q,$$

and V_q to be the set

$$\{q\} \cup \bigcup_{\gamma \in T_q} \gamma \cdot \hat{V}_q.$$

To see that W_q and V_q are open we just need to verify that they each contain a neighborhood of q. Since $\hat{V}_q$ and $\hat{W}_q$ each contain K, and $\Gamma_p \cdot K$ covers $\partial(\Gamma, \mathcal{H}) - \{p\}$, V_q and W_q each contain the set

$$\partial(\Gamma, \mathcal{H}) - \bigcup_{\gamma \in g\Gamma_p - T_q} \gamma K.$$

Since T_q is cofinite in $g\Gamma_p$ this is an open set containing q. □

6.1. **Construction of the relative automaton.** We will construct the relative automaton $\mathcal{G}$ satisfying the conditions of Proposition 6.1 by choosing a suitable open covering of $\partial(\Gamma, \mathcal{H})$, and then using compactness to take a finite subcover. The subsets of this subcover will be the vertices of $\mathcal{G}$.

We choose constants $\varepsilon > 0$, $\delta > 0$ so that $\varepsilon < \varepsilon_{\mathrm{con}}$, $\delta < \delta_{\mathrm{con}}$ (where $\varepsilon_{\mathrm{con}}$, δ_{con} are the constants coming from Lemma 6.5) and $\varepsilon < \varepsilon_p$, $\delta < \delta_p$ for each $p \in \Pi$ (where ε_p, δ_p are the constants coming from Lemma 6.7).

Then:

- For each $z \in \partial_{\mathrm{con}}\Gamma$, we define W_z, V_z, γ_z as in Lemma 6.5, with parameters ε, δ.
- For each $q \in \partial_{\mathrm{par}}\Gamma$, we define $V_q, W_q, \hat{V}_q, \hat{W}_q$, and T_q as in Lemma 6.7, again with parameters ε, δ.

The collections of sets $\{V_z : z \in \partial_{\mathrm{con}}\Gamma\}$ and $\{V_q : q \in \partial_{\mathrm{par}}\Gamma\}$ together give an open covering of the Bowditch boundary $\partial(\Gamma, \mathcal{H})$. So we choose a finite subcover $\mathcal{V}$, which we can write as

$$\mathcal{V} = \{V_a : a \in A\}$$

where A is a finite subset of $\partial(\Gamma, \mathcal{H})$. We can in particular ensure that A contains the finite set Π.

We identify the vertices of our Γ-graph $\mathcal{G}$ with A. For each $a \in A$, the set T_a is either $\{\gamma_a\}$ (if a is a conical limit point) or T_q (if $a = q$ for a parabolic point q). Then, for each $a \in A$, we define the open sets U_a by

$$U_a = N_M(\phi^{-1}(a), \varepsilon).$$

The edges of the Γ-graph $\mathcal{G}$ are defined as follows:

- For $a, b \in A$ with $a \in \partial_{\mathrm{con}}\Gamma$, there is an edge from a to b if $(\gamma_a^{-1} \cdot V_a) \cap V_b$ is nonempty.
- If $a, b \in A$ with $a \in \partial_{\mathrm{par}}\Gamma$, there is an edge from a to b if $\hat{V}_a \cap V_b$ is nonempty.

Since $\mathcal{V}$ is an open covering of $\partial(\Gamma, \mathcal{H})$, and the sets $\hat{V}_a$ and $\gamma_a^{-1}V_a$ are nonempty, every vertex of $\mathcal{G}$ has at least one outgoing edge. Moreover, for any parabolic point a, the set $\hat{V}_a$ depends only on the orbit of a in $\partial(\Gamma, \mathcal{H})$, so $\mathcal{G}$ must satisfy condition (2) in Proposition 6.1.

Proposition 6.9. *For each $a \in A$, we have*

$$\phi^{-1}(W_a) \subset U_a.$$

Proof. When a is not a parabolic vertex, Part (3) of Lemma 6.5 implies:

$$\phi^{-1}(W_a) = \gamma_a\gamma_a^{-1}\phi^{-1}(W_a) \subset \gamma_a N(\gamma_a^{-1}\phi^{-1}(W_a), 2\varepsilon) \subset N_M(\phi^{-1}(a), \varepsilon) = U_a.$$

When a is a parabolic vertex, then the claim follows directly from Part (1) of Lemma 6.7. □

Next we verify:

Proposition 6.10. *The collection of sets $\{W_v\}$ and $\{U_v\}$ are both $\mathcal{G}$-compatible systems of open sets for the Γ-graph $\mathcal{G}$.*

Proof. First fix an edge (a, b) with $a \in \partial_{\mathrm{con}}\Gamma$. Since $(\gamma_a^{-1}V_a) \cap V_b$ is nonempty, part 2 of Lemma 6.5 implies that $\gamma_a^{-1} \cdot W_a$ contains the δ-neighborhood of some point $z \in V_b$. Since $\mathrm{diam}(W_b) < \delta$ and $V_b \subset W_b$, we can find a small $\varepsilon' > 0$ so that $\gamma_a N_{\partial\Gamma}(W_b, \varepsilon') \subset W_a$.

In particular, $\gamma_a^{-1} \cdot W_a$ contains b, which means that $N_M(\gamma_a^{-1}\phi^{-1}(W_a), 2\varepsilon)$ contains $N_M(\phi^{-1}(b), 2\varepsilon)$, which contains $N_M(U_b, \varepsilon)$. Then, part 3 of Lemma 6.5 implies that $\gamma_a \cdot N_M(U_b, \varepsilon)$ is contained in $N_M(\phi^{-1}(a), \varepsilon) = U_a$.

Next fix an edge (q, b) with $q \in \partial_{\mathrm{par}}\Gamma$. From part 2 of Lemma 6.7, we know that $\hat{W}_q$ contains the δ-neighborhood of a point $z \in \hat{V}_q \cap V_b$. Since $\mathrm{diam}(W_b) < \delta$ and $V_b \subset W_b$, this means that $\hat{W}_q$ contains an ε'-neighborhood of W_b for some small $\varepsilon' > 0$. So part 4 of Lemma 6.7 implies that for any $\gamma \in T_q$, we have $\gamma \cdot N(W_b, \varepsilon') \subset W_q$.

In particular $\hat{W}_q$ contains b, so $N_M(\phi^{-1}(\hat{W}_q), 2\varepsilon)$ contains $N_M(\phi^{-1}(b), 2\varepsilon)$, which contains $N_M(U_b, \varepsilon)$. Then, part 4 of Lemma 6.7 implies that

$$\gamma N_M(U_b, \varepsilon) \subset N_M(\phi^{-1}(q), \varepsilon) = U_q$$

for any $\gamma \in T_q$. ☐

We observe:

Proposition 6.11. *The $\mathcal{G}$-compatible systems of open subsets $\{U_v\}$ and $\{W_v\}$ satisfy conditions (1) - (3) in Proposition 6.1.*

Proof. Part (1) follows directly from Proposition 6.9, and the fact that we defined each U_a to be the ε-neighborhood of $\phi^{-1}(a)$. Part (2) is true by the construction of the open covering $\mathcal{V}$ and the graph $\mathcal{G}$. Part (3) is true by construction and part (5) of Lemma 6.7. ☐

To finish the proof of Proposition 6.1, we now just need to show:

Proposition 6.12. *The Γ-graph $\mathcal{G}$ is a relative quasigeodesic automaton for the pair $(\Gamma, \mathcal{H})$.*

Proof. We apply Proposition 5.13, using the $\mathcal{G}$-compatible system $\{W_a\}$ and the sets $\{V_a\}$ we defined in the construction of $\mathcal{G}$.

The first three conditions of Proposition 5.13 are satisfied by construction. To see that conditions 4 and 5 hold, first observe that if $z \in V_a$ for a non-parabolic vertex a, then $\gamma_a^{-1} \cdot z$ lies in some V_b and (a, b) is an edge in $\mathcal{G}$. And if $z \in V_a - \{p_a\}$ for a parabolic vertex a, then part (3) of Lemma 6.7 says that there is some $\gamma \in T_a$ such that $\gamma^{-1} \cdot z \in \hat{V}_a$. If V_b contains $\gamma^{-1} \cdot z$, the edge (a, b) must be in $\mathcal{G}$.

It only remains to check that $\mathcal{G}$ is a divergent Γ-graph. Let $\{\alpha_n\}$ be an infinite $\mathcal{G}$-path, following a vertex path $\{v_n\}$. The $\mathcal{G}$-compatibility condition implies that $\gamma_n \overline{U}_{v_{n+1}}$ is a subset of $\gamma_{n-1} U_{v_n}$ for every n. Since M is connected and each U_v is a proper subset of M, this inclusion must be proper. This implies that in the sequence γ_n, no element can appear more than $\#A$ times and therefore γ_n is divergent. ☐

Remark 6.13. This last step is the only part of the proof of Proposition 6.1 which uses the connectedness of M. This hypothesis is likely unnecessary, but omitting it would involve introducing additional technicalities in the construction of the sets V_a, W_a—and as stated, the proposition is strong enough for our purposes.

Note that with this hypothesis removed, Proposition 6.1 would imply that any non-elementary relatively hyperbolic group has a relative quasigeodesic automaton (by taking $M = \partial(\Gamma, \mathcal{H})$). As stated, the proposition only shows that such an automaton exists when $\partial(\Gamma, \mathcal{H})$ is connected.

We conclude this section by observing that one can slightly refine the construction in Proposition 6.1 to obtain some stronger conditions on the resulting automaton.

Proposition 6.14. *Fix a compact subset Z of the Bowditch boundary $\partial(\Gamma, \mathcal{H})$. Then, for any open set $U \subset M$ containing $\phi^{-1}(Z)$, there is a relative quasigeodesic automaton $\mathcal{G}$ and a pair of $\mathcal{G}$-compatible systems of open sets $\{U_a\}$, $\{W_a\}$ as in Proposition 6.1, additionally satisfying the following: any $z \in Z$ is the limit of a $\mathcal{G}$-path $\{\alpha_n\}$ (with corresponding vertex path $\{v_n\}$) such that $U_{v_1} \subset U$.*

Proof. We choose $\varepsilon > 0$ so that U contains $N_M(\phi^{-1}(Z), \varepsilon)$. We then construct our relative quasigeodesic automaton $\mathcal{G}$ as in the proof of Proposition 6.1, but we also choose a finite subset $A_Z \subset Z$ so that the sets V_a for $a \in A_Z$ give a finite open covering of Z. We can ensure that the vertex set A of $\mathcal{G}$ contains A_Z.

Then, for any $z \in Z$, by the construction in Proposition 5.13, we can find a $\mathcal{G}$-path limiting to z whose first vertex is some $a \in A_Z$. The corresponding open set for this vertex is $U_a = N_M(\phi^{-1}(a), \varepsilon) \subset U$. ☐

Proposition 6.15. *For each parabolic point $p \in \Pi$, let K_p be a compact subset of $\partial(\Gamma, \mathcal{H}) - \{p\}$ such that $\Gamma_p \cdot K_p = \partial(\Gamma, \mathcal{H}) - \{p\}$. Then the relative quasigeodesic automaton in Proposition 6.1 can be chosen to satisfy the following:*

For every parabolic vertex w with $p_w = p \in \Pi$, and every $z \in K_p$, there is a $\mathcal{G}$-path limiting to z whose first vertex u is connected to w by an edge (w, u).

Proof. The proof of Lemma 6.7 shows that in our construction of the relative automaton, we can ensure that each set $\hat{V}_p$ contains K_p. So if $z \in K_p$, then by definition of the automaton, z lies in z lies in V_u with w connected to u by a directed edge. Then, following the proof of Proposition 5.13, we can find a $\mathcal{G}$-path limiting to z whose first vertex is u. ☐

7. Contracting paths in flag manifolds

Let $\Gamma \subset G$ be a discrete relatively hyperbolic group, and let $\mathcal{G}$ be a Γ-graph. Fix a pair of opposite parabolic subgroups P^+, P^-. Our goal in this section is to show that under certain conditions, if $\{U_v\}$ is a $\mathcal{G}$-compatible system of open subsets of G/P^+ for the action of Γ on G/P^+, then the sequence of group elements lying along an infinite $\mathcal{G}$-path is P^+-divergent.

7.1. **Contracting paths in Γ-graphs.**

Definition 7.1. Let Γ be a discrete subgroup of G, let $\mathcal{G}$ be a Γ-graph, and let $\{U_v\}_{v \in V(\mathcal{G})}$ be a $\mathcal{G}$-compatible system of open subsets of G/P^+. We say that a $\mathcal{G}$-path $\{\alpha_n\}_{n \in \mathbb{N}}$ is *contracting* if the decreasing intersection

$$\bigcap_{n=1}^{\infty} \alpha_1 \cdots \alpha_n \cdot U_{v_{n+1}} \tag{6}$$

is a singleton in G/P^+.

Definition 7.2. We say that an open set $\Omega \subset G/P^+$ is a *proper domain* if the closure of Ω lies in an affine chart $\mathrm{Opp}(\xi) \subset G/P^+$ for some $\xi \in G/P^-$.

Here is the main result in this section:

Proposition 7.3. *Let $\mathcal{G}$ be a Γ-graph for $(\Gamma, \mathcal{H})$, and let $\{U_v\}_{v \in V(\mathcal{G})}$ be $\mathcal{G}$-compatible system of open subsets of G/P^+.*

If the set U_v is a proper domain for each vertex v of the automaton, then every infinite $\mathcal{G}$-path is contracting.

7.2. **A metric property for bounded domains in flag manifolds.**

To prove Proposition 7.3, we consider a metric C_Ω defined by Zimmer [Zim18] on any proper domain $\Omega \subset G/P^+$. C_Ω is defined so that it is invariant under the action of G on G/P^+: for any x, y in some proper domain $\Omega \subset G/P^+$, and any $g \in G$, we have

$$C_\Omega(x, y) = C_{g\Omega}(gx, gy). \tag{7}$$

In general, C_Ω is not a complete metric. However, C_Ω induces the standard topology on Ω as an open subset of G/P. We will show that for a $\mathcal{G}$-path $\{\alpha_n\}$, the diameter of

$$\alpha_1 \cdots \alpha_n U_{v_{n+1}}$$

with respect to $C_{U_{v_1}}$ tends to zero as $n \to \infty$.

Zimmer's construction of C_Ω depends on an irreducible representation $\zeta : G \to \mathrm{PGL}(V)$ for some real vector space V. This is provided by a theorem of Guéritaud-Guichard-Kassel-Wienhard.

Theorem 7.4 ([GGKW17], see also [Zim18], Theorem 4.6)**.** *There exists a real vector space V, an irreducible representation $\zeta : G \to \mathrm{PGL}(V)$, a line $\ell \subset V$, and a hyperplane $H \subset V$ such that:*

(1) $\ell + H = V$.
(2) The stabilizer of ℓ in G is P^+ and the stabilizer of H in G is P^-.
(3) gP^+g^{-1} and hP^-h^{-1} are opposite if and only if $\zeta(g)\ell$ and $\zeta(h)H$ are transverse.

The representation ζ determines a pair of embeddings $\iota : G/P^+ \to \mathbb{P}(V)$ and $\iota^* : G/P^- \to \mathbb{P}(V^*)$ by

$$\iota(gP^+) = \zeta(g)\ell, \qquad \iota^*(gP^-) = \zeta(g)H.$$

In this section, we will identify $\mathbb{P}(V^*)$ with the space of projective hyperplanes in $\mathbb{P}(V)$, by identifying the projectivization of a functional $w \in V^*$ with the projectivization of its kernel.

Definition 7.5. Let Ω be an open subset of G/P^+. The *dual domain* $\Omega^* \subset G/P^-$ is

$$\Omega^* = \{\nu \in G/P^- : \nu \text{ is opposite to } \xi \text{ for every } \xi \in \overline{\Omega}\}.$$

Note that Ω^* is open if and only if Ω is a proper domain.

Definition 7.6. Let $w_1, w_2 \in \mathbb{P}(V^*)$, and let $z_1, z_2 \in \mathbb{P}(V)$. The *cross-ratio* $[w_1, w_2; z_1, z_2]$ is defined by

$$\frac{\tilde{w}_1(\tilde{z}_2)\tilde{w}_2(\tilde{z}_1)}{\tilde{w}_1(\tilde{z}_1)\tilde{w}_2(\tilde{z}_2)},$$

where $\tilde{w}_i$, $\tilde{z}_i$ are respectively lifts of w_i and z_i in V^* and V.

Remark 7.7. When V is two-dimensional, we can identify the projective line $\mathbb{P}(V^*)$ with $\mathbb{P}(V)$ by identifying each $[w] \in \mathbb{P}(V^*)$ with $[\ker(w)] \in \mathbb{P}(V)$. In that case, the cross-ratio defined above agrees with the standard four-point cross-ratio on $\mathbb{RP}^1$, given by

$$[a, b; c, d] := \frac{(d-a)(c-b)}{(c-a)(d-b)}. \tag{8}$$

The differences in (8) can be measured in any affine chart in $\mathbb{RP}^1$ containing a, b, c, d. Our convention is chosen so that if we identify $\mathbb{RP}^1$ with $\mathbb{R} \cup \{\infty\}$, we have $[0, \infty; 1, z] = z$.

Definition 7.8. Let $\Omega \subset G/P^+$ be a proper domain. We define the function $C_\Omega : \Omega \times \Omega \to \mathbb{R}$ by

$$C_\Omega(x, y) = \sup_{\xi_1, \xi_2 \in \Omega^*} \log |[\iota^*(\xi_1), \iota^*(\xi_2); \iota(x), \iota(y)]|.$$

For any $g \in G$ and any proper domain $\Omega \subset G/P^+$, we have $(g\Omega)^* = (g\Omega^*)$. So C_Ω must satisfy the G-invariance condition (7).

If Ω is a properly convex subset of $\mathbb{P}(V)$, and ζ, ι, ι^* are the identity maps on $\mathrm{PGL}(V)$, $\mathbb{P}(V)$, and $\mathbb{P}(V^*)$ respectively, then C_Ω agrees with the well-studied *Hilbert metric* on Ω. More generally we have:

Theorem 7.9 ([Zim18], Theorem 5.2)**.** *If Ω is open and bounded in an affine chart, then C_Ω is a metric on Ω which induces the standard topology on Ω as an open subset of G/P^+.*

Remark 7.10. This particular result in [Zim18] is stated only for noncompact simple Lie groups, but the proof only assumes that G is semisimple with no compact factor.

Since taking duals of proper domains reverses inclusions, it follows that if $\Omega_1 \subset \Omega_2$, then $C_{\Omega_1} \geq C_{\Omega_2}$. Our goal now is to sharpen this inequality, and show:

Proposition 7.11. *Let Ω_1, Ω_2 be proper domains in G/P^+, such that $\overline{\Omega_1} \subset \Omega_2$.*

There exists a constant $\lambda > 1$ (depending on Ω_1 and Ω_2) so that for all $x, y \in \Omega_1$,

$$C_{\Omega_1}(x, y) \geq \lambda \cdot C_{\Omega_2}(x, y).$$

A consequence is the following, which in particular implies Proposition 7.3.

Corollary 7.12. *Let $\mathcal{G}$ be a Γ-graph for a relatively hyperbolic group Γ, and let $\{U_v\}$ be a $\mathcal{G}$-compatible system of open subsets of G/P^+. If each U_v is a proper domain, then there are constants $\lambda_1, \lambda_2 > 0$ so that for any $\mathcal{G}$-path $\{\alpha_n\}$ in the Γ-graph $\mathcal{G}$, the diameter of*

$$\alpha_1 \cdots \alpha_n \cdot U_{v_{n+1}}$$

with respect to $C_{U_{v_1}}$ is at most

$$\lambda_1 \cdot \exp(-\lambda_2 \cdot n).$$

Proof. For any open set $U \subset G/P^+$ and $A \subset U$, we let $\mathrm{diam}_U(A)$ denote the diameter of A with respect to the metric C_U. We choose a uniform $\varepsilon > 0$ so that in some fixed metric on G/P^+, every edge (v, w) in $\mathcal{G}$, and every $\alpha \in T_v$, we have

$$\alpha N(U_w, \varepsilon) \subset U_v.$$

Then for each vertex set U_v, we write $U_v^\varepsilon = N(U_v, \varepsilon)$.

We take

$$\lambda_1 = \max\{\mathrm{diam}_{U_v^\varepsilon}(U_v)\}.$$

Proposition 7.11 implies that there exists $\lambda_v > 0$ such that for all $x, y \in U_v$, we have

$$C_{U_v}(x, y) \geq \exp(\lambda_v) \cdot C_{U_v^\varepsilon}(x, y).$$

Take $\lambda_2 = \min_v\{\lambda_v\}$. We claim that for all $n \geq 1$, we have

$$\mathrm{diam}_{U_1^\varepsilon}(\alpha_1 \cdots \alpha_n U_{v_n}) \leq \lambda_1 \exp(-\lambda_2 \cdot (n-1)).$$

We prove the claim via induction on the length of the $\mathcal{G}$-path $\{\alpha_n\}$. For $n = 1$, the claim is true because $\alpha_1 U_{v_2} \subset U_{v_1}$. For $n > 1$, we can assume

$$\lambda_1 \exp(-\lambda_2(n-2)) \geq \mathrm{diam}_{U_{v_2}^\varepsilon}(\alpha_2 \cdots \alpha_n \cdot U_{v_{n+1}}).$$

Then we have

$$\begin{aligned}\mathrm{diam}_{U_{v_2}^\varepsilon}(\alpha_2 \cdots \alpha_n \cdot U_{v_{n+1}}) &= \mathrm{diam}_{\alpha_1 U_{v_2}^\varepsilon}(\alpha_1 \cdots \alpha_n \cdot U_{v_{n+1}}) \\ &\geq \mathrm{diam}_{U_{v_1}}(\alpha_1 \cdots \alpha_n \cdot U_{v_{n+1}}) \\ &\geq \exp(\lambda_2) \cdot \mathrm{diam}_{U_{v_1}^\varepsilon}(\alpha_1 \cdots \alpha_n \cdot U_{v_{n+1}}).\end{aligned}$$

Finally, the claim implies the corollary because we know that

$$\begin{aligned}\mathrm{diam}_{U_1}(\alpha_1 \cdots \alpha_n U_{n+1}) &\leq \mathrm{diam}_{\alpha_1 U_2^\varepsilon}(\alpha_1 \cdots \alpha_n U_{n+1}) \\ &= \mathrm{diam}_{U_2^\varepsilon}(\alpha_2 \cdots \alpha_n U_{n+1}) \\ &\leq \lambda_1 \exp(\lambda_2(n-2)).\end{aligned}$$

So, we can replace λ_1 with $\lambda_1 \exp(-2\lambda_2)$ to get the desired result. ☐

We now proceed with the proof of Proposition 7.11. We first observe that in the special case where Ω_1, Ω_2 are properly convex subsets of real projective space, one can show the desired result essentially via the following:

Proposition 7.13. *Let a, b, c, d be points in $\mathbb{RP}^1$, arranged so that $a < b < c < d \leq a$ in a cyclic ordering on $\mathbb{RP}^1$. Then there exists a constant $\lambda > 1$, depending only on the cross-ratio $[a, b; c, d]$, so that for all distinct $x, y \in (b, c)$, we have*

$$|\log[b, c; x, y]| \geq \lambda \cdot |\log[a, d; x, y]|.$$

Proposition 7.13 is a standard fact in real projective geometry and can be verified by a computation. Note that we allow the degenerate case $a = d$: in this situation the right-hand side is identically zero for distinct $x, y \in (b, c)$. We allow no other equalities among a, b, c, d, so the cross-ratio $[a, b; c, d]$ lies in $\mathbb{R} - \{1\}$.

To apply Proposition 7.13 to our situation, we need to get some control on the behavior of the embeddings $\iota : G/P^+ \to \mathbb{P}(V)$ and $\iota^* : G/P^- \to \mathbb{P}(V^*)$. We do so in the next three lemmas below.

Lemma 7.14. *Let x, y be distinct points in G/P^+. There exists a one-parameter subgroup $g_t \subset G$ such that $\zeta(g_t)$ fixes $\iota(x)$ and $\iota(y)$, and acts nontrivially on the projective line L_{xy} spanned by $\iota(x)$ and $\iota(y)$.*

Proof. We can write $x = gP^+$ for some $g \in G$. Let $\mathfrak{a}$ denote an abelian subalgebra of the Lie algebra $\mathfrak{g}$ of G, such that for a maximal compact $K \subset G$, the exponential map $\mathfrak{a} \to G$ induces an isometric embedding $\mathfrak{a} \to G/K$ whose image is a maximal flat in G/K.

There is a conjugate $\mathfrak{a}'$ of $\mathfrak{a}$ such that the action of $\exp(\mathfrak{a}')$ on G/P^+ fixes both x and y (see [Ebe96], Proposition 2.21.14). So, up to the action of G on G/P^+, we can assume that x is fixed by a standard parabolic subgroup P_θ^+ conjugate to P^+, and that x, y are both fixed by the subgroup $\exp(\mathfrak{a})$.

We choose $Z \in \mathfrak{a}^+$ so that $\alpha(Z) \neq 0$ for all $\alpha \in \theta$. Then $g_t = \exp(tZ)$ is a 1-parameter subgroup of G fixing x. As $t \to +\infty$, g_t is P_θ^+-divergent, with unique attracting fixed point x.

Then [GGKW17], Lemma 3.7 implies that $\zeta(g_t)$ is P_1-divergent, where P_1 is the stabilizer of a line in V, and $\iota(x)$ is the unique one-dimensional eigenspace of $\zeta(g_t)$ whose eigenvalue has largest modulus. And, since $\zeta(g_t)$ fixes $\iota(x)$ and $\iota(y)$, $\zeta(g_t)$ preserves L_{xy}, and acts nontrivially since the eigenvalues of $\zeta(g_t)$ on $\iota(x)$ and $\iota(y)$ must be distinct. □

Lemma 7.15. *Let L be any projective line in $\mathbb{P}(V)$ tangent to the image of the embedding $\iota : G/P^+ \to \mathbb{P}(V)$ at a point $\iota(x)$ for $x \in G/P^+$. There exists a one-parameter subgroup g_t of G so that $\zeta(g_t)$ acts nontrivially on L with unique fixed point $\iota(x)$.*

Proof. Fix a sequence $y_n \in G/P^+$ such that $y_n \neq x$ and the projective line L_n spanned by $\iota(x)$ and $\iota(y_n)$ converges to L. By Lemma 7.14, there exists $Z_n \in \mathfrak{g}$ so that $\zeta(\exp(tZ_n))$ acts nontrivially on L_n, with fixed points $\iota(x)$ and $\iota(y_n)$.

In the projectivization $\mathbb{P}(\mathfrak{g})$, $[Z_n]$ converges to some $[Z]$. Since $\zeta : G \to \mathrm{PGL}(V)$ has finite kernel, there is an induced map $\zeta : \mathbb{P}(\mathfrak{g}) \to \mathbb{P}(\mathfrak{sl}(V))$, which satisfies

$$\zeta([Z_n]) \to \zeta([Z]).$$

A continuity argument shows that the one-parameter subgroup $\zeta(\exp(tZ))$ acts nontrivially on the line L, and has unique fixed point at $\iota(x)$. □

Lemma 7.16. *Let $\Omega \subset G/P^+$ be a proper domain, and let L be a projective line in $\mathbb{P}(V)$ which is either spanned by two points in $\iota(\Omega)$, or is tangent to $\iota(G/P^+)$ at a point $\iota(x)$ for $x \in \Omega$. Then*

$$W_L = \{v \in L : v = \iota^*(\xi) \cap L \text{ for } \xi \in \Omega^*\}$$

is a nonempty open subset of L.

Proof. W_L is nonempty since Ω^* is nonempty. So let $v \in W_L$, and choose $\xi \in \Omega^*$ so that $\iota^*(\xi) \cap L = v$. We need to show that an open interval $I \subset L$ containing v is contained in W_L.

If L is spanned by $x, y \in \iota(\Omega)$, then Lemma 7.14 implies that we can find a one-parameter subgroup $g_t \in G$ such that $\zeta(g_t)$ fixes x and y, and acts nontrivially on L. Since Ω^* is open, we can find $\varepsilon > 0$ so that $g_t \cdot \xi \in \Omega^*$ for $t \in (-\varepsilon, \varepsilon)$. Since x and y are in $\iota(\Omega)$, $\iota^*(\xi)$ is transverse to both x and y, so we have $v \neq x$, $v \neq y$. Then as t varies from $-\varepsilon$ to ε,

$$\iota^*(g_t \cdot \xi) \cap L = \zeta(g_t) \cdot v$$

gives an open interval in W_L containing v.

A similar argument using Lemma 7.15 shows that the claim also holds if L is tangent to $\iota(\Omega)$. □

We can now prove a slightly weaker version of Proposition 7.11, which we will then use to show the stronger version.

Lemma 7.17. *Let Ω_1, Ω_2 be proper domains in G/P^+, with $\overline{\Omega_1} \subset \Omega_2$, and let $K \subset \Omega_1$ be compact. There exists a constant $\lambda > 1$ such that for all $x, y \in K$,*

$$C_{\Omega_1}(x, y) \geq \lambda \cdot C_{\Omega_2}(x, y).$$

Proof. Since K is compact, it suffices to show that for fixed $x \in \Omega_1$, the ratio

$$\frac{C_{\Omega_1}(x, y)}{C_{\Omega_2}(x, y)}$$

is bounded below by some $\lambda > 1$ as y varies in $K - \{x\}$.

Let $y \in K - \{x\}$, and let L_{xy} denote the projective line spanned by $\iota(x)$ and $\iota(y)$. Choose $\xi, \eta \in \overline{\Omega_2^*}$ so that

$$C_{\Omega_2}(x, y) = \log |[\iota^*(\xi), \iota^*(\eta); \iota(x), \iota(y)]|.$$

That is, if $v = \iota^*(\xi) \cap L_{xy}$, $w = \iota^*(\eta) \cap L_{xy}$, we have

$$C_{\Omega_2}(x, y) = \log |[v, w; \iota(x), \iota(y)]| = \log \frac{|v - \iota(y)| \cdot |w - \iota(x)|}{|v - \iota(x)| \cdot |w - \iota(y)|},$$

where the distances are measured in any identification of L_{xy} with $\mathbb{R}\mathrm{P}^1 = \mathbb{R} \cup \{\infty\}$.

We can choose an identification of L_{xy} with $\mathbb{R} \cup \{\infty\}$ so that either $v < \iota(x) < \iota(y) < w$ or $v < \iota(x) < w < \iota(y)$. In either case, for any $v' \in (v, \iota(x)) \subset L_{xy}$, we have

$$\log |[v', w; \iota(x), \iota(y)]| > \log |[v, w; \iota(x), \iota(y)]|.$$

We know that $\overline{\Omega_2^*} \subset \Omega_1^*$, so ξ, η lie in Ω_1^*. Then Lemma 7.16 implies that there exists $\xi' \in \Omega_1^*$ so that $v' = \iota^*(\xi') \cap L_{xy}$ lies in the interval $(v, \iota(x)) \subset L_{xy}$. See Figure 8.

Then, we have

$$\begin{aligned} C_{\Omega_1}(x, y) &\geq \log |[\iota^*(\xi'), \iota^*(\eta); \iota(x), \iota(y)]| \\ &= \log |[v', w; \iota(x), \iota(y)] \\ &> \log |[v, w; \iota(x), \iota(y)] \\ &= C_{\Omega_2}(x, y). \end{aligned}$$

This shows that $\frac{C_{\Omega_1}(x,y)}{C_{\Omega_2}(x,y)} > 1$ for all $y \in K - \{x\}$. We still need to find some uniform $\lambda > 1$ so that $\frac{C_{\Omega_1}(x,y)}{C_{\Omega_2}(x,y)} \geq \lambda$ for all $y \in K - \{x\}$. To see this, suppose for the sake of a contradiction

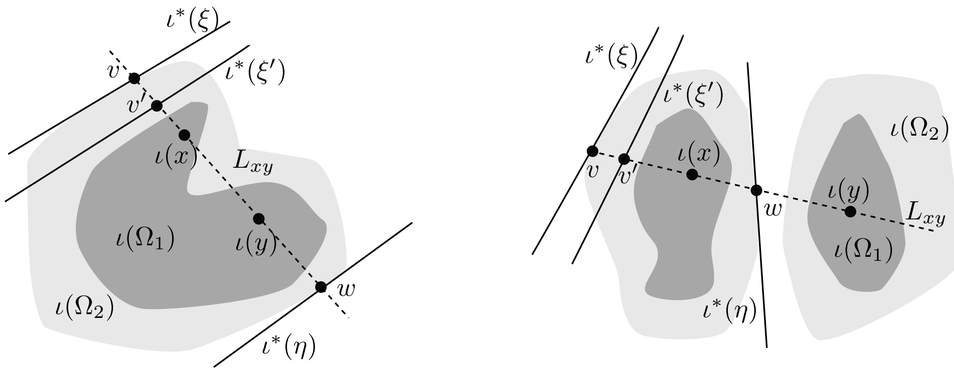

FIGURE 8. We can always find $\xi' \in \Omega_1^*$ close to ξ so that the absolute value of the cross-ratio $[\iota^*(\xi), \iota^*(\nu); \iota(x), \iota(y)]$ increases when we replace ξ with ξ'. In particular this is possible even when the sets $\iota(\Omega_1)$, $\iota(\Omega_2)$ fail to be convex (left) or even connected (right).

that for a sequence $y_n \in K - \{x\}$, we have

$$\frac{C_{\Omega_1}(x, y_n)}{C_{\Omega_2}(x, y_n)} \to 1. \tag{9}$$

Since K is compact, y_n must converge to x. Up to subsequence, the sequence of projective lines L_n spanned by $\iota(x)$ and $\iota(y_n)$ converges to a line L tangent to $\iota(G/P^+)$ at $\iota(x)$.

For each y_n, choose ξ_n, $\eta_n \in \overline{\Omega_2^*}$ so that

$$C_{\Omega_2}(x, y_n) = \log |[\iota^*(\xi_n), \iota^*(\eta_n); \iota(x), \iota(y_n)]|.$$

Let $v_n = \iota^*(\xi_n) \cap L_n$, $w_n = \iota^*(\eta_n) \cap L_n$. Then up to subsequence ξ_n converges to $\xi \in \overline{\Omega_2^*}$, η_n converges to $\eta \in \overline{\Omega_2^*}$, and v_n, w_n respectively converge to $v = \iota^*(\xi) \cap L$, $w = \iota^*(\eta) \cap L$.

Since x is in Ω_2, $\iota^*(\xi)$ and $\iota^*(\eta)$ are both transverse to $\iota(x)$—so in particular $x \neq w$ and $x \neq v$ (although a priori we could have $v = w$).

Since $\xi \in \overline{\Omega_2^*} \subset \Omega_1^*$, Lemma 7.16 implies that there exist $\xi', \eta' \in \Omega_1^*$ so that for some identification of L with $\mathbb{R} \cup \{\infty\}$, we have

$$v < \iota^*(\xi') \cap L < \iota(x) < \iota^*(\eta') \cap L < w.$$

Note that this is possible even if $v = w$, because then we can just identify both v and w with ∞. Let $v_n' = \iota^*(\xi') \cap L_n$, and let $w_n' = \iota^*(\eta') \cap L_n$. Respectively, v_n' and w_n' converge to $v' = \iota^*(\xi') \cap L$ and $w' = \iota^*(\eta') \cap L$.

This means that the cross-ratios $[v_n, v_n'; w_n', w_n]$ converge to $[v, v'; w', w] \in \mathbb{R} - \{1\}$, and in particular are bounded away from both ∞ and 1 for all n.

We choose identifications of L_n with $\mathbb{R} \cup \{\infty\}$ so that $v_n < v_n' < \iota(x) < w_n' < w_n$. Since y_n converges to x, for all sufficiently large n, we have $v_n' < \iota(y_n) < w_n'$. Then, Proposition 7.13 implies that for all n, we have

$$\log |[v_n', \iota(x), \iota(y_n), w_n']| \geq \lambda \cdot \log |[v_n, \iota(x), \iota(y_n), w_n]|$$

for some $\lambda > 1$ independent of n. But then since

$$C_{\Omega_1}(x, y_n) \geq \log |[\iota^*(\xi'), \iota^*(\eta'); \iota(x), \iota(y_n)]|,$$

we have $C_{\Omega_1}(x, y_n)/C_{\Omega_2}(x, y_n) \geq \lambda$ for all n, contradicting (9) above. □

Proof of Proposition 7.11. We fix an open set $\Omega_{1.5}$ such that $\overline{\Omega_1} \subset \Omega_{1.5}$ and $\overline{\Omega_{1.5}} \subset \Omega_2$. Since $C_{\Omega_1}(x,y) \geq C_{\Omega_{1.5}}(x,y)$ for all $x,y \in \Omega_1$, we just need to see that there is some $\lambda > 1$ so that

$$\frac{C_{\Omega_{1.5}}(x,y)}{C_{\Omega_2}(x,y)} \geq \lambda$$

for all $x,y \in \overline{\Omega_1}$. This follows from Lemma 7.17. ☐

7.3. Contracting paths are P^+-divergent.

Proposition 7.18. *Let $\mathcal{G}$ be a Γ-graph for a group $\Gamma \subset G$, and let $\{U_v\}$ be a $\mathcal{G}$-compatible system of open sets of G/P^+ with each U_v a proper domain.*

If α_n is a contracting $\mathcal{G}$-path, then the sequence

$$\gamma_n = \alpha_1 \cdots \alpha_n$$

is P^+-divergent with unique limit point ξ, where $\{\xi\} = \bigcap_{n=1}^{\infty} \gamma_n U_{n+1}$.

Proof. Consider the sequence of open sets

$$\gamma_n \cdot U_{v_{n+1}}.$$

Up to subsequence, $U_{v_{n+1}}$ is a fixed open set $U \subset G/P^+$. By assumption $\{\alpha_n\}$ is a contracting path, so $\gamma_n \cdot U_{v_{n+1}}$ converges to a singleton $\{\xi\}$. So, we apply Proposition 3.6. ☐

8. A weaker criterion for EGF representations

We have now developed enough tools to be able to prove our weaker characterization of EGF representations. We first prove a pair of lemmas.

Lemma 8.1. *Let $(\Gamma, \mathcal{H})$ be a relatively hyperbolic pair, let $\rho : \Gamma \to G$ be a representation, and let $P \subset G$ be a symmetric parabolic subgroup. Suppose there exists*

1. *a Γ-invariant closed set $\Lambda \subset G/P$ and a continuous equivariant surjective antipodal map $\phi : \Lambda \to \partial(\Gamma, \mathcal{H})$, and*
2. *a relative quasigeodesic automaton $\mathcal{G}$ and a $\mathcal{G}$-compatible system $\{U_v\}$ of open subsets of G/P, such that the U_v's cover Λ and each U_v is a proper domain intersecting Λ nontrivially.*

Then for every sequence $\gamma_n \in \Gamma$ which is unbounded in the coned-off Cayley graph $\mathrm{Cay}(\Gamma, S, \mathcal{P})$*, the sequence $\rho(\gamma_n)$ is P-divergent, and every P-limit point of $\rho(\gamma_n)$ lies in Λ.*

Proof. We will show that every subsequence of γ_n has a P-contracting subsequence, so take an arbitrary subsequence of γ_n. By Lemma 5.7, we may assume that for a bounded sequence $b_n \in \Gamma$, $\gamma_n b_n$ is the endpoint of a finite $\mathcal{G}$-path $\{\alpha_m^{(n)}\}_{m=1}^{M_n}$. Up to subsequence b_n is a constant b, independent of n.

Let $\{v_m^n\}$ be the vertex path associated to $\{\alpha_m^{(n)}\}$. Up to subsequence $v_{M_n+1}^n$ is a fixed vertex v, and v_1^n is a fixed vertex v'. Let $U_{v'}^{\varepsilon}$ be an ε-neighborhood of $U_{v'}$, with ε chosen sufficiently small so that $U_{v'}^{\varepsilon}$ is still a proper domain.

The sequence M_n must be unbounded, since the length of γ_n with respect to the coned-off Cayley graph metric is at most a fixed constant times M_n. Corollary 7.12 then implies that the diameter of

$$\rho(\gamma_n b) \cdot U_v = \rho(\alpha_1^{(n)}) \cdots \rho(\alpha_{M_n}^{(n)}) U_v$$

with respect to the metric $C_{U_{v'}^{\varepsilon}}$ tends to zero, exponentially in n. Since this sequence of sets lies in the compact set $\overline{U_{v'}} \subset U_{v'}^{\varepsilon}$, up to subsequence it must converge to a singleton $\{\xi\}$ in G/P. In fact ξ must lie in Λ, because Λ is compact and ξ is the limit of a sequence of points in the sequence of nonempty closed sets $(\rho(\gamma_n b)\cdot\overline{U_v})\cap\Lambda$. Then, since $\rho(\gamma_n)\cdot\rho(b)U_v$ converges to $\{\xi\}$, Proposition 3.6 implies that $\rho(\gamma_n)$ is P-divergent with unique P-limit ξ. □

Lemma 8.2. *Let $(\Gamma, \mathcal{H})$ be a relatively hyperbolic pair, let $\rho : \Gamma \to G$ be a representation, let $\Lambda \subset G/P$ be a closed Γ-invariant set, and let $\phi : \Lambda \to \partial(\Gamma, \mathcal{H})$ be a continuous equivariant surjective antipodal map.*

Suppose that $\gamma_n \in \Gamma$ is a sequence converging to $z_+ \in \partial_{\mathrm{con}}(\Gamma, \mathcal{H})$, such that $\rho(\gamma_n)$ is P-divergent and every P-limit point of $\rho(\gamma_n^{\pm1})$ lies in Λ. If γ_n^{-1} converges to $z_- \in \partial(\Gamma, \mathcal{H})$, then for every compact set $K \subset \mathrm{Opp}(\phi^{-1}(z_-))$ and every open U containing $\phi^{-1}(z_+)$, for large enough n, we have $\rho(\gamma_n)K \subset U$.

Proof. It suffices to show that every subsequence of γ_n has a further subsequence satisfying the desired property. So, we can freely extract subsequences throughout this proof.

We assume P symmetric, so $\rho(\gamma_n^{-1})$ is also P-divergent and has nonempty P-limit set. Let $\xi_\pm$ be a pair of flags in the P-limit sets of $\rho(\gamma_n^{\pm1})$, respectively; by assumption we have $\xi_\pm \in \Lambda$. By Lemma 3.7, we have a subsequence so that $\rho(\gamma_n)$ converges to ξ_+ uniformly on compacts in $\mathrm{Opp}(\xi_-)$.

Antipodality of ϕ implies that every compact subset of $\partial(\Gamma, \mathcal{H}) - \{\phi(\xi_-)\}$ is contained in $\phi(\mathrm{Opp}(\xi_-) \cap \Lambda)$. Then, by equivariance and continuity of ϕ, we see that γ_n must converge to $\phi(\xi_+)$ on compacts in $\partial(\Gamma, \mathcal{H}) - \{\phi(\xi_-)\}$. This uniquely characterizes the points $\phi(\xi_\pm)$ as the limits of $\gamma_n^{\pm1}$ in $\partial(\Gamma, \mathcal{H})$. So, we see that $\rho(\gamma_n)$ converges uniformly to $\xi_+ \in \phi^{-1}(z_+)$ on every compact in $\mathrm{Opp}(\phi^{-1}(z_-)) \subset \mathrm{Opp}(\xi_-)$, as required. □

We recall the statement of our weaker characterization of EGF representations here:

Proposition 4.6. *Let $\rho : \Gamma \to G$ be a representation of a relatively hyperbolic group, and let $\Lambda \subset G/P$ be a closed $\rho(\Gamma)$-invariant set, where $P \subset G$ is a symmetric parabolic subgroup. Suppose that $\phi : \Lambda \to \partial(\Gamma, \mathcal{H})$ is a continuous surjective ρ-equivariant antipodal map.*

Then ρ is an EGF representation with EGF boundary extension ϕ if and only if both of the following conditions hold:

(a) For any sequence $\gamma_n \in \Gamma$ limiting conically to some point in $\partial(\Gamma, \mathcal{H})$, $\rho(\gamma_n^{\pm1})$ is P-divergent and every P-limit point of $\rho(\gamma_n^{\pm1})$ lies in Λ.

(b) For every parabolic point $p \in \partial_{\mathrm{par}}(\Gamma, \mathcal{H})$, there exists an open set $C_p \subset G/P$, with $\Lambda - \phi^{-1}(p) \subset C_p$, such that for any compact $K \subset C_p$ and any open set U containing $\phi^{-1}(p)$, for all but finitely many $\gamma \in \Gamma_p$, we have $\rho(\gamma) \cdot K \subset U$.

Proof. To see the "only if" part, observe that if we know that ϕ is an EGF boundary extension, we can use the results of Section 6 to construct an automaton satisfying the hypotheses of Lemma 8.1, which immediately implies that the first condition holds. The second condition is immediate from the fact that ϕ extends convergence dynamics.

So, we focus on the "if" part. For each conical limit point $z \in \partial_{\mathrm{con}}(\Gamma, \mathcal{H})$, we let $C_z = \mathrm{Opp}(\phi^{-1}(z))$. Each C_z contains $\Lambda - \phi^{-1}(z)$ by antipodality of ϕ. For each $p \in \partial_{\mathrm{par}}(\Gamma, \mathcal{H})$, we can replace C_p with $\mathrm{Opp}(\phi^{-1}(p)) \cap C_p$: this set is still open, and it again contains $\Lambda - \{\phi^{-1}(p)\}$ by antipodality.

Observe that if γ_n is a sequence limiting conically to $z_+ \in \partial(\Gamma, \mathcal{H})$, with γ_n^{-1} converging to z_-, then Lemma 8.2, together with part (b) of our hypotheses, implies that the map $\phi : \Lambda \to G/P$ satisfies both conditions (1) and (2) given at the beginning of Section 6. So, by Proposition 6.1, we know that there is a relative quasigeodesic automaton $\mathcal{G}$ satisfying the hypotheses of Lemma 8.1.

We now want to show that parts (a) and (b) of our hypotheses show that ϕ is an EGF boundary extension, so let $\gamma_n \in \Gamma$ be a sequence with $\gamma_n^{\pm1} \to z_\pm \in \partial(\Gamma, \mathcal{H})$. We fix an open set $U \subset G/P$ containing $\phi^{-1}(z_+)$ and a compact $K \subset C_{z_-}$. Our goal is to show that for large enough n, we have $\rho(\gamma_n)K \subset U$.

We consider two cases:

Case 1: γ_n is unbounded in the coned-off Cayley graph $\mathrm{Cay}(\Gamma, S, \mathcal{P})$. By Lemma 8.1, $\rho(\gamma_n^{\pm 1})$ is P-divergent, and every P-limit point of $\rho(\gamma_n^{\pm 1})$ lies in Λ. Then we are done by Lemma 8.2.

Case 2: γ_n is bounded in $\mathrm{Cay}(\Gamma, S, \mathcal{P})$. We can write γ_n as an alternating product

$$\gamma_n = g_1^{(n)} h_1^{(n)} \cdots g_k^{(n)} h_k^{(n)} g_{k+1}^{(n)},$$

where $g_i^{(n)}$ is bounded in Γ, and $h_i^{(n)}$ lies in $\Gamma_{p_i^n}$ for a parabolic point $p_i^n \in \Pi$. Without loss of generality, the $h_i^{(n)}$ are unbounded in Γ as $n \to \infty$. Up to subsequence we can assume that $g_i^{(n)} = g_i$ and $p_i^n = p_i$ (independent of n). Since Π contains exactly one representative of each parabolic orbit, we can also assume that $g_{i+1}p_{i+1} \neq p_i$ for any i.

We claim that γ_n converges to $z_+ = g_1 p_1$, γ_n^{-1} converges to $z_- = g_{k+1}^{-1} p_k$, and for any compact $K \subset C_{z_-}$ and open U containing $\phi^{-1}(z_+)$, for large n, we have $\gamma_n \cdot K \subset U$.

Fix such a compact K and open U. We will prove the claim by inducting on k. When $k = 1$, then $p = p_1 = p_k$, and $\gamma_n = g_1 h_n g_2$ for $h_n \in \Gamma_p$ and $g_1, g_2 \in \Gamma$ fixed. The distance between $h_n g_2$ and h_n is bounded in any word metric on Γ, so $h_n g_2$ converges to p in $\overline{\Gamma}$ and $g_1 h_n g_2$ converges to $g_1 p = z_+$. We also know that $K \subset C_{z_-} = C_{g_2^{-1}p}$, so $h_n g_2 K$ eventually lies in a small neighborhood of $\phi^{-1}(p)$ by part (b) of our hypotheses. Then $g_1 h_n g_2 K$ lies in any small neighborhood of $\phi^{-1}(g_1 p) = \phi^{-1}(z_+)$.

When $k > 1$, we consider the sequence

$$\gamma_n' = g_2 h_2^{(n)} \cdots g_k h_k^{(n)} g_{k+1}.$$

Inductively we can assume that for large n, $\gamma_n' \to g_2 p_2$ and $\rho(\gamma_n') \cdot K$ is a subset of an arbitrarily small neighborhood of $\phi^{-1}(g_2 p_2)$. Then since $p_1 \neq g_2 p_2$, for large enough n, $\rho(\gamma_n') \cdot K$ is a compact subset of C_{p_1}. So our hypotheses imply that for large n,

$$\rho(\gamma_n) \cdot K = \rho(g_1 h_1^{(n)}) \rho(\gamma_n') \cdot K \subset U.$$

□

The arguments above also imply the following characterization of EGF representations. This result is not needed anywhere else in the paper.

Proposition 8.3. *Let $(\Gamma, \mathcal{H})$ be a relatively hyperbolic pair, let $\rho : \Gamma \to G$ be a representation, and let $P \subset G$ be a symmetric parabolic subgroup. Suppose that there exists a closed Γ-invariant subset $\Lambda \subseteq G/P$ and a surjective equivariant antipodal map $\phi : \Lambda \to \partial(\Gamma, \mathcal{H})$.*

Then ρ is EGF with boundary extension ϕ if and only if for every $z \in \partial(\Gamma, \mathcal{H})$, there exists an open $C_z \subset G/P$ containing $\Lambda - \phi^{-1}(z)$, such that:

(a) For any sequence $\gamma_n \in \Gamma$ limiting conically to some point z in $\partial(\Gamma, \mathcal{H})$, with $\gamma_n^{-1} \to z_-$, any open set U containing $\phi^{-1}(z)$, and any compact $K \subset C_{z_-}$, we have $\rho(\gamma_n) \cdot K \subset U$ for all sufficiently large n.

(b) For any parabolic point $p \in \partial(\Gamma, \mathcal{H})$, any compact $K \subset C_p$, and any open set U containing $\phi^{-1}(p)$, for all but finitely many $\gamma \in \Gamma_p$, we have $\rho(\gamma) \cdot K \subset U$.

Proof. The "only if" direction is immediate, so suppose we have a representation satisfying the hypotheses above. The results of Section 6 imply that there is a relative quasigeodesic automaton satisfying the hypotheses of Lemma 8.1. We then apply this lemma together with Proposition 4.6 to obtain the desired result. □

9. Relative stability

In this section we prove the main *relative stability property* for EGF representations (Theorem 1.4).

9.1. Deformations of EGF representations.

In general, the set of EGF representations is not an open subset of $\mathrm{Hom}(\Gamma, G)$. However, it is relatively open in a subspace of $\mathrm{Hom}(\Gamma, G)$ where we restrict the deformations of the peripheral subgroups appropriately. Roughly speaking, we want to consider subspaces $\mathcal{W} \subset \mathrm{Hom}(\Gamma, G)$ where the *dynamical* behavior of the peripheral subgroups changes continuously under deformation. That is, if ρ_t is a small deformation of a representation ρ_0, where $\rho_0(\Gamma_p)$ attracts points towards Λ_p at a particular "speed," then we want $\rho_t(\Gamma_p)$ to attract points towards a small deformation of Λ_p at a similar "speed."

The precise condition is the following:

Definition 9.1. Let $\rho_0 : \Gamma \to G$ be an EGF representation with boundary extension $\phi : \Lambda \to \partial(\Gamma, \mathcal{H})$, and let $\mathcal{W} \subset \mathrm{Hom}(\Gamma, G)$ contain ρ_0.

We say that $\mathcal{W}$ is *peripherally stable at* (ρ_0, ϕ) if for every $p \in \partial_{\mathrm{par}}(\Gamma, \mathcal{H})$, every open set U containing $\phi^{-1}(p)$, every compact set $K \subset C_p$, and every cofinite set $T \subset \Gamma_p$ such that

$$\rho_0(T) \cdot K \subset U,$$

there is an open set $\mathcal{W}' \subset \mathcal{W}$ containing ρ_0, such that for every $\rho' \in \mathcal{W}'$, we have

$$\rho'(T) \cdot K \subset U.$$

We restate the main result of the paper below:

Theorem 1.4. *Let $\rho : \Gamma \to G$ be EGF with respect to P, let $\phi : \Lambda \to \partial(\Gamma, \mathcal{H})$ be a boundary extension, and let $\mathcal{W} \subseteq \mathrm{Hom}(\Gamma, G)$ be peripherally stable at (ρ, ϕ). For any compact subset Z of $\partial(\Gamma, \mathcal{H})$ and any open set $V \subset G/P$ containing $\phi^{-1}(Z)$, there is an open subset $\mathcal{W}' \subset \mathcal{W}$ containing ρ such that each $\rho' \in \mathcal{W}'$ is EGF with respect to P, and has an EGF boundary extension ϕ' satisfying $\phi'^{-1}(Z) \subset V$.*

Remark 9.2. In [Bow98], Bowditch explored the deformation spaces of geometrically finite groups $\Gamma \subset \mathrm{PO}(d, 1)$, and gave an explicit discription of semialgebraic subspaces of $\mathrm{Hom}(\Gamma, \mathrm{PO}(d, 1))$ in which small deformations of Γ are still geometrically finite.

Bowditch's deformation spaces are peripherally stable, so it seems desirable to find a general algebraic description of peripherally stable subspaces.

Even in $\mathrm{PO}(d, 1)$, the question is subtle, however. Bowditch also gives examples of geometrically finite representations $\rho : \Gamma \to \mathrm{PO}(d, 1)$ (for $d \geq 4$) and deformations ρ_t of ρ in $\mathrm{Hom}(\Gamma, \mathrm{PO}(d, 1))$ such that the restriction of ρ_t to each cusp group in Γ is discrete, faithful, and parabolic, but ρ_t is not even discrete; further examples exist where the deformation is discrete, but not geometrically finite.

Example 9.3. Let $B \in \mathrm{SL}(d, \mathbb{R})$ be a d-dimensional Jordan block with eigenvalue 1 and eigenvector v, and let $A \in \mathrm{SL}(d + 2, \mathbb{R})$ be the block matrix $\begin{pmatrix} B & & \\ & 1 & \\ & & 1 \end{pmatrix}$.

Although $[v]$ is not quite an attracting fixed point of A, it is still an "attracting subspace" in the sense that if K is any compact subset of $\mathbb{RP}^{d+1}$ which does not intersect a fixed hyperplane of A, then $A^n \cdot K$ converges to $\{[v]\}$. Via a ping-pong argument, one can use this "attracting" behavior to show that for some $k \geq 1$ and some $M \in \mathrm{SL}(d + 2, \mathbb{R})$, the group Γ generated by $\alpha = A^k$ and $\beta = MA^kM^{-1}$ is a discrete free group with free generators α, β. The group Γ is hyperbolic relative to the subgroups $\langle\alpha\rangle$, $\langle\beta\rangle$, and the inclusion $\Gamma \hookrightarrow \mathrm{SL}(d+2,\mathbb{R})$ is EGF with respect to $P_{1,d+1}$ (the stabilizer of a line in a hyperplane in $\mathbb{R}^{d+2}$).

Here, there are peripherally stable deformations of Γ which change the Jordan block decomposition of A. For instance, consider a continuous path $A_t : [0,1] \to \mathrm{SL}(d+2,\mathbb{R})$ given by $A_t = \begin{pmatrix} B_t & & \\ & 1 & \\ & & 1 \end{pmatrix}$, where $B_0 = B$ and B_t is a *diagonalizable* matrix in $\mathrm{SL}(d,\mathbb{R})$. For small values of t, the group Γ_t generated by $\alpha_t = A_t^k$ and β is still discrete and freely generated by α_t and β—since the "attracting" fixed points of A_t deform continuously with t, the same exact ping-pong setup works for all small $t \geq 0$. And indeed the path in $\mathrm{Hom}(\Gamma, \mathrm{SL}(d+2,\mathbb{R}))$ determined by the path A_t is a peripherally stable subspace.

On the other hand, consider the path $A_t' = \begin{pmatrix} B & & \\ & e^t & \\ & & e^{-t} \end{pmatrix}$, and let $\alpha_t' = A_t'^k$. In this case the corresponding subspace of $\mathrm{Hom}(\Gamma, \mathrm{SL}(d+2,\mathbb{R}))$ is *not* peripherally stable: while the group generated by α_t' is still discrete, the attracting fixed points of A_t' do *not* deform continuously in t. So, there is no way to use the ping-pong setup for Γ to ensure that $\Gamma_t' = \langle \alpha_t', \beta \rangle$ is a discrete group.

Example 9.4. Here is a somewhat more interesting example of a *non*-peripherally stable deformation. Let M be a finite-volume noncompact hyperbolic 3-manifold, with holonomy representation $\rho : \pi_1 M \to \mathrm{PSL}(2,\mathbb{C})$ (so there is an identification $M = \mathbb{H}^3/\rho(\pi_1 M)$). Then $\pi_1 M$ is hyperbolic relative to the collection $\mathcal{C}$ of conjugates of cusp groups (each of which is isomorphic to $\mathbb{Z}^2$), and the representation ρ is geometrically finite (in particular, EGF).

In this case, for *any* sufficiently small nontrivial deformation ρ' of ρ in the character variety $\mathrm{Hom}(\pi_1 M, \mathrm{PSL}(2,\mathbb{C}))/\mathrm{PSL}(2,\mathbb{C})$, the restriction of ρ' to some cusp group $C \in \mathcal{C}$ either fails to be discrete or has infinite kernel. So $\mathrm{Hom}(\pi_1 M, \mathrm{PSL}(2,\mathbb{C}))$ is *not* peripherally stable, because any sufficiently small deformation of ρ inside of a peripherally stable subspace must have discrete image and finite kernel on each $C \in \mathcal{C}$. This is true despite the fact that arbitrarily small deformations of ρ are holonomy representations of complete hyperbolic structures on Dehn fillings of M (so in particular, they are discrete).

The main ingredient in the proof of Theorem 1.4 is the relative quasigeodesic automaton $\mathcal{G}$ and the associated $\mathcal{G}$-compatible system of open sets $\{U_v\}$ we constructed in Proposition 6.1. The following proposition is immediate from the definition of peripheral stability:

Proposition 9.5. *Let $\rho : \Gamma \to G$ be an EGF representation with boundary extension ϕ, and let $\mathcal{W} \subset \mathrm{Hom}(\Gamma, G)$ be a subspace which is peripherally stable at (ρ, ϕ).*

If $\mathcal{G}$ is a relative quasigeodesic automaton for Γ, and $\{U_v\}$ is a $\mathcal{G}$-compatible system of open subsets of G/P for $\rho(\Gamma)$, then there is an open subset $\mathcal{W}' \subset \mathcal{W}$ containing ρ such that for every $\rho' \in \mathcal{W}'$, $\{U_v\}$ is also a $\mathcal{G}$-compatible system of open sets for $\rho'(\Gamma)$.

Theorem 1.4 then follows from a kind of converse to Proposition 6.1: we will show that we can reconstruct a map extending the convergence dynamics of Γ from the $\mathcal{G}$-compatible system $\{U_v\}$.

9.2. **An equivariant map on conical limit points.**

For the rest of this section, we let $\rho : \Gamma \to G$ be a representation which is EGF with respect to a symmetric parabolic subgroup $P \subset G$. We let $\phi : \Lambda \to \partial(\Gamma, \mathcal{H})$ be a boundary extension for ρ, and assume that $\mathcal{W} \subset \mathrm{Hom}(\Gamma, G)$ is peripherally stable at (ρ, ϕ). We also let Z be a compact subset of $\partial(\Gamma, \mathcal{H})$, and let $V \subset G/P$ be an open subset containing $\phi^{-1}(Z)$. We again fix a finite subset $\Pi \subset \partial_{\mathrm{par}}(\Gamma, \mathcal{H})$, containing one point from every Γ-orbit in $\partial_{\mathrm{par}}(\Gamma, \mathcal{H})$.

Using Proposition 6.1, we can find a relative quasigeodesic automaton $\mathcal{G}$ and $\mathcal{G}$-compatible system $\{U_v\}$ of open subsets of G/P for $\rho(\Gamma)$. Using Proposition 6.14, we can ensure that for any $z \in Z$, there is a $\mathcal{G}$-path $\{\alpha_n\}$ limiting to z (with vertex path $\{v_n\}$) so that U_{v_1} is contained in V.

For each $p \in \Pi$, we also fix a compact set $K_p \subset \partial(\Gamma, \mathcal{H}) - \{p\}$ such that $\Gamma_p \cdot K_p = \partial(\Gamma, \mathcal{H}) - \{p\}$, and assume that the automaton $\mathcal{G}$ has been constructed to satisfy Proposition 6.15.

Antipodality of the map ϕ implies that for each $z \in \partial(\Gamma, \mathcal{H})$, each fiber $\phi^{-1}(z)$ is a closed subset of some affine chart in G/P. So, we can also assume that U_v is a proper domain for each vertex v of $\mathcal{G}$. In fact, by way of the following lemma, we can assume even more:

Lemma 9.6. *Let ρ be an EGF representation with boundary map $\phi : \Lambda \to \partial(\Gamma, \mathcal{H})$.*

For any $\delta > 0$, we can find a relative quasigeodesic automaton $\mathcal{G}$ with $\mathcal{G}$-compatible system $\{U_v\}$ of open sets in G/P as in Proposition 6.1, so that for any $x, y \in \partial(\Gamma, \mathcal{H})$ with $d(x, y) > \delta$, if $\phi^{-1}(x) \subset U_v$ and $\phi^{-1}(y) \subset U_w$, then $\overline{U_v}$ and $\overline{U_w}$ are opposite.

Proof. We choose $\varepsilon > 0$ so that if $d(v, w) > \delta/2$ for $v, w \in \partial(\Gamma, \mathcal{H})$, then the closed ε-neighborhoods of

$$\phi^{-1}(v), \qquad \phi^{-1}(w)$$

are opposite. This is possible for a fixed pair $v, w \in \partial(\Gamma, \mathcal{H})$ since antipodality is an open condition, and $\phi^{-1}(v)$, $\phi^{-1}(w)$ are opposite compact sets. Then we can pick a uniform ε for all pairs since the the subset $\{(u, v) \in (\partial(\Gamma, \mathcal{H}))^2 : d(u, v) > \delta/2\}$ is compact.

Consider $\mathcal{G}$-compatible systems of open subsets $\{U_v\}$ and $\{W_v\}$ for the action of Γ on G/P and $\partial(\Gamma, \mathcal{H})$, coming from Proposition 6.1. We can ensure that for each vertex a, the diameter of W_a is at most $\delta/4$, and $U_a \subset N(\phi^{-1}(w), \varepsilon)$ for some $w \in W_a$.

If $x, y \in \partial(\Gamma, \mathcal{H})$ satisfy $d(x, y) > \delta$, and $x \in W_a$, $y \in W_b$, we have

$$d(v, w) > \delta/2$$

for all $v \in W_a$, $w \in W_b$.

Then, if $\phi^{-1}(x) \subset U_a$ and $\phi^{-1}(y) \subset U_b$, we have

$$U_a \subset N(\phi^{-1}(v), \varepsilon), \qquad U_b \subset N(\phi^{-1}(w), \varepsilon)$$

for $v \in W_a$, $w \in W_b$ with $d(v, w) > \delta/2$. By our choice of ε, the closures of $N(\phi^{-1}(w), \varepsilon)$ and $N(\phi^{-1}(v), \varepsilon)$ are opposite. ☐

Using cocompactness of the action of Γ on the space of distinct pairs in $\partial(\Gamma, \mathcal{H})$, we know that there exists some fixed $\delta > 0$ such that for any distinct $z_1, z_2 \in \partial(\Gamma, \mathcal{H})$, we can find some $\gamma \in \Gamma$ such that $d(\gamma z_1, \gamma z_2) > \delta$. Then, in light of Lemma 9.6, we can make the following assumption:

Assumption 9.7. For any $z_1, z_2 \in \partial(\Gamma, \mathcal{H})$ satisfying $d(z_1, z_2) > \delta$, if $\phi^{-1}(z_1) \subset U_v$ and $\phi^{-1}(z_2) \subset U_w$ for v, w vertices of $\mathcal{G}$, then $\overline{U_v}$ and $\overline{U_w}$ are opposite.

With our relative quasigeodesic automaton $\mathcal{G}$ and compatible system of open sets $\{U_v\}$ fixed, we now choose an open subset $\mathcal{W}' \subset \mathcal{W}$ so that for any $\rho' \in \mathcal{W}'$, $\{U_v\}$ is also a $\mathcal{G}$-compatible system for the action of $\rho'(\Gamma)$ on G/P. Our main goal for the rest of this section is to show that any $\rho' \in \mathcal{W}'$ is an EGF representation. So, we fix some $\rho' \in \mathcal{W}'$.

Let $\mathrm{Path}(\mathcal{G})$ denote the set of infinite $\mathcal{G}$-paths. Proposition 7.3 implies that every path in $\mathrm{Path}(\mathcal{G})$ is contracting for the ρ'-action, so we have a map

$$\psi_{\rho'} : \mathrm{Path}(\mathcal{G}) \to G/P,$$

where the path $\{\alpha_n\}$ maps to the unique element of

$$\bigcap_{n=1}^{\infty} \rho'(\alpha_1)\cdots\rho'(\alpha_n)U_{v_{n+1}}.$$

Lemma 9.8. *The map* $\psi_{\rho'} : \mathrm{Path}(\mathcal{G}) \to G/P$ *induces an equivariant map*

$$\psi_{\rho'} : \partial_{\mathrm{con}}\Gamma \to G/P.$$

Proof. We first need to see that $\psi_{\rho'}$ is well-defined, i.e. that if z is a conical limit point and $\{\alpha_n\}$, $\{\beta_n\}$ are $\mathcal{G}$-paths limiting to z, then $\psi_{\rho'}(\{\alpha_n\}) = \psi_{\rho'}(\{\beta_n\})$.

Let

$$\gamma_n = \alpha_1 \cdots \alpha_n, \qquad \eta_m = \beta_1 \cdots \beta_m.$$

We can use Proposition 5.11 to see that γ_n and η_m lie within bounded Hausdorff distance of a geodesic in $\mathrm{Cay}(\Gamma, S, \mathcal{P})$ limiting to z, so there is a fixed D so that for infinitely many pairs m, n,

$$d(\gamma_n, \eta_m) < D$$

in the Cayley graph of Γ. Proposition 7.18 implies that $\rho'(\gamma_n)$ and $\rho'(\eta_n)$ are both P-divergent sequences and each have a unique P-limit point in G/P, given by $\psi_{\rho'}(\{\alpha_n\})$, $\psi_{\rho'}(\{\beta_m\})$, respectively. Then, Lemma 4.23 in [KLP17] implies that because $\rho'(\gamma_n) = \rho'(\eta_n)g_n$ for a bounded sequence $g_n \in G$, the P-limit points of $\rho'(\gamma_n)$ and $\rho'(\eta_n)$ must agree and therefore $\psi_{\rho'}(\{\alpha_n\}) = \psi_{\rho'}(\{\beta_m\})$.

Next we observe that $\psi_{\rho'}$ is equivariant. Fix a finite generating set S for Γ. It suffices to show that $\psi_{\rho'}(s \cdot z) = \rho'(s) \cdot \psi_{\rho'}(z)$ for all $s \in S$.

Let $\{\alpha_n\}$ be a $\mathcal{G}$-path limiting to some $z \in \partial_{\mathrm{con}}\Gamma$, and consider the sequence

$$\gamma'_n = s\alpha_1 \cdots \alpha_n.$$

Again, Proposition 5.11 implies that γ'_n lies bounded Hausdorff distance from a geodesic in $\mathrm{Cay}(\Gamma, S, \mathcal{P})$, which must limit to $s \cdot z$. So if we fix a $\mathcal{G}$-path β_n limiting to $s \cdot z$, the same argument as above shows that $\psi_{\rho'}(\{\beta_n\}) = \rho'(s) \cdot \psi_{\rho'}(\{\alpha_n\})$. ☐

It will turn out that $\psi_{\rho'}$ is also both continuous and injective. However, we do not prove this directly.

9.3. **Extending $\psi_{\rho'}$ to parabolic points.** We want to extend the map $\psi_{\rho'} : \partial_{\mathrm{con}}\Gamma \to G/P$ to the entire Bowditch boundary $\partial(\Gamma, \mathcal{H})$. To do so, we need to view $\psi_{\rho'}$ as a map to the set of closed subsets of G/P.

The first step is to define $\psi_{\rho'}$ on the finite set $\Pi \subset \partial_{\mathrm{par}}\Gamma$. For any vertex v in $\mathcal{G}$, we consider the set

$$B_v = \bigcup_{(v,y) \text{ edge in } \mathcal{G}} U_y.$$

Then, for each $p \in \Pi$, we pick a parabolic vertex w so that $p_w = p$. We define Λ'_p to be the closure of the set of accumulation points of sequences of the form $\rho'(\gamma_n) \cdot x$, for $x \in B_w$ and γ_n distinct elements of Γ_p. Part (3) of Proposition 6.1 guarantees that $B_w \subset C_p$, and $\mathcal{G}$-compatiblity of the system $\{U_v\}$ for the $\rho'(\Gamma)$-action on G/P implies that $\Lambda'_p \subset U_w$. By construction, Λ'_p is invariant under the action of $\rho(\Gamma_p)$.

Next, given a parabolic point $q \in \partial_{\mathrm{par}}\Gamma$, we write $q = g \cdot p$ for $p \in \Pi$, and then define

$$\psi_{\rho'}(q) := \rho'(g)\Lambda'_p.$$

Since Λ'_p is Γ_p-invariant and Γ_p is exactly the stabilizer of p, this does not depend on the choice of coset representative in $g\Gamma_p$. Moreover $\psi_{\rho'}$ is still ρ'-equivariant.

In addition, if v is any parabolic vertex with parabolic point $p_v = g \cdot p$ for $p \in \Pi$, part (2) of Proposition 6.1 ensures that $B_v = B_w$ for any parabolic vertex w with $p_w = p$. So, $\rho'(g) \cdot \Lambda'_p$ is exactly the closure of the set of accumulation points of the form $\rho'(g\gamma_n) \cdot x$ for sequences $\gamma_n \in \Gamma_p$ and $x \in B_v$. Then $\mathcal{G}$-compatibility implies that $\psi_{\rho'}(p_v) = \rho(g)\Lambda'_p$ is a subset of U_v.

Remark 9.9. There is a natural topology on the space of closed subsets of G/P, induced by the Hausdorff distance arising from some (any) choice of metric on G/P. We emphasize that the map $\psi_{\rho'}$ is *not* necessarily continuous with respect to this topology.

Ultimately we want to use $\psi_{\rho'}$ to define a map extending the convergence dynamics of Γ, so we will need to also define the sets C'_z for each $z \in \partial(\Gamma, \mathcal{H})$. For now, we only define C'_p for $p \in \Pi$: this will be the set

$$\bigcup_{\gamma \in \Gamma_p} \rho'(\gamma) B_w.$$

We can immediately observe:

Proposition 9.10. *C'_p is $\rho'(\Gamma_p)$-invariant. Moreover, for any infinite sequence $\gamma_n \in \Gamma_p$, any compact $K \subset C'_p$, and any open $U \subset G/P$ containing Λ'_p, for sufficiently large n, $\rho'(\gamma_n) \cdot K$ lies in U.*

Proof. Γ_p-invariance follows directly from the definition.

Fix a compact $K \subset C'_p$ and an open $U \subset G/P$ containing Λ'_p. K is contained in finitely many sets $\rho'(\gamma)B_w$ for $\gamma \in \Gamma_p$, so any accumulation point of $\rho'(\gamma_n)x$ for $x \in K$ and $\gamma_n \in \Gamma_p$ lies in Λ'_p. In particular, for sufficiently large n, $\gamma_n x$ lies in U, and since K is compact we can pick n large enough so that $\gamma_n x \in U$ for all $x \in K$. ☐

We next want to use $\psi_{\rho'}$ to define an antipodal extension from a subset of G/P to $\partial(\Gamma, \mathcal{H})$.

Lemma 9.11. *For any $z \in \partial(\Gamma, \mathcal{H})$, if $\{\alpha_n\}$ is a $\mathcal{G}$-path limiting to z with corresponding vertex path $\{v_n\}$, then $\phi^{-1}(z)$ and $\psi_{\rho'}(z)$ are both subsets of U_{v_1}.*

Proof. If z is a conical limit point, then this follows immediately from Proposition 5.11 and the definition of $\psi_{\rho'}$. On the other hand, if z is a parabolic point, then $z = \alpha_1 \cdots \alpha_N p_v$, where v is a parabolic vertex at the end of the vertex path $\{v_n\}$. By part (3) of Proposition 6.1, we have $p_v \in W_v$ and thus $\phi^{-1}(p_v) \subset U_v$. By ρ-equivariance of ϕ we have

$$\phi^{-1}(z) = \rho(\alpha_1 \cdots \alpha_N)\phi^{-1}(p_v),$$

so by $\mathcal{G}$-compatibility we have $\phi^{-1}(z) \subset U_{v_1}$. On the other hand, we have constructed $\psi_{\rho'}$ so that $\psi_{\rho'}(p_v) \subset U_v$, so ρ'-equivariance of $\psi_{\rho'}$ and $\mathcal{G}$-compatibility also show that $\psi_{\rho'}(z) \subset U_{v_1}$. ☐

Lemma 9.12. *For any two distinct points z_1, z_2 in $\partial(\Gamma, \mathcal{H})$, the sets*

$$\psi_{\rho'}(z_1), \qquad \psi_{\rho'}(z_2)$$

are opposite (in particular disjoint).

Proof. We know that for any distinct $z_1, z_2 > 0$, we can find $\gamma \in \Gamma$ so that $d(\gamma z_1, \gamma z_2) > \delta$. So, since $\psi_{\rho'}$ is ρ'-equivariant, we just need to show that if $z_1, z_2 \in \partial(\Gamma, \mathcal{H})$ satisfy $d(z_1, z_2) > \delta$, then $\psi_{\rho'}(z_1)$ is opposite to $\psi_{\rho'}(z_2)$.

Let $\{\alpha_n\}$, $\{\beta_n\}$ be $\mathcal{G}$-paths respectively limiting to points $z_1, z_2 \in \partial(\Gamma, \mathcal{H})$ with $d(z_1, z_2) > \delta$, with corresponding vertex paths $\{v_n\}$ and $\{w_n\}$. By Lemma 9.11, we must have $\phi^{-1}(z_1) \subset U_{v_1}$ and $\phi^{-1}(z_2) \subset U_{w_2}$, so under Assumption 9.7, we know that U_{v_1} and U_{w_1} are opposite.

But then we are done since Lemma 9.11 also implies that $\psi_{\rho'}(z_1) \subset U_{v_1}$ and $\psi_{\rho'}(z_2) \subset U_{w_1}$. ☐

9.4. **The boundary set of the deformed representation.** We define our candidate boundary set $\Lambda' \subset G/P$ by

$$\Lambda' = \bigcup_{z \in \partial(\Gamma, \mathcal{H})} \psi_{\rho'}(z).$$

We then have an equivariant map

$$\phi' : \Lambda' \to \partial(\Gamma, \mathcal{H}),$$

where $\phi'(x) = z$ if $x \in \psi_{\rho'}(z)$. Lemma 9.12 implies that ϕ' is well-defined and antipodal. It is necessarily both surjective and ρ'-equivariant, and its fibers are either singletons or translates of the sets Λ'_p for $p \in \Pi$.

It now remains to verify the properties of the candidate set Λ' and the map ϕ' needed to show that ϕ' is an EGF boundary extension.

Lemma 9.13. *For every vertex v of $\mathcal{G}$, the intersection $\Lambda' \cap U_v$ is nonempty.*

Proof. The construction in Section 6 ensures that every vertex of the automaton $\mathcal{G}$ has at least one outgoing edge. In particular this means that for a given vertex v, there is an infinite $\mathcal{G}$-path whose first vertex is v. This $\mathcal{G}$-path limits to a conical limit point z, and Lemma 9.11 implies that $\psi_{\rho'}(z)$ is a (nonempty) subset of both U_v and Λ'. ☐

Lemma 9.14. *For any $z \in Z$, we have $\phi'^{-1}(z) \subset V$.*

Proof. Recall that we used Proposition 6.14 to construct our automaton so that for any $z \in Z$, there is a $\mathcal{G}$-path limiting to z with vertex path $\{v_n\}$ such that $U_{v_1} \subset V$. Then Lemma 9.11 implies $\phi'^{-1}(z) \subset V$. ☐

Lemma 9.15. *Λ' is compact.*

Proof. Fix a sequence $y_n \in \Lambda'$, and let $x_n = \phi'(y_n)$. Since $\partial(\Gamma, \mathcal{H})$ is compact, up to subsequence $x_n \to x$. We want to see that a subsequence of y_n converges to some $y \in \Lambda'$. We consider two possibilities:

Case 1: x is a parabolic point. We can write $x = g \cdot p$, where $p \in \Pi$. Let w be a parabolic vertex with $p_w = p$, and consider the compact set $K_p \subset \partial(\Gamma, \mathcal{H}) - \{p\}$, chosen so that $\Gamma_p \cdot K = \partial(\Gamma, \mathcal{H}) - \{p\}$. If $x_n = q$ for infinitely many n, we are done, so assume otherwise, and choose $\gamma_n \in \Gamma_p$ so that $z_n = \gamma_n^{-1} g^{-1} x_n \in K_p$.

We have assumed (using Proposition 6.15) that the automaton $\mathcal{G}$ has been constructed so that there is always a $\mathcal{G}$-path limiting to z_n whose first vertex v_n is connected to w by an edge (w, v_n). Lemma 9.11 implies that $\phi'^{-1}(z_n)$ lies in U_n, which is contained in C'_p by definition.

Then using Proposition 9.10, we know that up to subsequence,

$$\rho'(\gamma_n)\phi'^{-1}(z_n) = \rho'(\gamma_n)\phi'^{-1}(\gamma_n^{-1} g^{-1} x_n)$$

converges to a compact subset of Λ'_p, which means that

$$y_n \in \rho'(g)\rho'(\gamma_n)\phi'^{-1}(\gamma_n^{-1} g^{-1} x_n)$$

subconverges to a point in $\rho'(g)\Lambda'_p$.

Case 2: x is a conical limit point. We want to show that any sequence in $\phi'^{-1}(x_n)$ limits to $\phi'^{-1}(x)$, so fix any $\varepsilon > 0$. Using Corollary 7.12, we can choose N so that if $\{\alpha_m\}$ is any $\mathcal{G}$-path limiting to x, with corresponding vertex path $\{v_m\}$, then the diameter of

$$\rho'(\alpha_1 \cdots \alpha_N)U_{v_{N+1}}$$

is less than ε with respect to a metric on U_{v_1}. We fix such a $\mathcal{G}$-path $\{\alpha_m\}$. Then, we use Lemma 5.15 to see that for sufficiently large n, there is a $\mathcal{G}$-path $\{\beta_m^n\}$ limiting to x_n with $\beta_i = \alpha_i$ for $i \leq N$. Thus the Hausdorff distance (with respect to $C_{U_{v_1}}$) between $\phi'^{-1}(x_n)$ and $\phi'^{-1}(x)$ is at most ε. Since $\phi'^{-1}(x_n)$ and $\phi'^{-1}(x)$ both lie in the compact set $\rho'(\alpha_1)\overline{U_{v_2}} \subset U_{v_1}$, this proves the claim.

□

Lemma 9.16. *ϕ' is continuous and proper.*

Proof. Since Λ' is compact, we just need to show continuity. Fix $y \in \Lambda'$ and a sequence $y_n \in \Lambda'$ approaching y. We want to show that $\phi'(y_n)$ approaches $\phi'(y) = x$.

Suppose otherwise. Since $\partial(\Gamma, \mathcal{H})$ is compact, up to a subsequence $z_n = \phi'(y_n)$ approaches $z \neq x$. Using the equivariance of ϕ', and cocompactness of the Γ-action on distinct pairs in $\partial(\Gamma, \mathcal{H})$, we may assume that $d(x, z) > \delta$. For sufficiently large n, we have $d(x, z_n) > \delta$ as well. Then, as in the proof of Lemma 9.12, by Assumption 9.7 we know that for any vertices v, w in $\mathcal{G}$ such that U_v contains $\psi_{\rho'}(x)$ and U_w contains $\psi_{\rho'}(z_n)$, the intersection $\overline{U_v} \cap \overline{U_w}$ is empty.

But by definition of ϕ', we have

$$y \in \psi_{\rho'}(x) \subset U_v, \qquad y_n \in \psi_{\rho'}(z_n) \subset U_w$$

for vertices v, w in $\mathcal{G}$. This contradicts the fact that $y_n \to y$. □

9.5. **Dynamics on the deformation.** To complete the proof of Theorem 1.4, we just need to show:

Proposition 9.17. *The map ϕ' extends the convergence group action of Γ.*

Proof. We will apply Proposition 4.6. The preceding arguments show that the relative quasi-geodesic automaton $\mathcal{G}$, the map $\phi' : \Lambda' \to \partial(\Gamma, \mathcal{H})$, and the $\mathcal{G}$-compatible system $\{U_v\}$ satisfy the hypotheses of Lemma 8.1. This immediately implies that the first condition of Proposition 4.6 is satisfied.

To see that the second condition is also satisfied, let q be a parabolic point in $\partial(\Gamma, \mathcal{H})$, and write $q = g \cdot p$ for $p \in \Pi$, and then take $C'_q = \rho'(g) \cdot C'_p$. Proposition 9.10 says that for any $p \in \Pi$, any compact $K \subset C'_p$, and any open $U \subset G/P$ containing Λ'_p, if γ_n is an infinite sequence in Γ_p, then $\rho'(\gamma_n) \cdot K \subset U$ for sufficiently large n. Then, since $\Gamma_q = g\Gamma_p g^{-1}$, the same is true for any parabolic point q.

So, we just need to check that for each $p \in \Pi$, C'_p contains $\Lambda' - \Lambda'_p$. We consider the compact set $K_p \subset \partial(\Gamma, \mathcal{H}) - \{p\}$ satisfying $\Gamma_p K_p = \partial(\Gamma, \mathcal{H}) - \{p\}$. We observed in the proof of Lemma 9.15 that C'_p contains $\phi'^{-1}(K_p)$. But then since C'_p is $\rho'(\Gamma_p)$-invariant (by Proposition 9.10), we have

$$\rho'(\Gamma_p) \cdot \phi'^{-1}(K_p) = \phi'^{-1}(\partial(\Gamma, \mathcal{H}) - \{p\}) \subset C'_p.$$

□

Remark 9.18. The definition of the set Λ' and the map ϕ' immediately imply that the *fibers* of the deformed boundary extension $\phi' : \Lambda' \to \partial(\Gamma, \mathcal{H})$ satisfy the conclusions of Proposition 4.8: the fiber over each conical limit point is a singleton, and the fiber over each parabolic point p is the closure of the accumulation sets of Γ_p-orbits in C'_p. So, we obtain Proposition 4.8 by taking $\mathcal{W}$ to be the singleton $\{\rho\}$, and following the proof of Theorem 1.4 (using C_p for C'_p throughout).

Appendix A. Contraction dynamics on flag manifolds

Let V be a real vector space, and let A_n be a sequence of elements of $\mathrm{PGL}(V)$. It is sometimes possible to determine the global dynamical behavior of A_n on $\mathbb{P}(V)$ by considering the action of A_n on a small open subset of $\mathbb{P}(V)$: if there is an open subset $U \subset \mathbb{P}(V)$ such that $A_n \cdot U$ converges to a point in $\mathbb{P}(V)$, then in fact there is a dense open subset $U_- \subset \mathbb{P}(V)$ (the complement of a hyperplane) on which A_n converges to the same point, uniformly on compacts.

A similar statement holds for the action of A_n on Grassmannians $\mathrm{Gr}(k, V)$. These claims can be proved by considering the behavior of the singular value gaps of A_n as $n \to \infty$.

In this appendix we give a general result along these lines, where we take sequences of group elements $g_n \in G$ for a semisimple Lie group G with no compact factor and trivial center, and consider the limiting behavior of g_n on open subsets of some flag manifold G/P^+, where P^+ is a parabolic subgroup.

Proposition 3.6. *Let g_n be a sequence in G, and suppose that for some nonempty open subset $U \subset G/P^+$, we have $g_n \cdot U \to \{\xi\}$ for $\xi \in G/P^+$. Then g_n is P^+-divergent, and has a unique P^+-limit point $\xi \in G/P^+$.*

We will prove Proposition 3.6 by reducing it to the case where $G = \mathrm{PGL}(d, \mathbb{R})$ and $P^+ = P_1$ is the stabilizer of $[e_1] \in \mathbb{RP}^{d-1} \simeq G/P_1$. In this situation, P^+-divergence can be understood in terms of the behavior of the *singular value gaps* of the sequence g_n:

Proposition A.1. *Suppose that $G = \mathrm{PGL}(d, \mathbb{R})$, and let $P^+ = P_1 \subset G$ be the stabilizer of a line in $\mathbb{R}^d$. A sequence $g_n \in G$ is P_1-divergent if and only if*

$$\frac{\sigma_1(g_n)}{\sigma_2(g_n)} \to \infty,$$

where $\sigma_i(g_n)$ is the ith-largest singular value of g_n.

For convenience, we give a proof of Proposition 3.6 in this special case.

Lemma A.2. *Let g_n be a sequence in $\mathrm{PGL}(d, \mathbb{R})$, and suppose that for a nonempty open subset $U \subset \mathbb{RP}^{d-1}$, $g_n U$ converges to a point in $\mathbb{RP}^{d-1}$. Then, the singular value gap*

$$\frac{\sigma_1(g_n)}{\sigma_2(g_n)}$$

tends to ∞ as $n \to \infty$.

Proof. It suffices to show that any subsequence of g_n has a subsequence which satisfies the property. Using the Cartan decomposition of $\mathrm{PGL}(d, \mathbb{R})$, we can write

$$g_n = k_n a_n k'_n,$$

for $k_n, k'_n \in K = \mathrm{PO}(d)$ and a_n a diagonal matrix whose diagonal entries are $\sigma_1, \ldots, \sigma_d$. Up to subsequence k_n and k'_n converge respectively to $k, k' \in K$. For sufficiently large n, $k'_n U \cap k'U$ is nonempty, so by replacing U with $k'U$ we can assume that $k'_n = \mathrm{id}$ for all n. Furthermore, if $k_n a_n U$ converges to a point $z \in \mathbb{RP}^{d-1}$, then $a_n U$ converges to $k^{-1} z$.

So, $a_n U$ converges to a point, and since a_n is a diagonal matrix, the gap between the moduli of its largest and second-largest eigenvalues must be unbounded. ☐

To prove the general case of Proposition 3.6, we take an irreducible representation $\zeta : G \to \mathrm{PGL}(V)$ coming from Theorem 7.4, so that P^+ maps to the stabilizer of a line ℓ in V, P^- maps to the stabilizer of a hyperplane H in V, and gP^+g^{-1}, hP^-h^{-1} are opposite if and only if $\zeta(g)\ell$, $\zeta(h)H$ are transverse. As in section 7, this determines embeddings $\iota : G/P \to \mathbb{P}(V)$ and $\iota^* : G/P^- \to \mathbb{P}(V^*)$ by

$$\iota(gP^+) = \zeta(g)\ell, \qquad \iota^*(gP^-) = \zeta(g)H.$$

The representation ζ additionally has the property that for any sequence $g_n \in G$, the singular value gaps

$$\sigma_1(\zeta(g_n))/\sigma_2(\zeta(g_n))$$

are unbounded if and only if g_n is P^+-divergent (see [GGKW17], Lemma 3.7).

Proof of Proposition 3.6. By [Zim18], Lemma 4.7, there exist flags $\xi_1, \ldots, \xi_D \in U$ so that lifts of $\iota(\xi_i)$ give a basis of V. Since $g_n \cdot U$ converges to a point in G/P, the set

$$\{\zeta(g_n) \cdot \iota(\xi_i) : 1 \le i \le D\}$$

converges to a single point in $\mathbb{P}(V)$.

This means that we can fix lifts $\tilde{\iota(\xi_i)} \in V$ so that, up to a subsequence, $\zeta(g_n)$ takes the projective $(D-1)$-simplex

$$\left[\sum_{i=1}^{D} \lambda_i \tilde{\iota(\xi_i)} : \lambda_i > 0\right] \subset \mathbb{P}(V)$$

to a point. This simplex is an open subset of $\mathbb{P}(V)$. Now we can apply Lemma A.2 to see that the sequence g_n is P^+-divergent.

We now just need to check that ξ is the unique P^+-limit point of g_n. Choose any subsequence of g_n. Then any P^+-contracting subsequence g_m of this subsequence satisfies

$$g_m|_{\mathrm{Opp}(\xi_-)} \to \xi'$$

uniformly on compacts for some $\xi_- \in G/P^-$ and $\xi' \in G/P^+$. But since $\mathrm{Opp}(\xi_-)$ is open and dense, it intersects U nontrivially and thus $\xi' = \xi$. ☐

References

[Bal14] Samuel A. Ballas. Deformations of noncompact projective manifolds. *Algebr. Geom. Topol.*, 14(5):2595–2625, 2014.

[Bal21] Samuel A. Ballas. Constructing convex projective 3-manifolds with generalized cusps. *J. Lond. Math. Soc. (2)*, 103(4):1276–1313, 2021.

[BCL20] Samuel A. Ballas, Daryl Cooper, and Arielle Leitner. Generalized cusps in real projective manifolds: classification. *J. Topol.*, 13(4):1455–1496, 2020.

[BDL15] Samuel Ballas, Jeffrey Danciger, and Gye-Seon Lee. Convex projective structures on non-hyperbolic three-manifolds. *Geometry & Topology*, 22, 08 2015.

[Ben04] Yves Benoist. Convexes divisibles. I. In *Algebraic groups and arithmetic*, pages 339–374. Tata Inst. Fund. Res., Mumbai, 2004.

[Ben06a] Yves Benoist. Convexes divisibles. IV. Structure du bord en dimension 3. *Invent. Math.*, 164(2):249–278, 2006.

[Ben06b] Yves Benoist. Convexes hyperboliques et quasiisométries. *Geom. Dedicata*, 122:109–134, 2006.

[BH99] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.

[BM20] Samuel Ballas and Ludovic Marquis. Properly convex bending of hyperbolic manifolds. *Groups Geom. Dyn.*, 14(2):653–688, 2020.

[Bob19] Martin D. Bobb. Convex projective manifolds with a cusp of any non-diagonalizable type. *J. Lond. Math. Soc. (2)*, 100(1):183–202, 2019.

[Bog97] O. V. Bogopol'skiĭ. Infinite commensurable hyperbolic groups are bi-Lipschitz equivalent. *Algebra i Logika*, 36(3):259–272, 357, 1997.

[Bow98] Brian H. Bowditch. Spaces of geometrically finite representations. *Ann. Acad. Sci. Fenn. Math.*, 23(2):389–414, 1998.

[Bow99] B. H. Bowditch. Convergence groups and configuration spaces. In *Geometric group theory down under (Canberra, 1996)*, pages 23–54. de Gruyter, Berlin, 1999.

[Bow12] B. H. Bowditch. Relatively hyperbolic groups. *Internat. J. Algebra Comput.*, 22(3):1250016, 66, 2012.

[BPS19] Jairo Bochi, Rafael Potrie, and Andrés Sambarino. Anosov representations and dominated splittings. *J. Eur. Math. Soc. (JEMS)*, 21(11):3343–3414, 2019.

[BV23] Pierre-Louis Blayac and Gabriele Viaggi. Divisible convex sets with properly embedded cones. *arXiv e-prints*, page arXiv:2302.07177, February 2023.

[Cho10] Suhyoung Choi. The convex real projective orbifolds with radial or totally geodesic ends: The closedness and openness of deformations. *arXiv e-prints*, page arXiv:1011.1060, November 2010.

[CLM20] Suhyoung Choi, Gye-Seon Lee, and Ludovic Marquis. Convex projective generalized Dehn filling. *Ann. Sci. Éc. Norm. Supér. (4)*, 53(1):217–266, 2020.

[CLM22] Suhyoung Choi, Gye-Seon Lee, and Ludovic Marquis. Deformation spaces of Coxeter truncation polytopes. *J. Lond. Math. Soc. (2)*, 106(4):3822–3864, 2022.

[CLT18] Daryl Cooper, Darren Long, and Stephan Tillmann. Deforming convex projective manifolds. *Geom. Topol.*, 22(3):1349–1404, 2018.

[CM14] Mickaël Crampon and Ludovic Marquis. Finitude géométrique en géométrie de Hilbert. *Ann. Inst. Fourier (Grenoble)*, 64(6):2299–2377, 2014.

[CZZ21] Richard Canary, Tengren Zhang, and Andrew Zimmer. Cusped Hitchin representations and Anosov representations of geometrically finite Fuchsian groups. *arXiv e-prints*, page arXiv:2103.06588, March 2021.

[DGK17] Jeffrey Danciger, François Guéritaud, and Fanny Kassel. Convex cocompact actions in real projective geometry. *arXiv e-prints*, page arXiv:1704.08711, April 2017.

[DGK18] Jeffrey Danciger, François Guéritaud, and Fanny Kassel. Convex cocompactness in pseudo-Riemannian hyperbolic spaces. *Geom. Dedicata*, 192:87–126, 2018.

[DGK+21] Jeffrey Danciger, François Guéritaud, Fanny Kassel, Gye-Seon Lee, and Ludovic Marquis. Convex cocompactness for Coxeter groups. *arXiv e-prints*, page arXiv:2102.02757, February 2021.

[DS05] Cornelia Druţu and Mark Sapir. Tree-graded spaces and asymptotic cones of groups. *Topology*, 44(5):959–1058, 2005. With an appendix by Denis Osin and Mark Sapir.

[Ebe96] Patrick B. Eberlein. *Geometry of nonpositively curved manifolds*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1996.

[Ger12] Victor Gerasimov. Floyd maps for relatively hyperbolic groups. *Geom. Funct. Anal.*, 22(5):1361–1399, 2012.

[GGKW17] François Guéritaud, Olivier Guichard, Fanny Kassel, and Anna Wienhard. Anosov representations and proper actions. *Geom. Topol.*, 21(1):485–584, 2017.

[GM08] Daniel Groves and Jason Fox Manning. Dehn filling in relatively hyperbolic groups. *Israel J. Math.*, 168:317–429, 2008.

[GP13] Victor Gerasimov and Leonid Potyagailo. Quasi-isometric maps and Floyd boundaries of relatively hyperbolic groups. *J. Eur. Math. Soc. (JEMS)*, 15(6):2115–2137, 2013.

[GW12] Olivier Guichard and Anna Wienhard. Anosov representations: domains of discontinuity and applications. *Invent. Math.*, 190(2):357–438, 2012.

[GW24] Antonin Guilloux and Theodore Weisman. Limits of limit sets in rank-one symmetric spaces. 2024.

[Hel01] Sigurdur Helgason. *Differential geometry, Lie groups, and symmetric spaces*, volume 34 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001. Corrected reprint of the 1978 original.

[IZ22] Mitul Islam and Andrew Zimmer. The structure of relatively hyperbolic groups in convex real projective geometry. *arXiv e-prints*, page arXiv:2203.16596, March 2022.

[Kap07] Michael Kapovich. Convex projective structures on Gromov-Thurston manifolds. *Geom. Topol.*, 11:1777–1830, 2007.

[KKL19] Michael Kapovich, Sungwoon Kim, and Jaejeong Lee. Structural stability of meandering-hyperbolic group actions. *arXiv e-prints*, page arXiv:1904.06921, April 2019.

[KL23] Michael Kapovich and Bernhard Leeb. Relativizing characterizations of Anosov subgroups, I. *Groups Geom. Dyn.*, 17(3):1005–1071, 2023. With an appendix by Gregory A. Soifer.

[KLP17] Michael Kapovich, Bernhard Leeb, and Joan Porti. Anosov subgroups: dynamical and geometric characterizations. *Eur. J. Math.*, 3(4):808–898, 2017.

[Kna02] Anthony W. Knapp. *Lie groups beyond an introduction*, volume 140 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, second edition, 2002.

[Lab06] François Labourie. Anosov flows, surface groups and curves in projective space. *Invent. Math.*, 165(1):51–114, 2006.

[LLS21] Gye-Seon Lee, Jaejeong Lee, and Florian Stecker. Anosov triangle reflection groups in SL(3,R). *arXiv e-prints*, page arXiv:2106.11349, June 2021.

[MMW22] Kathryn Mann, Jason Fox Manning, and Theodore Weisman. Stability of hyperbolic groups acting on their boundaries. Preprint, arXiv:2206.14914, 2022.

[MMW24] Kathryn Mann, Jason Fox Manning, and Theodore Weisman. Topological stability of relatively hyperbolic groups acting on their boundaries. *arXiv e-prints*, page arXiv:2402.06144, February 2024.

[Rie21] J. Maxwell Riestenberg. A quantified local-to-global principle for Morse quasigeodesics. *arXiv e-prints*, page arXiv:2101.07162, January 2021.

[Tra13] Hung Cong Tran. Relations between various boundaries of relatively hyperbolic groups. *Internat. J. Algebra Comput.*, 23(7):1551–1572, 2013.

[Tuk94] Pekka Tukia. Convergence groups and Gromov's metric hyperbolic spaces. *New Zealand J. Math.*, 23(2):157–187, 1994.

[Tuk98] Pekka Tukia. Conical limit points and uniform convergence groups. *J. Reine Angew. Math.*, 501:71–98, 1998.

[Wan23a] Tianqi Wang. Anosov representations over closed subflows. *Trans. Amer. Math. Soc.*, 376(9):6177–6214, 2023.

[Wan23b] Tianqi Wang. Notions of Anosov representation of relatively hyperbolic groups. *arXiv e-prints*, page arXiv:2309.15636, September 2023.

[Wei23a] Theodore Weisman. Dynamical properties of convex cocompact actions in projective space. *Journal of Topology*, 16(3):990–1047, 2023.

[Wei23b] Theodore Weisman. Examples of extended geometrically finite representations. 2023.

[Wol20] Adva Wolf. *Convex Projective Geometrically Finite Structures*. ProQuest LLC, Ann Arbor, MI, 2020. Thesis (Ph.D.)–Stanford University.

[Yam04] Asli Yaman. A topological characterisation of relatively hyperbolic groups. *J. Reine Angew. Math.*, 566:41–89, 2004.

[Zhu21] Feng Zhu. Relatively dominated representations. *Ann. Inst. Fourier (Grenoble)*, 71(5):2169–2235, 2021.

[Zim18] Andrew M. Zimmer. Proper quasi-homogeneous domains in flag manifolds and geometric structures. *Ann. Inst. Fourier (Grenoble)*, 68(6):2635–2662, 2018.

[Zim21] Andrew Zimmer. Projective Anosov representations, convex cocompact actions, and rigidity. *J. Differential Geom.*, 119(3):513–586, 2021.

[ZZ22] Feng Zhu and Andrew Zimmer. Relatively Anosov representations via flows I: theory. *arXiv e-prints*, page arXiv:2207.14737, July 2022.

Department of Mathematics, University of Michigan, Ann Arbor MI 48109, USA
Email address: tjwei@umich.edu