SINGULAR VALUE GAP ESTIMATES FOR FREE PRODUCTS OF SEMIGROUPS

KONSTANTINOS TSOUVALAS AND THEODORE WEISMAN

Abstract. We establish lower estimates for singular value gaps of free products of 1-divergent semigroups $\Gamma_1, \Gamma_2 \subset \mathsf{GL}_d(\mathbb{K})$ which are in ping-pong position. As an application, we prove that if Γ_1 and Γ_2 are quasi-isometrically embedded subgroups in ping pong position, then the group they generate $\langle \Gamma_1, \Gamma_2 \rangle$ is also quasi-isometrically embedded. In addition, we establish that the class of linear finitely generated groups, admitting a faithful linear representation over $\mathbb{R}$ which is a quasi-isometric embedding, is closed under free products.

Contents

1. Introduction

When a group G acts on a space X, combination theorems relying on "ping-pong" inclusions of open subsets of X give a robust means of producing examples of subgroups of G satisfying some desired property. A typical setup for such a combination theorem is to consider a pair of sub(semi)groups Γ_1, Γ_2 of the ambient group G, and some configuration of sets in X mapped into each other by nontrivial elements of Γ_1, Γ_2. One then wishes to show that the (semi)group Γ generated by Γ_1, Γ_2:

(a) is naturally isomorphic to the free product $\Gamma_1 * \Gamma_2$, and
(b) also satisfies some property shared by both Γ_1 and Γ_2.

A simple application of the ping-pong lemma often suffices to prove (a), but proving that (b) holds depends on the precise property in question.

Theorems of this type date back to Klein's original 19th-century combination theorems for discrete groups of isometries of hyperbolic space [12], later developed further by Maskit [16] and carried through in numerous different contexts by other authors, see e.g. [13, 14, 15, 17].

In the present paper, we are concerned with proving ping-pong type combination theorems for *discrete subgroups of higher-rank semisimple Lie groups.* We focus on the case where Γ_1, Γ_2 are sub(semi)groups of $\mathsf{GL}_d(\mathbb{K})$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Our main results (Theorem 1.2 and Theorem 1.6 below) give estimates for certain *singular value gaps* of elements in the semigroup $\Gamma = \langle \Gamma_1, \Gamma_2 \rangle$, in terms of singular value gaps of elements in Γ_1 and Γ_2. In particular, these estimates can be used to show that singular value gap growth properties of Γ_1 and Γ_2 can be passed to the combination Γ, and we prove several corollaries of this type below.

Our theorems are strong enough to recover combination theorems regarding free products of *Anosov subgroups* proved by Dey-Kapovich [5] and Danciger-Guéritaud-Kassel [3] (see also [7,

Date: September 30, 2024.

Thm. 1.3]), but they also apply to groups outside of this context. Theorem 1.2 applies to any pair of 1-*divergent semigroups* (a class which in particular includes all subgroups of 1-Anosov and relatively 1-Anosov groups), and Theorem 1.6 can be applied to essentially arbitrary discrete subgroups of semisimple Lie groups.

1.1. **Singular value gap estimates for 1-divergent semigroups.** Before stating our first main theorem, we set up some notation and terminology. For a matrix $g \in \mathsf{GL}_d(\mathbb{K})$ we denote by $\sigma_1(g) \geqslant \cdots \geqslant \sigma_d(g)$ (resp. $\ell_1(g) \geqslant \cdots \geqslant \ell_d(g)$) the singular values (resp. moduli of eigenvalues) of g. For $1 \leqslant i, j \leqslant d$, we write $\frac{\sigma_i}{\sigma_j}(g)$ for $\frac{\sigma_i(g)}{\sigma_j(g)}$ and $\frac{\ell_i}{\ell_j}(g)$ for $\frac{\ell_i(g)}{\ell_j(g)}$. A semigroup $\Gamma \subset \mathsf{GL}_d(\mathbb{K})$ is called k-*divergent* $(1 \leqslant k < d)$ if, for every sequence γ_n of pairwise distinct elements in Γ, we have $\frac{\sigma_k}{\sigma_{k+1}}(\gamma_n) \to \infty$.

For semigroups $\Gamma_1, \Gamma_2 \subset \mathsf{GL}_d(\mathbb{K})$ let $\langle \Gamma_1, \Gamma_2 \rangle$ denote the semigroup they generate. An expression $\gamma_1 \cdots \gamma_n \in \langle \Gamma_1, \Gamma_2 \rangle$, $n \geqslant 2$, is called a *reduced word* if all γ_i are non-trivial and no consecutive γ_i lie in the same semigroup.

We use $\mathbb{P}(\mathbb{K}^d)$ to denote the projective space over the (real or complex) vector space $\mathbb{K}^d$, and for each $1 \leqslant k < d$ we let $\mathsf{Gr}_k(\mathbb{K}^d)$ denote the Grassmannian of k-planes in $\mathbb{K}^d$. Recall that the space of hyperplanes $\mathsf{Gr}_{d-1}(\mathbb{K}^d)$ is canonically identified with the dual projective space $\mathbb{P}((\mathbb{K}^d)^*)$.

Definition 1.1. Let $\Gamma_1, \Gamma_2 \subset \mathsf{GL}_d(\mathbb{K})$ be semigroups, and let $U_1, U_2 \subset \mathbb{P}(\mathbb{K}^d)$ and $V_1, V_2 \subset \mathsf{Gr}_{d-1}(\mathbb{K}^d)$ be open subsets. We say that Γ_1, Γ_2 are *in ping-pong position relative to* U_1, U_2 and V_1, V_2 if, whenever $\{i, j\} = \{1, 2\}$, the following conditions hold:

(1) The sets $\overline{U_i}$ and $\overline{V_j}$ are transverse.
(2) For every $\gamma \in \Gamma_i \setminus \{I_d\}$, we have $\gamma\overline{U_j} \subset U_i$, and $\gamma^{-1}\overline{V_j} \subset V_i$.

Our first main theorem is below:

Theorem 1.2. *Suppose that $\Gamma_1, \Gamma_2 \subset \mathsf{GL}_d(\mathbb{K})$ are 1-divergent semigroups, in ping-pong position relative to subsets $U_1, U_2 \subset \mathbb{P}(\mathbb{K}^d), V_1, V_2 \subset \mathsf{Gr}_{d-1}(\mathbb{K}^d)$. Then:*

(i) There exists $C_1 > 0$ and $\lambda > 1$ such that, for any reduced word $\gamma_1 \cdots \gamma_n \in \langle \Gamma_1, \Gamma_2 \rangle$, we have
$$\frac{\sigma_1}{\sigma_2}(\gamma_1 \cdots \gamma_n) \geqslant C_1 \lambda^n.$$
(ii) There exists $C_2 > 0$ such that, for any reduced word $\gamma_1 \cdots \gamma_n \in \langle \Gamma_1, \Gamma_2 \rangle$, we have
$$\sigma_1(\gamma_1 \cdots \gamma_n) \geqslant C_2^n \sigma_1(\gamma_1) \cdots \sigma_1(\gamma_n).$$
(iii) There exists $C_3 > 0$ such that, for any reduced word $\gamma_1 \cdots \gamma_n \in \langle \Gamma_1, \Gamma_2 \rangle$, we have
$$\frac{\sigma_1}{\sigma_2}(\gamma_1 \cdots \gamma_n) \geqslant C_3^n \frac{\sigma_1}{\sigma_2}(\gamma_1) \cdots \frac{\sigma_1}{\sigma_2}(\gamma_n).$$
(iv) There exists $C_4 > 0$ such that, for every $n \in \mathbb{N}$ even and any reduced word $\gamma_1 \cdots \gamma_n \in \langle \Gamma_1, \Gamma_2 \rangle$, we have
$$\frac{\ell_1}{\ell_2}(\gamma_1 \cdots \gamma_n) \geqslant C_4^n \frac{\sigma_1}{\sigma_2}(\gamma_1) \cdots \frac{\sigma_1}{\sigma_2}(\gamma_n).$$

In a typical application of Theorem 1.2, the semigroup $\langle \Gamma_1, \Gamma_2 \rangle$ is naturally isomorphic to the free product of semigroups $\Gamma_1 * \Gamma_2$, and in fact this always holds whenever Γ_1 and Γ_2 are both groups. See Proposition 3.4.

Recall that a representation $\rho : H \to \mathsf{GL}_d(\mathbb{K})$ is a *quasi-isometric embedding* if there exist $C \geqslant 0$ and $c \geqslant 1$ such that for every $h \in H$,
$$c^{-1}|h|_{\mathsf{H}} - C \leqslant \log \frac{\sigma_1}{\sigma_d}(\rho(h)) \leqslant c|h|_{\mathsf{H}} + C,$$
where $|h|_H$ is the length of h with respect to a choice of finite generating set for H. One consequence of the estimates in Theorem 1.2 is the following.

Corollary 1.3. *Let $\Gamma_1, \Gamma_2 < \mathsf{GL}_d(\mathbb{K})$ be two 1-divergent subgroups which are in ping-pong position relative to subsets U_1, U_2, V_1, V_2, so that $\langle \Gamma_1, \Gamma_2 \rangle$ is naturally identified with the free product $\Gamma_1 * \Gamma_2$. If both $\Gamma_1, \Gamma_2 < \mathsf{GL}_d(\mathbb{K})$ are quasi-isometrically embedded, then so is $\langle \Gamma_1, \Gamma_2 \rangle$.*

The notion of an Anosov semigroup was introduced in [11, §5]. As another corollary of Theorem 1.2 we obtain the following result.

Corollary 1.4. *Let $\Gamma < \mathsf{GL}_d(\mathbb{K})$ be an 1-Anosov subgroup. Suppose that the proximal limit set of Γ and its dual Γ^* are contained (in possibly different) affine charts of $\mathbb{P}(\mathbb{K}^d)$. There is an element $g \in \mathsf{GL}_d(\mathbb{K})$ and a finite-index subgroup $\Gamma' < \Gamma$ such that the semigroup $\langle \Gamma', g \rangle$ is 1-Anosov and isomorphic to the free product of the semigroups Γ' and $\{g^n : n \geqslant 0\}$.*

This corollary allows us to produce non-elementary examples of 1-Anosov semigroups in $\mathsf{GL}_d(\mathbb{R})$ which do *not* generate 1-Anosov subgroups; see Section 3.3.1.

Remark 1.5. Using techniques similar to the proof of Corollary 1.4, one can also use Theorem 1.2 to recover special cases of combination theorems for Anosov subgroups due to Dey-Kapovich [5, 4] and Danciger-Guéritaud-Kassel [3]. Since Anosov subgroups have uniform eigenvalue and singular value gap growth properties, these combination theorems themselves recover estimates as in Theorem 1.2—but *only* in the case where the combined group is Anosov, which need not be the case for general applications of Theorem 1.2.

We remark that the techniques in [4] and [3] also establish combination theorems for non-trivial amalgams and HNN extensions of Anosov subgroups. We do not provide estimates as in Theorem 1.2 for such combinations in this paper, although we expect that our methods also apply in this case.

1.2. Singular value gap estimates for arbitrary free products.

Our second main theorem applies more broadly, to arbitrary subgroups of $\mathsf{SL}_d(\mathbb{R})$. We do not explicitly assume any ping-pong type configuration of open subsets in the statement of this theorem; such a configuration instead arises in the course of the proof.

Theorem 1.6. *For any $d \in \mathbb{N}_{\geqslant 2}$, there exists a representation $\rho : \mathsf{SL}_d(\mathbb{R}) * \mathsf{SL}_d(\mathbb{R}) \to \mathsf{GL}_m(\mathbb{R})$, $m = \frac{d^2(d+1)^2}{4} + 1$, with the following property. If $\Gamma_1, \Gamma_2 < \mathsf{SL}_d(\mathbb{R})$ are discrete finitely generated subgroups, then there exist finite-index subgroups $H_1 < \Gamma_1$ and $H_2 < \Gamma_2$ such that the restriction of ρ to $H_1 * H_2$ is discrete and faithful. In addition, for any $\varepsilon \in (0,1)$, the finite-index subgroups H_1, H_2 can be chosen so that for every reduced word $\gamma_1 \cdots \gamma_n \in H_1 * H_2$, the following estimate holds:*

$$\frac{\sigma_1}{\sigma_m}\left(\rho(\gamma_1 \cdots \gamma_n)\right) \geqslant \prod_{i=1}^{n} \left(\frac{\sigma_1}{\sigma_d}(\gamma_i)\right)^{2-\varepsilon}.$$

Remark 1.7. The first part of Theorem 1.6 (concerning the existence of finite-index subgroups so that the representation $\rho : H_1 * H_2 \to \mathsf{GL}_m(\mathbb{R})$ is faithful and discrete) follows from work of Danciger-Guéritaud-Kassel [3], and when we prove Theorem 1.6 we will use their techniques to show that this part holds. Thus our main contribution in Theorem 1.6 is the second part, giving the singular value gap estimate.

As a consequence of the estimate in Theorem 1.6, we establish that the class of linear groups admitting representations which are quasi-isometric embeddings are closed under free products.

Corollary 1.8. *Suppose that $\Gamma_1, \Gamma_2 < \mathsf{GL}_d(\mathbb{R})$ are finitely generated and quasi-isometrically embedded subgroups. Then there exists $r \in \mathbb{N}$ and a faithful representation $\rho : \Gamma_1 * \Gamma_2 \to \mathsf{GL}_r(\mathbb{R})$ which is a quasi-isometric embedding.*

1.3. Other semisimple Lie groups.

In this paper we only explicitly consider discrete subgroups of $\mathsf{GL}_d(\mathbb{K})$ for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. However, it is still possible to indirectly apply our results towards discrete subgroups of an arbitrary semisimple Lie group G, by mapping these discrete subgroups into $\mathsf{GL}_d(\mathbb{K})$ via some representation $G \to \mathsf{GL}_d(\mathbb{K})$. We give a rough idea of this below.

If P is a parabolic subgroup of G, then one can consider P-*divergent* groups $\Gamma \subset G$; for an appropriate choice of representation $G \to \mathsf{GL}_d(\mathbb{K})$, a group Γ is P-divergent if and only if its image in $\mathsf{GL}_d(\mathbb{K})$ is 1-divergent (see e.g. [8, Sec. 3]). One can consider P-divergent subgroups Γ_1, Γ_2 of G which are in ping-pong position with respect to open subsets of the *flag manifold* G/P, as well as its natural opposite. One can show that in this situation, it is possible to pass to finite-index subgroups Γ_1', Γ_2' of Γ_1, Γ_2, whose images in $\mathsf{GL}_d(\mathbb{K})$ are in ping-pong position with respect to open subsets of $\mathbb{P}(\mathbb{K}^d)$ and $\mathsf{Gr}_{d-1}(\mathbb{K}^d)$. Thus, one may apply Theorem 1.2 to obtain estimates for the behavior of the Jordan and Cartan projections of elements in the free product $\Gamma_1' * \Gamma_2'$, with respect to the roots of G determined by the parabolic subgroup P.

Acknowledgements. The authors would like to thank Subhadip Dey for several helpful comments and corrections. The second author was partially supported by NSF grant DMS-2202770.

2. Preliminaries

2.1. **Geometry of Grassmannians over $\mathbb{K}^d$.** Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. For $d \geqslant 2$, $(e_1, \ldots, e_d)$ is the canonical basis of $\mathbb{K}^d$ equipped with the standard (Hermitian) inner product $\langle \cdot, \cdot \rangle$, and $\mathsf{S}^1(\mathbb{K}^d)$ is the unit sphere in $\mathbb{K}^d$ with respect to this inner product. We denote by K_d the maximal compact subgroup of $\mathsf{GL}_d(\mathbb{K})$ preserving the inner product $\langle \cdot, \cdot \rangle$ (i.e. $\mathsf{K}_d = \mathsf{O}(d)$ if $\mathbb{K} = \mathbb{R}$ and $\mathsf{K}_d = \mathsf{U}(d)$ if $\mathbb{K} = \mathbb{C}$).

Given a unit vector $v \in \mathbb{K}^d$, $v^\perp = \{\omega \in \mathbb{K}^d : \langle \omega, v \rangle = 0\}$ denotes the orthogonal complement of the vector space spanned by v. We equip the projective space $\mathbb{P}(\mathbb{K}^d)$ with the metric

$$d_{\mathbb{P}}([u],[v]) = \sqrt{1 - \frac{|\langle u, v \rangle|^2}{||u||^2||v||^2}}.$$

For any $u, v \in \mathbb{K}^d$, a straightforward computation gives us the following formula for $d_{\mathbb{P}}([u],[v])$ in terms of the canonical basis $(e_1, \ldots, e_d)$:

$$d_{\mathbb{P}}([u],[v])^2 = \frac{\sum_{1 \leqslant i < j \leqslant k} |\langle u, e_i \rangle \langle v, e_j \rangle - \langle u, e_j \rangle \langle v, e_i \rangle|^2}{\left(\sum_{i=1}^d |\langle u, e_i \rangle|^2\right)\left(\sum_{i=1}^d |\langle v, e_i \rangle|^2\right)}. \tag{1}$$

We equip the Grassmannian $\mathsf{Gr}_{d-1}(\mathbb{K}^d)$ with the metric d_{Gr} defined by viewing elements of $\mathsf{Gr}_{d-1}(\mathbb{K}^d)$ as subsets of $\mathbb{P}(\mathbb{K}^d)$, and taking Hausdorff distance with respect to $d_{\mathbb{P}}$. Equivalently, if $u^\perp$ and $v^\perp$ are hyperplanes in $\mathsf{Gr}_{d-1}(\mathbb{K}^d)$ for $u, v \in \mathsf{S}^1(\mathbb{K}^d)$, then

$$d_{\mathsf{Gr}}(u^\perp, v^\perp) = d_{\mathbb{P}}([u],[v]).$$

For a point $x \in \mathbb{P}(\mathbb{K}^d)$ (resp. $V \in \mathsf{Gr}_{d-1}(\mathbb{K}^d)$) and $\varepsilon > 0$, let $B_\varepsilon(x)$ and $B_\varepsilon(V)$ respectively denote the open balls of radius ε about x and V, with respect to the metrics $d_{\mathbb{P}}$ and d_{Gr}.

If x is an element of $\mathsf{Gr}_k(\mathbb{K}^d)$, and y is an element of $\mathsf{Gr}_j(\mathbb{K}^d)$, with $j + k \leqslant d$, then we let $\mathrm{dist}(x, y)$ denote the minimum distance between x and y when both are viewed as subsets of projective space $\mathbb{P}(\mathbb{K}^d)$. In the special case where X is a singleton $\{[u]\}$ for $u \in \mathsf{S}^1(\mathbb{K}^d)$, and Y is a k-dimensional projective subspace $\mathbb{P}(V)$, then $\mathrm{dist}([u], \mathbb{P}(V))$ is the length of the projection of u onto $V^\perp$. In particular, if $u = k_1 e_1$ and $V = k_2 e_1^\perp$ for $k_1, k_2 \in \mathsf{K}_d$, then

$$\mathrm{dist}([u], \mathbb{P}(V)) = |\langle k_1 e_1, k_2 e_1 \rangle|. \tag{2}$$

For $[u],[v] \in \mathbb{P}(\mathbb{K}^d)$ and $V, W \in \mathsf{Gr}_{d-1}(\mathbb{K}^d)$, we shall use repeatedly the following estimates from the triangle inequality:

$$\left|\mathrm{dist}([u], \mathbb{P}(V)) - \mathrm{dist}([v], \mathbb{P}(V))\right| \leqslant d_{\mathbb{P}}\left([u],[v]\right)$$
$$\left|\mathrm{dist}([u], \mathbb{P}(V)) - \mathrm{dist}([v], \mathbb{P}(W))\right| \leqslant d_{\mathsf{Gr}}\left(V, W\right).$$

2.2. **Singular values and the Cartan projection.** For a matrix $g \in \mathsf{GL}_d(\mathbb{K})$ let $\sigma_1(g) \geqslant \sigma_2(g) \geqslant \ldots \geqslant \sigma_d(g)$ (resp. $\ell_1(g) \geqslant \ell_2(g) \geqslant \ldots \geqslant \ell_d(g)$) be the singular values (resp. moduli of eigenvalues) of g in non-increasing order. Recall that for each $i \in \{1, \ldots, d-1\}$, $\sigma_i(g) = \sqrt{\ell_i(gg^*)}$, where $g^* = \overline{g}^t$ is the conjugate transpose of g.

Recall that $\mathsf{GL}_d(\mathbb{K})$ has a *Cartan* or *KAK decomposition*

$$\mathsf{GL}_d(\mathbb{K}) = \mathsf{K}_d \exp(\mathfrak{a}^+)\mathsf{K}_d,$$

where $\mathfrak{a}^+ = \{\mathrm{diag}(a_1, \ldots, a_d) : a_1 \geqslant a_2 \geqslant \cdots \geqslant a_d\}$. This means that for any $g \in \mathsf{GL}_d(\mathbb{K})$, there exist $k_g, k_g' \in \mathsf{K}_d$ so that

$$g = k_g \exp(\mu(g))k_g',$$

where $\mu : \mathsf{GL}_d(\mathbb{K}) \to \mathbb{R}^d$ is the *Cartan projection* defined by

$$\mu(g) := (\log \sigma_1(g), \ldots, \log \sigma_d(g)).$$

Given $i \in \{1, \ldots, d-1\}$, we say that a matrix $g \in \mathsf{GL}_d(\mathbb{K})$ has a *gap of index* i, if $\sigma_i(g) > \sigma_{i+1}(g)$. In this case, if we write $g = k_g \mathrm{diag}(\sigma_1(g), \ldots, \sigma_d(g))k_g'$ for $k_g, k_g' \in \mathsf{K}_d$, then the subspace

$$\Xi_i(g) := k_g\langle e_1, \ldots, e_i\rangle$$

is well-defined and does not depend on the choice of $k_g, k_g' \in \mathsf{K}_d$.

A semigroup $\Gamma \subset \mathsf{GL}_d(\mathbb{K})$ is *i-divergent* if for every infinite sequence of distinct elements $(\gamma_n)_{n\in\mathbb{N}} \subset \Gamma$ we have

$$\lim_{n\to\infty} \frac{\sigma_i}{\sigma_{i+1}}(\gamma_n) = +\infty.$$

Note that if the semigroup Γ is i-divergent, the inverse semigroup Γ^{-1} is $(d-i)$-divergent. For an i-divergent semigroup $\Gamma \subset \mathsf{GL}_d(\mathbb{K})$ denote by $\Lambda_i(\Gamma)$ its *limit set* in $\mathsf{Gr}_i(\mathbb{K}^d)$, i.e. the set of all accumulation points of the sequence $(\Xi_i(\gamma_n))_{n\in\mathbb{N}}$ as $(\gamma_n)_{n\in\mathbb{N}} \subset \Gamma$ runs over all infinite sequences of distinct elements in Γ. For more details on definitions we refer to [8, 10].

2.3. **Initial estimates.** We now prove some basic lemmas that we will use throughout the proofs of our main results. Some (but not all) of these lemmas are almost identical to results appearing in the appendix of [1]. However, in our case, we work in $\mathbb{K}^d$, rather than $\mathbb{R}^d$, and we do not need to assume that the matrices we consider have any gap between their singular values. For the reader's convenience we provide short proofs for all of our estimates.

The first two lemmas we need are contraction results, relating singular value gaps of elements in $\mathsf{GL}_d(\mathbb{K})$ to radii of nested balls in $\mathbb{P}(\mathbb{K}^d)$. Lemma 2.1 shows that, any element $g \in \mathsf{GL}_d(\mathbb{K})$ always maps some ball B_1 in projective space $\mathbb{P}(\mathbb{K}^d)$ into some other ball B_2, with the ratio of the radii of B_1 and B_2 depending on the singular value gap $\frac{\sigma_1}{\sigma_2}(g)$ and the distance from B_1 to the space $\Xi_{d-1}(g^{-1})$. Lemma 2.2 gives a converse result: if we know that some element $g \in \mathsf{GL}_d(\mathbb{K})$ maps some ball B_1 in $\mathbb{P}(\mathbb{K}^d)$ into some other ball B_2, then we have an estimate on $\frac{\sigma_1}{\sigma_2}(g)$ in terms of the radii of B_1 and B_2, and the distance from B_1 to $\Xi_{d-1}(g^{-1})$.

Lemma 2.1. *Let $g \in \mathsf{GL}_d(\mathbb{K})$ and write $g := k_g \exp(\mu(g))k_g'$ for $k_g, k_g' \in \mathsf{K}_d$. Fix $0 < \varepsilon < 1$ and $x \in \mathbb{P}(\mathbb{K}^d)$ with $\mathrm{dist}\big(x, \mathbb{P}((k_g')^{-1}e_1^\perp)\big) \geqslant \varepsilon$. Then for every $0 < \delta \leqslant \frac{1}{2}\varepsilon$ we have that*

$$gB_\delta(x) \subset B_{\delta'}(gx),$$

where $\delta' := \frac{2\delta}{\varepsilon}\frac{\sigma_2}{\sigma_1}(g)$.

Proof. First observe that since $\mathrm{dist}\big(x, \mathbb{P}((k_g')^{-1}e_1^\perp)\big) \geqslant \varepsilon$, for every $y \in B_\delta(x)$ we have that

$$\mathrm{dist}\big(y, \mathbb{P}((k_g')^{-1}e_1^\perp)\big) \geqslant \varepsilon - d_{\mathbb{P}}(x, y) \geqslant \frac{\varepsilon}{2}.$$

Write $x = [u_x], y = [u_y]$ for $u_x, u_y \in \mathsf{S}^1(\mathbb{K}^d)$. We can use (1) to see that

$$d_{\mathbb{P}}(gx, gy)^2 = \frac{\sum_{1 \leqslant i < j \leqslant d} \sigma_i(g)^2 \sigma_j(g)^2 \left|\langle k'_g u_x, e_i\rangle\langle k'_g u_y, e_j\rangle - \langle k'_g u_x, e_j\rangle\langle k'_g u_y, e_i\rangle\right|^2}{\left(\sum_{i=1}^d \sigma_i(g)^2 \left|\langle k'_g u_x, e_i\rangle\right|^2\right)\left(\sum_{i=1}^d \sigma_i(g)^2 \left|\langle k'_g u_y, e_i\rangle\right|^2\right)}$$
$$\leqslant \frac{\sigma_1(g)^2\sigma_2(g)^2}{\sigma_1(g)^4} \frac{d_{\mathbb{P}}(x,y)^2}{\left|\langle k'_g u_x, e_1\rangle\right|^2 \left|\langle k'_g u_y, e_1\rangle\right|^2}.$$

Using (2), we know that $\varepsilon/2 \leqslant \operatorname{dist}\big(y, \mathbb{P}((k'_g)^{-1} e_1^{\perp})\big) = \left|\langle k'_g u_y, e_1\rangle\right|^2$, so we can bound the above beneath

$$\frac{4\sigma_2(g)^2}{\sigma_1(g)^2}\frac{1}{\varepsilon^2} d_{\mathbb{P}}(x,y)^2 \leqslant \frac{4\sigma_2(g)^2}{\sigma_1(g)^2}\frac{\delta^2}{\varepsilon^2}.$$

The lemma follows. □

Lemma 2.2. *Let $g \in \mathsf{GL}_d(\mathbb{K})$ written as $g = k_g \exp(\mu(g)) k'_g$ for $k_g, k'_g \in \mathsf{K}_d$, and let $x \in \mathbb{P}(\mathbb{K}^d)$. Suppose that $0 < \delta, r < 1$ satisfy $gB_\delta(x) \subset B_r(gx)$. Then the following estimate holds:*

$$\frac{\sigma_1}{\sigma_2}(g) \geqslant \frac{\delta}{4r} \operatorname{dist}\big(x, \mathbb{P}((k'_g)^{-1} e_1^{\perp})\big)$$

Proof. After conjugation by k'_g, we may assume without loss of generality that $k'_g = \mathrm{id}$. Fix a unit vector $u_x \in \mathsf{S}^1(\mathbb{K}^d)$ with $x = [u_x]$. If $\operatorname{dist}\big(x, \mathbb{P}(e_1^{\perp})\big) = \left|\langle u_x, e_1\rangle\right| = 0$, there is nothing to prove. So we may assume that $\left|\langle u_x, e_1\rangle\right| > 0$ and $\left|\langle u_x, e_2\rangle\right| < 1$.

For each $t \in (0,1)$, consider the unit vector

$$v(t) := \frac{u_x + te_2}{\left|\left|u_x + te_2\right|\right|}.$$

A computation shows that

$$d_{\mathbb{P}}([u_x], [v(t)]) = \sqrt{1 - \left|\langle u_x, v(t)\rangle\right|^2} = \frac{t\sqrt{1 - \left|\langle u_x, e_2\rangle\right|^2}}{\left|\left|u_x + te_2\right|\right|}$$
$$\leqslant \frac{t}{1-t}\sqrt{1 - \left|\langle u_x, e_2\rangle\right|^2}.$$

In particular, we have $[v(t)] \in B_\delta(x)$ if t satisfies

$$t = \frac{\delta}{\delta + \sqrt{1 - |\langle u_x, e_2\rangle|^2}}.$$

So, now set $u_y = v(t)$ for some t as above. Suppose that $gB_\delta(x) \subset B_r(x)$, so that $d_{\mathbb{P}}([gu_x], [gu_y]) < r$. Using the formula (1), we have the estimates

$$d_{\mathbb{P}}([gu_x], [gu_y])^2 = \frac{\sum_{1 \leqslant i < j \leqslant d} \sigma_i(g)^2 \sigma_j(g)^2 \left|\langle u_x, e_i\rangle\langle u_y, e_j\rangle - \langle u_x, e_j\rangle\langle u_y, e_i\rangle\right|^2}{\left(\sum_{i=1}^d \sigma_i(g)^2 \left|\langle u_x, e_i\rangle\right|^2\right)\left(\sum_{i=1}^d \sigma_i(g)^2 \left|\langle u_y, e_i\rangle\right|^2\right)}$$
$$\geqslant \frac{\sigma_2(g)^2}{\sigma_1(g)^2}\left|\langle u_x, e_1\rangle\langle u_y, e_2\rangle - \langle u_x, e_2\rangle\langle u_y, e_1\rangle\right|^2$$
$$= \frac{\sigma_2(g)^2}{\sigma_1(g)^2}\frac{t^2\left|\langle u_x, e_1\rangle\right|^2}{\left|\left|u_x + te_2\right|\right|^2}$$
$$\geqslant \frac{\sigma_2(g)^2}{\sigma_1(g)^2}\frac{\delta^2}{16}\operatorname{dist}\big(x, \mathbb{P}(e_1^{\perp})\big)^2.$$

We deduce that

$$\frac{\sigma_1}{\sigma_2}(g) \geqslant \frac{\delta}{4r}\operatorname{dist}\big(x, \mathbb{P}(e_1^{\perp})\big),$$

which finishes the proof of the lemma. □

The next lemma also captures the idea that a group element $g \in \mathsf{GL}_d(\mathbb{K})$ attracts a point $x \in \mathbb{P}(\mathbb{K}^d)$ towards the subspace $\Xi_1(g)$, by an amount related to the singular value gap $\frac{\sigma_1}{\sigma_2}(g)$ and the distance from x to $\Xi_{d-1}(g^{-1})$.

Lemma 2.3 (Compare [1, Lem. A.6])**.** *Let $g \in \mathsf{GL}_d(\mathbb{K})$, and write $g_1 = k_g \exp(\mu(g))k'_g$. For any $x \in \mathbb{P}(\mathbb{K}^d) \smallsetminus \mathbb{P}((k'_g)^{-1}e_1^\perp)$ we have the estimate*

$$d_{\mathbb{P}}\left(gx, [k_g e_1]\right) \leqslant \frac{\sigma_2}{\sigma_1}(g) \cdot \frac{1}{\operatorname{dist}\left(x, \mathbb{P}((k'_g)^{-1}e_1^\perp)\right)}.$$

Proof. Let $v \in \mathbb{K}^d$ be a unit vector with $x = [v]$. By the definition of the metric $d_{\mathbb{P}}$ we have that

$$\begin{aligned} d_{\mathbb{P}}(gx, [k_g e_1]) &= 1 - \frac{\left|\langle \exp(\mu(g))k'_g v, e_1\rangle\right|^2}{||\exp(\mu(g))k'_g v||} = \frac{\sum_{i=2}^d \sigma_i(g)^2 |k'_g v, e_i|^2}{\sum_{i=1}^d \sigma_i(g)^2 |\langle k_g v, e_i\rangle|^2} \\ &\leqslant \frac{\sigma_2(g)^2}{\sigma_1(g)^2}\frac{1}{|\langle k'_g v, e_1\rangle|^2} = \frac{\sigma_2(g)^2}{\sigma_1(g)^2}\operatorname{dist}\left(x, \mathbb{P}((k'_g)^{-1}e_1^\perp)\right)^{-2}. \end{aligned}$$

□

The next lemma allows us to relate the singular value gaps of a *product* $g_1 g_2$ of elements in $\mathsf{GL}_d(\mathbb{K})$ to the singular values of g_1 and g_2.

Lemma 2.4. *Let $w_1, w_2 \in \mathsf{GL}_d(\mathbb{K})$. Then for $i \in \{1, 2\}$ we have the estimate*

$$\frac{\sigma_1}{\sigma_2}(w_1 w_2) \geqslant \frac{\sigma_d(w_i)^2}{\sigma_1(w_i)^2}\frac{\sigma_1}{\sigma_2}(w_1)\frac{\sigma_1}{\sigma_2}(w_2).$$

Proof. Since $\sigma_1(\wedge^2 g) = \sigma_1(g)\sigma_2(g)$ for all $g \in \mathsf{GL}_d(\mathbb{K})$, we know that

$$\frac{\sigma_1(w_1 w_2)}{\sigma_2(w_1 w_2)} = \frac{\sigma_1^2(w_1 w_2)}{\sigma_1(\wedge^2 w_1 w_2)}.$$

For $i = 1, 2$, we know that $\sigma_1^2(w_1 w_2) \geqslant \sigma_1(w_{3-i})^2 \sigma_d(w_i)^2$, meaning we have

$$\frac{\sigma_1^2(w_1 w_2)}{\sigma_1(\wedge^2 w_1 w_2)} \geqslant \frac{\sigma_1(w_i)\sigma_1(w_{3-i})^2\sigma_d(w_i)^2}{\sigma_1(w_i)\sigma_1(\wedge^2 w_1)\sigma_1(\wedge^2 w_2)} = \frac{\sigma_d(w_i)^2}{\sigma_1(w_i)^2}\frac{\sigma_1(w_1)\sigma_1(w_2)}{\sigma_2(w_1)\sigma_2(w_2)}.$$

□

The next two lemmas also give us an estimate on the singular value gaps of a product $g_1 g_2$, but this time our estimates also involve the relative positions of certain subspaces $\Xi_k(g_1)$, $\Xi_k(g_2)$, and $\Xi_k(g_1 g_2)$.

Lemma 2.5 (Compare [1, Lem. A.7])**.** *Let $g_1, g_2 \in \mathsf{GL}_d(\mathbb{K})$ and for $i \in \{1, 2\}$ write $g_i = k_{g_i}\exp(\mu(g_i))k'_{g_i}$, with $k_{g_i}, k'_{g_i} \in \mathsf{K}_d$. Then the following estimate holds:*

$$\frac{\sigma_1(g_1 g_2)}{\sigma_1(g_1)\sigma_1(g_2)} \geqslant \operatorname{dist}\left([k_{g_2}e_1], \mathbb{P}((k_{g_1})^{-1}e_1^\perp)\right).$$

Proof. A computation shows that

$$\begin{aligned}\frac{\sigma_1(g_1g_2)}{\sigma_1(g_1)\sigma_1(g_2)} &= \left\|k_{g_1}\mathrm{diag}\left(1,\frac{\sigma_2}{\sigma_1}(g_1),\ldots,\frac{\sigma_d}{\sigma_1}(g_1)\right)k'_{g_1}k_{g_2}\mathrm{diag}\left(1,\frac{\sigma_2}{\sigma_1}(g_2),\ldots,\frac{\sigma_d}{\sigma_1}(g_2)\right)k'_{g_2}\right\| \\ &= \left\|\mathrm{diag}\left(1,\frac{\sigma_2}{\sigma_1}(g_1),\ldots,\frac{\sigma_d}{\sigma_1}(g_1)\right)k'_{g_1}k_{g_2}\mathrm{diag}\left(1,\frac{\sigma_2}{\sigma_1}(g_2),\ldots,\frac{\sigma_d}{\sigma_1}(g_2)\right)\right\| \\ &\geqslant \left\|\mathrm{diag}\left(1,\frac{\sigma_2}{\sigma_1}(g_1),\ldots,\frac{\sigma_d}{\sigma_1}(g_1)\right)k'_{g_1}k_{g_2}e_1\right\| \\ &\geqslant \left|\left\langle k'_{g_2}k_{g_1}e_1,e_1\right\rangle\right| = \mathrm{dist}\big([k_{g_2}e_1],\mathbb{P}((k_{g_1})^{-1}e_1^{\perp})\big).\end{aligned}$$

□

Lemma 2.6 (Compare [1, Lem. A.9])**.** *Let $g_1,g_2\in\mathsf{GL}_d(\mathbb{K})$ and write*

$$\begin{aligned} g_1g_2 &= k_{g_1g_2}\exp\left(\mu(g_1g_2)\right)k'_{g_1g_2}, \qquad \textit{for } k_{g_1g_2},k'_{g_1g_2}\in\mathsf{K}_d,\\ g_i &= k_{g_i}\exp\left(\mu(g_i)\right)k'_{g_i}, \quad \textit{for } k_{g_i},k'_{g_i}\in\mathsf{K}_d, \quad \textit{and } i=1,2.\end{aligned}$$

Then the following estimates hold:

(i)

$$d_{\mathbb{P}}\left([k_{g_1g_2}e_1],[k_{g_1}e_1]\right)\leqslant\sqrt{d-1}\frac{\sigma_2(g_1)\sigma_1(g_2)}{\sigma_1(g_1g_2)}\leqslant\frac{\sigma_2}{\sigma_1}(g)\cdot\frac{\sqrt{d-1}}{\mathrm{dist}\big([k_{g_2}e_1],\mathbb{P}\left((k'_{g_1})^{-1}e_1^{\perp}\right)\big)},$$

(ii)

$$d_{\mathsf{Gr}}\left(k_{g_1g_2}e_d^{\perp},k_{g_1}e_d^{\perp}\right)\leqslant\sqrt{d-1}\frac{\sigma_d(g_1g_2)}{\sigma_d(g_2)\sigma_{d-1}(g_1)}\leqslant\frac{\sigma_d}{\sigma_{d-1}}(g_1)\cdot\frac{\sqrt{d-1}}{\mathrm{dist}\big([(k'_{g_1})^{-1}e_d],\mathbb{P}(k_{g_2}e_d^{\perp})\big)}.$$

Proof. To prove (i), note that

$$k_{g_1}^{-1}k_{g_1g_2}\exp(\mu(g_1g_2))=\exp(\mu(g_1))k'_{g_1}k_{g_2}\exp(\mu(g_2))k'_{g_2}(k'_{g_1g_2})^{-1}$$

and hence for $2\leqslant i\leqslant d$ we have

$$\begin{aligned}\sigma_1(g_1g_2)\left|\left\langle k_{g_1}^{-1}k_{g_1g_2}e_1,e_i\right\rangle\right| &= \left|\left\langle k_{g_1}^{-1}k_{g_1g_2}\exp(\mu(g_1g_2))e_1,e_i\right\rangle\right| \\ &= \left|\left\langle \exp(\mu(g_1))k'_{g_1}k_{g_2}\exp(\mu(g_2))k'_{g_2}(k'_{g_1g_2})^{-1}e_1,e_i\right\rangle\right| \\ &= \sigma_i(g_1)\left|\left\langle k'_{g_1}k_{g_2}\exp(\mu(g_2))k'_{g_2}(k'_{g_1g_2})^{-1}e_1,e_i\right\rangle\right| \\ &\leqslant \sigma_i(g_1)\sigma_1(g_2).\end{aligned}$$

Then as

$$1=||k_{g_1}^{-1}k_{g_1g_2}e_1||^2=\sum_{i=1}^d\left|\left\langle k_{g_1}^{-1}k_{g_1g_2}e_1,e_i\right\rangle\right|^2=\left|\left\langle k_{g_1}^{-1}k_{g_1g_2}e_1,e_1\right\rangle\right|^2+\sum_{i=2}^d\left|\left\langle k_{g_1}^{-1}k_{g_1g_2}e_1,e_i\right\rangle\right|^2,$$

we have that $d_{\mathbb{P}}\left([k_{g_1g_2}e_1],[k_{g_1}e_1]\right)\leqslant\sqrt{d-1}\frac{\sigma_2(g_1)\sigma_1(g_2)}{\sigma_1(g_1g_2)}$. This proves the left-hand inequality in (i); the right-hand inequality follows from Lemma 2.5. The proof of (ii) is analogous. □

Finally we close this subsection with the following elementary observation:

Lemma 2.7. *Let $x\in\mathbb{P}(\mathbb{K}^d)$, $0<\theta<1$ and $V\in\mathsf{Gr}_{d-1}(\mathbb{K}^d)$. There exists $y\in B_\theta(x)$ such that* $\mathrm{dist}(y,\mathbb{P}(V))\geqslant\frac{\theta}{2}$.

Proof. Without loss of generality assume $V=e_1^{\perp}$. Let us write $x=[v_0]$, $v_0=\sum_{i=1}^d x_ie_i$. If $|x_1|\geqslant\frac{\theta}{2}$ then $\mathrm{dist}\big([v_0],\mathbb{P}(e_1^{\perp})\big)\geqslant\frac{\theta}{2}$ and the statement holds with $y=x$. So, suppose that $|x_1|<\frac{\theta}{2}$. Let w_0 be the vector $\sum_{i=2}^d x_ie_i$, so that $v_0=x_1e_1+w_0$, and consider the unit vector

$$v'_0=\frac{\theta}{2}e_1+\frac{\sqrt{1-\theta^2/4}}{||w_0||}w_0.$$

Note that $\mathrm{dist}([v_0'], \mathbb{P}(e_1^\perp)) = \frac{\theta}{2}$ and also

$$\begin{aligned}
|\langle v_0', v_0\rangle| &\geqslant \left|\left\langle w_0, \frac{\sqrt{1-\theta^2/4}}{||w_0||} w_0\right\rangle\right| - |x_1|\frac{\theta}{2} \\
&\geqslant ||w_0||\sqrt{1-\frac{\theta^2}{4}} - \frac{\theta^2}{4} \\
&= \sqrt{1-|x_1|^2}\sqrt{1-\frac{\theta^2}{4}} - \frac{\theta^2}{4} \\
&\geqslant 1 - \frac{\theta^2}{2} \geqslant \sqrt{1-\theta^2},
\end{aligned}$$

hence $d_{\mathbb{P}}[v_0'], [v_0] \leqslant \theta$. □

2.4. **Caratheodory metrics on subsets of projective space.** We need to introduce one more technical tool before we can turn to the proof of Theorem 1.2, namely a metric defined by Zimmer [19] on certain open subsets of $\mathbb{P}(\mathbb{K}^d)$. The definition of the metric relies on a pair of embeddings

$$\iota : \mathbb{P}(\mathbb{K}^d) \hookrightarrow \mathbb{P}(\mathbb{R}^n), \ \iota^* : \mathsf{Gr}_{d-1}(\mathbb{K}^d) \hookrightarrow \mathsf{Gr}_{n-1}(\mathbb{R}^n),$$

satisfying the property that $x \in \mathbb{P}(\mathbb{K}^d)$, $W \in \mathsf{Gr}_{d-1}(\mathbb{K}^d)$ are transverse if and only if $\iota(x)$, $\iota^*(W)$ are transverse. When $\mathbb{K} = \mathbb{R}$, we can take ι, ι^* to be the identity maps. When $\mathbb{K} = \mathbb{C}$, we identify $\mathbb{C}^d$ with $\mathbb{R}^{2d}$, which realizes $\mathbb{P}(\mathbb{C}^d)$ as a submanifold of $\mathsf{Gr}_2(\mathbb{R}^{2d})$, and then take ι to be the restriction of the Plücker embedding $\mathsf{Gr}_2(\mathbb{R}^{2d}) \to \mathbb{P}(\wedge^2\mathbb{R}^{2d})$. The map ι^* is defined dually, using the natural identifications $\mathsf{Gr}_{d-1}(\mathbb{C}^d) \simeq \mathbb{P}((\mathbb{C}^d)^*)$ and $\mathsf{Gr}_{n-1}(\mathbb{R}^n) \simeq \mathbb{P}((\mathbb{R}^n)^*)$.

Let U be any nonempty open subset of $\mathbb{P}(\mathbb{K}^d)$. We say that U is a *proper domain* if there exists some $W \in \mathsf{Gr}_{d-1}(\mathbb{K}^d)$ which is transverse to every $x \in \overline{U}$. When U is a proper domain, let $U^* \subset \mathsf{Gr}_{d-1}(\mathbb{K}^d)$ denote the (nonempty) set

$$\left\{W \in \mathsf{Gr}_{d-1}(\mathbb{K}^d) : v \notin W, \ \forall [v] \in \overline{U}\right\}.$$

Recall that the *cross-ratio* $\chi : \mathbb{P}(\mathbb{R}^n)^2 \times \mathsf{Gr}_{n-1}(\mathbb{R}^n)^2 \to \mathbb{R}$ can be defined by the formula

$$\chi([v_1], [v_2]; u_1^\perp, u_2^\perp) := \frac{\langle v_1, u_1\rangle\langle v_2, u_2\rangle}{\langle v_1, u_2\rangle\langle v_2, u_1\rangle},$$

whenever $[v_1], [v_2]$ are both transverse to $u_1^\perp, u_2^\perp$. Now, when U is a proper domain, define a function $\mathsf{d}_U : U \times U \to \mathbb{R}_{\geqslant 0}$ as follows: for $[v_1], [v_2] \in U$, take

$$\mathsf{d}_U([v_1], [v_2]) := \sup_{u_1^\perp, u_2^\perp \in U^*} \log\left|\chi(\iota([v_1]), \iota([v_2]); \iota^*(u_1^\perp), \iota^*(u_2^\perp))\right|.$$

When U is a *properly convex domain* in real projective space (see Section 4.1), then d_U is, up to a constant multiplicative factor, precisely the well-known *Hilbert metric* on U. More generally, d_U satisfies the following properties, which we will use in the sequel:

Theorem 2.8. *For proper domains U, U_1, U_2 in $\mathbb{P}(\mathbb{K}^d)$:*

(1) *The function d_U defines a proper metric on U which induces the subspace topology on U as an open subset of $\mathbb{P}(\mathbb{K}^d)$.*

(2) *The metric d_U is $\mathsf{GL}_d(\mathbb{K})$-invariant, in the sense that for any $g \in \mathsf{GL}_d(\mathbb{K})$ and $x, y \in U$, we have*
$$\mathsf{d}_U(x, y) = \mathsf{d}_{gU}(gx, gy).$$

(3) *If $U_1 \subseteq U_2$, then $\mathsf{d}_{U_1} \geqslant \mathsf{d}_{U_2}$.*

(4) *If $\overline{U_1} \subset U_2$, then there exists a constant $\lambda > 1$ depending only on U_1, U_2 so that for any $x, y \in U_1$, we have*
$$\mathsf{d}_{U_1}(x, y) \geqslant \lambda \mathsf{d}_{U_2}(x, y).$$

Proof. (1) is [19, Thm. 5.2], and (2) and (3) are immediate from the definition of d_U. (4) is [18, Prop. 7.11]. ☐

Remark 2.9.

(a) In general, the metric d_U need not be complete; in the case $\mathbb{K} = \mathbb{R}$, d_U is complete if and only if U is a properly convex open subset of $\mathbb{P}(\mathbb{R}^d)$.

(b) The construction of d_U in [19] also makes sense in the situation where U is an open subset of some Grassmannian $\mathsf{Gr}_k(\mathbb{K}^d)$, for any $1 \leqslant k < d$; however, in this paper we only need to work with d_U in the situation where $k = 1$.

3. Estimates for 1-divergent semigroups

In this section we prove Theorem 1.2, as well as its corollaries Corollary 1.3 and Corollary 1.4. Throughout this section we assume that we have two semigroups $\Gamma_1, \Gamma_2 \subset \mathsf{GL}_d(\mathbb{K})$ which are in ping-pong position relative to subsets $U_1, U_2 \subset \mathbb{P}(\mathbb{K}^d)$, $V_1, V_2 \subset \mathsf{Gr}_{d-1}(\mathbb{K}^d)$. Recall that this means that whenever $\{i, j\} = \{1, 2\}$, the following conditions hold:

(P1) The sets $\overline{U_i}$ and $\overline{V_j}$ are transverse.

(P2) For every $\gamma \in \Gamma_i \smallsetminus \{I_d\}$, we have $\gamma\overline{U_j} \subset U_i$, and $\gamma^{-1}\overline{V_j} \subset V_i$.

3.1. **Setup for the proof.** By condition (P1) above, we may fix $\varepsilon > 0$ so that for $\{i, j\} = \{1, 2\}$, we have

$$\operatorname{dist}\left(\overline{U}_i, \overline{V_j}\right) := \inf\left\{\operatorname{dist}(x, y) : x \in \overline{U_i}, y \in \overline{V_j}\right\} \geqslant \varepsilon. \tag{3}$$

We also fix $x_i \in U_i$ (resp. $y_i \in V_i$) and $0 < \theta < \varepsilon^2$, depending only on U_1, U_2, V_1, V_2, such that $B_\theta(x_i) \subset U_i$ (resp. $B_\theta(y_i) \subset V_i$) for $i = 1, 2$.

Let Δ be the semigroup $\langle \Gamma_1, \Gamma_2 \rangle$. We may view each element $g \in \Delta$ as a reduced alternating word $\gamma_1 \cdots \gamma_n$. The correspondence between such alternating words and elements of Δ need not be bijective. However, this does not matter for any of the arguments below, so we will frequently identify elements in Δ with such reduced words.

Let $M = \max\left\{\frac{16}{\varepsilon\theta}, \frac{8\sqrt{d-1}}{\varepsilon^2}\right\}$, and let $F \subset \Delta$ be the set of all elements $h \in \Delta$ satisfying $\frac{\sigma_1}{\sigma_2}(h) < M$; in other words F is a minimal subset such that $\Delta \smallsetminus F$ satisfies

$$\frac{\sigma_1}{\sigma_2}(h) \geqslant \max\left\{\frac{16}{\varepsilon\theta}, \frac{8\sqrt{d-1}}{\varepsilon^2}\right\} \quad \forall h \in \Delta \smallsetminus F. \tag{4}$$

Later we will show that Δ is 1-divergent, which means that F is actually a finite set. For now, since we know that Γ_1 and Γ_2 are 1-divergent, we know that each of $\Gamma_1 \cap F$ and $\Gamma_2 \cap F$ is finite.

Lemma 3.1. *Let $g \in \Delta \smallsetminus F$ and $g = g_1 \cdots g_n$, $n \geqslant 1$ be a reduced word in $\Delta \smallsetminus F$, with $g_i \in \Gamma_1 \cup \Gamma_2 \smallsetminus \{I_d\}$.*

(i) if $i(g_1) \in \{1, 2\}$ is the unique index such that $g_1 \in \Gamma_{i(g_1)}$, we have

$$d_{\mathbb{P}}\left(\Xi_1(g), U_{i(g_1)}\right) \leqslant \frac{\varepsilon}{8}.$$

(ii) if $i(g_n) \in \{1, 2\}$ is the unique index such that $g_n \in \Gamma_{i(g_n)}$, we have

$$d_{\mathsf{Gr}}\left(\Xi_{d-1}(g^{-1}), V_{i(g_n)}\right) \leqslant \frac{\varepsilon}{8}.$$

Proof. By the ping-pong conditions (P1) and (P2) we have that $gU_j \subset U_i$, where $i = i(g_n) \in \{1, 2\}$ is the unique index such that $g_1 \in \Gamma_i$ and $j = j(g_n) \in \{1, 2\}$ is the unique index such that $g_n \in \Gamma_{3-j}$. In particular, by Lemma 2.7 we may choose $x \in B_\theta(x_j)$ such that $\operatorname{dist}(x, \Xi_{d-1}(g^{-1})) \geqslant \frac{\theta}{2}$. Thus, since $\frac{\sigma_1}{\sigma_2}(g) > 16(\varepsilon\theta)^{-1}$, by Lemma 2.3 we have that

$$d_{\mathbb{P}}\left(gx, \Xi_1(g)\right) \leqslant \frac{2}{\theta}\frac{\sigma_2}{\sigma_1}(g) \leqslant \frac{\varepsilon}{8}.$$

Since $gx \in U_{i(g_1)}$, part (i) follows. The proof of (ii) is analogous. □

3.2. **Proof of Theorem 1.2.** We begin with the proof of Theorem 1.2 part (i). First, we observe:

Lemma 3.2. *Let $\{i, j\} = \{1, 2\}$. There exists some $\delta > 0$ so that $N_\delta(U_i)$ and $N_\delta(V_j)$ are proper domains (see Section 2.4), and so that for every $\gamma \in \Gamma_i \setminus \{I_d\}$, we have*

$$\gamma N_\delta(U_j) \subset N_{\delta/2}(U_i),$$
$$\gamma^{-1} N_\delta(V_j) \subset N_{\delta/2}(V_i).$$

Proof. The ping-pong conditions imply that both U_i and V_i are proper domains. Then since $\overline{U_i}$ and $\overline{V_i}$ are compact, and transversality is an open condition, there exists a δ so that $N_\delta(U_i)$ and $N_\delta(V_i)$ are proper.

So, we just need to show that there is some δ so that the desired inclusions hold. We will only show this for the sets U_i, U_j, since the inclusions for V_i, V_j are analogous. The first step is to show that there is some $\delta > 0$ and a finite subset $F' \subset \Gamma_i$ so that so that the desired inclusions hold for all $\gamma \in \Gamma_i \setminus F'$. Let $\delta = \varepsilon/2$, where ε is the constant from (3), and let $F \subset \Delta$ be the subset defined in (4). We choose a finite set $F' \subset \Gamma_i$ large enough to contain $F \cap \Gamma_i$, and so that $\frac{\sigma_1}{\sigma_2}(g) \geqslant 64/3\varepsilon^2$ for all $g \in \Gamma_i \setminus F'$.

Fix $x \in N_\delta(U_j)$ and $g \in \Gamma_i \setminus F'$. By Lemma 3.1 we know that

$$d_{\mathsf{Gr}}\left(\Xi_{d-1}(g^{-1}), V_i\right) \leqslant \frac{\varepsilon}{8},$$

and since $d_{\mathbb{P}}(x, U_j) \leqslant \varepsilon/2$ and $\operatorname{dist}\big(\overline{U_j}, \overline{V_i}\big) \geqslant \varepsilon$, we have therefore

$$\operatorname{dist}\big(\Xi_{d-1}(g^{-1}), x\big) \geqslant \frac{3\varepsilon}{8}.$$

Then, as $\frac{\sigma_1}{\sigma_2}(g) \geqslant 64/3\varepsilon^2$, Lemma 2.3 implies that

$$d_{\mathbb{P}}(gx, \Xi_1(g)) \leqslant \frac{\varepsilon}{8}.$$

Then we can apply Lemma 3.1 again to see that $d_{\mathbb{P}}(gx, U_i) \leqslant \varepsilon/4 = \delta/2$.

This shows that for all $\gamma \in \Gamma_i \setminus F'$, we have $\gamma N_\delta(U_j) \subset N_{\delta/2}(U_i)$. Now, since $\overline{U_j}$ is compact, the ping-pong condition (P2) implies that for every $\gamma \in F' \setminus \{\mathrm{id}\}$, there is some $\delta(\gamma) > 0$ so that $\gamma N_{\delta(\gamma)}(U_j) \subset U_i$. So, by replacing δ with the minimum of δ and $\min_{\gamma \in F' \setminus \mathrm{id}} \delta(\gamma)$, we obtain the desired inclusions for every $\gamma \in \Gamma_i \setminus \{I_d\}$. □

Proof of Theorem 1.2 (i). Consider a reduced alternating word $\gamma_1 \cdots \gamma_n$ in $\langle \Gamma_1, \Gamma_2 \rangle$, with $\gamma_1 \in \Gamma_i$ and $\gamma_n \in \Gamma_j$.

For any $r > 0$ and any $U \subset \mathbb{P}(\mathbb{K}^d)$, let U^r denote the r-neighborhood $N_r(U)$. Fix $\delta > 0$ from Lemma 3.2. The previous lemma 3.2 implies that the U_i^δ, U_j^δ are both proper domains, so we may let D_i, D_j respectively denote the diameters of the sets $U_i^{\delta/2}, U_j^{\delta/2}$ with respect to the Caratheodory metrics $\mathsf{d}_{U_i^\delta}, \mathsf{d}_{U_j^\delta}$ from Section 2.4.

We may now inductively apply Lemma 3.2 and Theorem 2.8 (4) to see that there is a uniform constant $\lambda > 1$ so that, with respect to the metric U_i^δ, the diameter of the set $\gamma_1 \cdots \gamma_n U_j$ is at most $\lambda^{-(n-1)} D_j$. Now, observe that the function $\mathsf{d}_{U_i^\delta} : U_i \times U_i \to \mathbb{R}_{\geqslant 0}$ is continuously differentiable. So, since $\overline{U_i}$ is a compact subset of U_i^δ, the metrics $\mathsf{d}_{U_i^\delta}$ and $\mathsf{d}_{\mathbb{P}}$ are bi-Lipschitz equivalent on U_i, and thus the diameter of $\gamma_1 \cdots \gamma_n U_j$ (with respect to the metric $d_{\mathbb{P}}$) is at most $\lambda^{-n} A$ for a uniform constant $A < \infty$.

Let $g = \gamma_1 \cdots \gamma_n$, and pick a Cartan decomposition $g = k_g \exp(\mu(g)) k_g'$. Recall that we have chosen $x_j \in U_j$ and $\theta > 0$ so that $B_\theta(x_j) \subset U_j$. Using Lemma 2.7, fix $x \in B_\theta(x_j)$ so that $\mathsf{d}_{\mathbb{P}}(x, \mathbb{P}(k_g'^{-1} e_1^\perp)) > \theta/2$. Then apply Lemma 2.2 to the balls $B_{\theta/2}(x)$ and $B_r(gx)$ with $r = \lambda^{-n} A$ to obtain the desired estimate. □

The next step is to use the initial estimate above to prove the following:

Proposition 3.3. *The semigroup $\Delta = \langle \Gamma_1, \Gamma_2 \rangle$ is 1-divergent. In particular, the set F defined above is finite.*

For a similar result to the above, see Theorem 3.1 in [5]. We provide a proof of this result since our ping-pong setup is slightly different; in particular we do not assume disjointness of the ping-pong sets U_1, U_2.

Note that we could also get a sharper version of Proposition 3.3 as an immediate consequence of parts (i) and (iii) of Theorem 1.2. However, we need the version above to establish the rest of Theorem 1.2.

Proof of Proposition 3.3. Suppose that the statement of the proposition is false. Then there is an infinite sequence g_n of pairwise distinct elements in Δ such that $\frac{\sigma_1}{\sigma_2}(g_n)$ is bounded. We may write each of these elements as an alternating word

$$g_n = \gamma_{1,n} \cdots \gamma_{m_n,n},$$

so that each $\gamma_{i,n}$ is a nontrivial element in some Γ_j.

Part (i) of Theorem 1.2 immediately implies that m_n is uniformly bounded in n, so we may pass to a subsequence and assume that $m_n = m$ for some fixed m, independent of n. By passing to a further subsequence, we may assume that for every n, we have $\gamma_{1,n} \in \Gamma_j$ and $\gamma_{m,n} \in \Gamma_k$ for some fixed j, k; without loss of generality $j = 1$.

To complete the proof, we use an iterative argument exploiting the relationship between 1-divergence and expansion/contraction in $\mathbb{P}(\mathbb{K}^d)$; this is similar to the approach in e.g. [5, Lem. 3.3]. We first claim that the elements in the sequence $\gamma_{1,n}$ must lie in a fixed finite subset of Γ_1. If not, then after further extraction we may assume that $\gamma_{1,n} \in \Gamma_{1,n} \smallsetminus F$ for all n. Then by Lemma 3.1, we have

$$d_{\mathsf{Gr}}(\Xi_{d-1}\left(\gamma_{1,n}^{-1}\right), V_1) < \frac{\varepsilon}{8},$$

which implies that $\operatorname{dist}\left(\Xi_{d-1}(\gamma_{1,n}^{-1}), x\right) > \frac{7\varepsilon}{8}$ for every $x \in U_2$. Then, by Lemma 2.3, since we assume Γ_1 is 1-divergent, the diameter of the set $\gamma_{1,n}U_2$ tends to zero as $n \to \infty$. By the ping-pong setup, the diameter of

$$g_nU_k = \gamma_{1,n} \ldots \gamma_{m,n}U_k$$

also tends to zero. But, because of Lemma 2.2, this contradicts the fact that $\frac{\sigma_1}{\sigma_2}(g_n)$ is bounded. This shows our claim, meaning that the elements appearing in the sequence $\gamma_{1,n}$ lie in some finite subset of Γ_1.

Then, since the elements in g_n are pairwise distinct, a subsequence of $g_n' := \gamma_{1,n}^{-1}g_n = \gamma_{2,n} \ldots \gamma_{m,n}$ is pairwise distinct; further, by Lemma 2.4, we also know that $\frac{\sigma_1}{\sigma_2}(\gamma_{2,n} \ldots \gamma_{m,n})$ is bounded. Arguing iteratively as above, we eventually see that (after further extraction) each of the m sequences $\gamma_{\ell,n}, 1 \leqslant \ell \leqslant m$ lies in a finite subset of some Γ_j, contradicting the fact that the g_n are pairwise distinct. ☐

The proof of the rest of Theorem 1.2 essentially follows the same rough outline as the proof of Proposition 3.3, but with more precise estimates throughout.

Proof of Theorem 1.2 (ii). Via Proposition 3.3, we can define $C_0 := \max_{f \in F} \frac{\sigma_1}{\sigma_d}(f)$ and set

$$C_2 := \max\left\{\frac{2}{\varepsilon}, C_0^2\right\}.$$

We shall prove, using induction on $n \in \mathbb{N}$, that for every reduced word $g_1 \cdots g_n \in \Delta$, $g_i \in \Gamma_i \smallsetminus \{I_d\}$, we have that

$$\sigma_1\left(g_1 \cdots g_n\right) \geqslant C_2^{-n}\sigma_1(g_1) \cdots \sigma_1(g_n). \tag{5}$$

Before we proceed with the induction, we observe the following. For any $h_1, h_2 \notin F$ and any reduced word $h_i \in \Delta$ of the form $h_i = h_{i1} \cdots h_{im_i}$, such that $h_{1m_1} \in \Gamma_{i_1}$, $h_{2m_2} \in \Gamma_{i_2}$, $i_1 \neq i_2$, by Lemma 3.1 (i) and (ii) and (3) we have that

$$\mathrm{dist}\big(\Xi_1(h_2), \Xi_{d-1}(h_1^{-1})\big) \geqslant \inf_{y \in V_{i_1}} \mathrm{dist}(\Xi_1(h_2), y) - \frac{\varepsilon}{8} \geqslant \mathrm{dist}(U_{i_2}, V_{i_1}) - \frac{3\varepsilon}{8} > \frac{\varepsilon}{2}. \tag{6}$$

Note that the desired statement holds trivially when $n = 1$, so assume that (5) holds for reduced words in n elements, and fix a reduced word $g_1 \cdots g_n g_{n+1} \in \Delta$. If either $g_1 \cdots g_n \in F$ or $g_{n+1} \in F$, then note that

$$\sigma_1\left(g_1 \cdots g_n g_{n+1}\right) \geqslant C_0^{-1} \sigma_1(g_1 \cdots g_n)\sigma_1(g_{n+1}) \geqslant C_2^{-n-1}\sigma_1(g_1) \cdots \sigma_1(g_{n+1})$$

and the statement holds true. So, suppose that $g_1 \cdots g_n \notin F$ and $g_{n+1} \notin F$.

Since $g_1 \cdots g_n g_{n+1} \in \Delta$ is reduced, by (6) and Lemma 2.5 we have that

$$\frac{\sigma_1(g_1 \cdots g_n g_{n+1})}{\sigma_1(g_1 \cdots g_n)\sigma_1(g_{n+1})} \geqslant \mathrm{dist}\big(\Xi_1(g_{n+1}), \Xi_{d-1}((g_1 \cdots g_n)^{-1})\big) \geqslant \frac{\varepsilon}{2}.$$

In particular, by the inductive hypothesis, since $\frac{\varepsilon}{2} \geqslant C_2^{-1}$, we have that

$$\sigma_1\left(g_1 \cdots g_n g_{n+1}\right) \geqslant C_2^{-n-1}\sigma_1(g_1) \cdots \sigma_1(g_{n+1}).$$

This completes the proof of the induction and (5) follows. □

Proof of Theorem 1.2 (iii). Again due to Proposition 3.3, we can define

$$C_3 := \max\left\{\max_{f \in F}\left(\frac{\sigma_1}{\sigma_d}(g)\right)^2, \frac{2^9}{7\varepsilon^3}\right\}.$$

We shall use induction to prove the following bound: every reduced word $g_1 \cdots g_n \in \Delta$ satisfies the estimate

$$\frac{\sigma_1}{\sigma_2}\left(g_1 \cdots g_n\right) \geqslant C_3^{-n}\frac{\sigma_1}{\sigma_2}(g_1) \cdots \frac{\sigma_1}{\sigma_2}(g_n). \tag{7}$$

The bound holds trivially for $n = 1$. Suppose now that we know that (7) holds for $n \in \mathbb{N}$. Fix a reduced word $g_1 \ldots, g_n g_{n+1} \in \Delta$, such that $g_{n+1} \in \Gamma_{j_1}$, $g_n \in \Gamma_{j_2}$, $j_1 \neq j_2$. If either $g_{n+1} \in F$ or $g_1 \cdots g_n \in F$, by Lemma 2.4, we have that

$$\frac{\sigma_1}{\sigma_2}(g_1 \cdots g_n g_{n+1}) \geqslant C_3^{-1}\frac{\sigma_1}{\sigma_2}(g_1 \cdots g_n)\frac{\sigma_1}{\sigma_2}(g_{n+1}) \geqslant C_3^{-n-1}\frac{\sigma_1}{\sigma_2}(g_1) \cdots \frac{\sigma_1}{\sigma_2}(g_{n+1})$$

and (7) holds for $g_1 \cdots g_n g_{n+1} \in \Delta$.

Now suppose that $h_n := g_1 \cdots g_n \notin F$ and $g_{n+1} \notin F$. Consider the ball $B_\theta(x_{j_2}) \subset U_{j_2}$ and note that by (3) and Lemma 3.1 (ii) that $\mathrm{dist}(x_{j_2}, \Xi_{d-1}(g_{n+1}^{-1})) \geqslant \frac{7\varepsilon}{8}$. In particular, Lemma 2.1 implies that

$$g_{n+1}B_\theta(x_{j_2}) \subset B_{\epsilon_{n+1}}(g_{n+1}x_{j_2}),$$

where $\epsilon_{n+1} := \frac{16\theta}{7\varepsilon}\frac{\sigma_2}{\sigma_1}(g_{n+1})$. In addition observe that

$$d_{\mathbb{P}}\left(g_{n+1}x_{j_2}, U_{j_1}\right) \leqslant d_{\mathbb{P}}\left(g_{n+1}x_{j_2}, \Xi_1(g_{n+1})\right) + d_{\mathbb{P}}\left(\Xi_1(g_{n+1}), U_{j_1}\right) \leqslant \frac{8}{7\varepsilon}\frac{\sigma_2}{\sigma_1}(g_{n+1}) + \frac{\varepsilon}{8} < \frac{3\varepsilon}{8}.$$

Since $g_1 \cdots g_n g_{n+1} \in \Delta$ is reduced and $h_n = g_1 \cdots g_n \notin F$, by Lemma 3.1 (ii) we have that

$$\mathrm{dist}\big(g_{n+1}x_{j_2}, \Xi_{d-1}(h_n^{-1})\big) \geqslant \mathrm{dist}(U_{j_1}, V_{j_2}) - \frac{3\varepsilon}{8} - \frac{\varepsilon}{8} \geqslant \frac{\varepsilon}{2}.$$

By applying Lemma 2.1 for the action of h_n on the ball $B_{\epsilon_{n+1}}(g_{n+1}x_{j_2})$ and the previous estimate, we conclude that

$$h_n B_{\epsilon_{n+1}}(g_{n+1}x_{j_2}) \subset B_{\epsilon'_{n+1}}(h_n g_{n+1}x_{j_2}), \text{ where}$$

$$\epsilon'_{n+1} := \frac{4}{\varepsilon}\frac{\sigma_2}{\sigma_1}(h_n)\epsilon_{n+1} = \frac{64\theta}{7\varepsilon^2}\frac{\sigma_2}{\sigma_1}(g_{n+1})\frac{\sigma_2}{\sigma_1}(h_n).$$

At this point note that we do not necessarily know that $h_n g_{n+1} \notin F$, nor that $h_n g_{n+1} \in \Delta$ has a gap of index 1. However, we have assumed $g_{n+1}, h_n \notin F$, so if we write $h_n g_{n+1} = k_n \exp(\mu(h_n g_{n+1}))k_n'$ for some $k_n, k_n' \in \mathsf{K}_d$, by Lemma 2.6 (ii), we have that

$$d_{\mathsf{Gr}}\left((k_n')^{-1}e_d^{\perp}, \Xi_{d-1}(g_{n+1}^{-1})\right) \leqslant \frac{\sigma_2}{\sigma_1}(g_{n+1}) \cdot \frac{\sqrt{d-1}}{\mathrm{dist}\left(\Xi_1(g_{n+1}), \Xi_{d-1}(h_n^{-1})\right)}. \tag{8}$$

At this point, since $g_{n+1} \notin F$, by (4), we have $\frac{\sigma_1}{\sigma_2}(g_{n+1}) > \frac{8\sqrt{d-1}}{\varepsilon^2}$. In addition, by Lemma 3.1 we know that $d_{\mathsf{Gr}}\left(\Xi_{d-1}(h_n^{-1}), V_{j_2}\right) \leqslant \frac{\varepsilon}{8}$, hence

$$\mathrm{dist}\left(\Xi_1(g_{n+1}), \Xi_{d-1}(h_n^{-1})\right) \geqslant \mathrm{dist}\left(U_{j_1}, V_{j_2}\right) - \frac{\varepsilon}{8} - \frac{\varepsilon}{8} \geqslant \frac{3\varepsilon}{4}.$$

The previous bound and (8) imply the bound

$$d_{\mathsf{Gr}}\left((k_n')^{-1}e_d^{\perp}, V_{j_1}\right) \leqslant d_{\mathsf{Gr}}\left((k_n')^{-1}e_d^{\perp}, \Xi_{d-1}(g_{n+1}^{-1})\right) + d_{\mathsf{Gr}}\left(\Xi_{d-1}(g_{n+1}^{-1}), V_{j_1}\right) < \frac{\varepsilon}{8} + \frac{\varepsilon}{4}.$$

In particular, since $x_{j_2} \in U_{j_2}$

$$\mathrm{dist}(x_{j_2}, \mathbb{P}((k_n')^{-1}e_d^{\perp})) \geqslant \mathrm{dist}\left(U_{j_2}, V_{j_1}\right) - \frac{3\varepsilon}{8} > \frac{\varepsilon}{2}.$$

Then, since $g_1 \cdots g_n g_{n+1} B_\theta(x_{j_2}) \subset B_{\epsilon'_{n+1}}(g_1 \cdots g_{n+1} x_{j_2})$, by the previous estimate, Lemma 2.2 and the inductive step, we conclude that

$$\begin{aligned}\frac{\sigma_1}{\sigma_2}(g_1 \cdots g_n g_{n+1}) &\geqslant \frac{\theta}{4\epsilon'_{n+1}} \mathrm{dist}(x_{j_2}, \mathbb{P}((k_n')^{-1}e_d^{\perp})) \geqslant \frac{7\varepsilon^3}{2^9}\frac{\sigma_1}{\sigma_2}(g_1 \cdots g_n)\frac{\sigma_1}{\sigma_2}(g_{n+1}) \\ &\geqslant C_3^{-(n+1)}\frac{\sigma_1}{\sigma_2}(g_1) \cdots \frac{\sigma_1}{\sigma_2}(g_{n+1}).\end{aligned}$$

This concludes the proof of part (iii). ☐

Proof of Theorem 1.2 (iv). Let $n \in \mathbb{N}$ be an even integer and a reduced word $\gamma_1\gamma_2\cdots\gamma_n \in \langle \Gamma_1, \Gamma_2 \rangle$. Since $\gamma_1, \gamma_n \in \Gamma_1 \cup \Gamma_2$ do not lie in the same semigroup, for every $m \in \mathbb{N}$ the word $(\gamma_1 \cdots \gamma_n)^m \in \langle \Gamma_1, \Gamma_2 \rangle$ is reduced and by Theorem 1.2 (ii) we have that

$$\sigma_1\left((\gamma_1\gamma_2\cdots\gamma_n)^m\right) \geqslant C_2^{-nm}\sigma_1(\gamma_1)^m\sigma_1(\gamma_2)^m\cdots\sigma_1(\gamma_n)^m.$$

Therefore, since $\ell_1(\gamma_1\gamma_2\cdots\gamma_n) = \lim_{m\to\infty}\sigma_1\left((\gamma_1\gamma_2\cdots\gamma_n)^m\right)^{\frac{1}{m}}$, we obtain the estimate:

$$\ell_1\left(\gamma_1\gamma_2\cdots\gamma_n\right) \geqslant C_2^{-n}\sigma_1(\gamma_1)\sigma_1(\gamma_2)\cdots\sigma_1(\gamma_n). \tag{9}$$

Using (9) we finally obtain:

$$\begin{aligned}\frac{\ell_1}{\ell_2}\left(\gamma_1\gamma_2\cdots\gamma_n\right) &= \frac{\ell_1(\gamma_1\gamma_2\cdots\gamma_n)^2}{\ell_1(\wedge^2(\gamma_1\gamma_2\cdots\gamma_n))} \geqslant C_2^{-2n}\frac{\sigma_1(\gamma_1)^2\sigma_1(\gamma_2)^2\cdots\sigma_1(\gamma_n)^2}{\sigma_1(\wedge^2\gamma_1\gamma_2\cdots\gamma_n)} \\ &\geqslant C_2^{-2n}\frac{\sigma_1(\gamma_1)^2\sigma_1(\gamma_2)^2\cdots\sigma_1(\gamma_n)^2}{\sigma_1(\wedge^2\gamma_1)\sigma_1(\wedge^2\gamma_2)\cdots\sigma_1(\wedge^2\gamma_n)} \\ &= C_2^{-2n}\frac{\sigma_1}{\sigma_2}(\gamma_1)\frac{\sigma_1}{\sigma_2}(\gamma_2)\cdots\frac{\sigma_1}{\sigma_2}(\gamma_n).\end{aligned}$$

☐

3.3. **Consequences of Theorem 1.2.** Below we prove Corollary 1.3 and Corollary 1.4, and then use Corollary 1.4 to construct an interesting example of a 1-Anosov semigroup mentioned in the introduction. We first make a basic observation:

Proposition 3.4. *In the context of Theorem 1.2, if any of the conditions below hold, then the group $\langle \Gamma_1, \Gamma_2 \rangle$ is naturally isomorphic to the free product $\Gamma_1 * \Gamma_2$:*

(a) the ping-pong sets U_1, U_2 are disjoint,
(b) the ping-pong sets V_1, V_2 are disjoint,

(c) Γ_1 *and* Γ_2 *are both groups.*

Proof. If either (a) or (b) holds, then the result is an immediate consequence of the ping-pong lemma for semigroups. If (c) holds, then combining (i) and (iii) in Theorem 1.2 we see that the kernel of the map $\rho : \Gamma_1 * \Gamma_2 \to \mathsf{GL}_d(\mathbb{K})$ is a finite normal subgroup of the free product $\Gamma_1 * \Gamma_2$. Since Γ_1 and Γ_2 are both included into $\mathsf{GL}_d(\mathbb{K})$, the kernel of ρ is trivial and $\Gamma_1 * \Gamma_2 \to \langle \Gamma_1, \Gamma_2 \rangle$ is an isomorphism. □

Remark 3.5. The conclusion of Proposition 3.4 can fail if all three of (a), (b), and (c) do not hold. For a simple example, consider the semigroup Γ generated by a nontrivial loxodromic element g in $\mathsf{SL}(2,\mathbb{R})$, and take $\Gamma_1 = \Gamma_2 = \Gamma$. Then, if $U_1 = U_2 = U$ is a small neighborhood of the attracting point of g in $\mathbb{P}(\mathbb{R}^2)$, and $V_1 = V_2 = V$ is a small neighborhood of the repelling point in $\mathsf{Gr}_1(\mathbb{R}^2) = \mathbb{P}(\mathbb{R}^2)$, the ping-pong conditions are satisfied, but clearly $\Gamma_1 * \Gamma_2$ is not isomorphic to $\langle \Gamma_1, \Gamma_2 \rangle = \Gamma$.

Proof of Corollary 1.3. Fix finite generating sets for Γ_1, Γ_2, which induce a word metric $|\cdot|$ on Γ_1, Γ_2, and the abstract free product $\Gamma_1 * \Gamma_2$. We have assumed that Γ_1, Γ_2 are quasi-isometrically embedded, which means that there is a uniform constant $L > 0$ such that

$$\sigma_1(\gamma_i) \geqslant L|\gamma_i| \tag{10}$$

for every $\gamma_i \in \Gamma_1 \cup \Gamma_2$.

Let $\gamma_1 \cdots \gamma_n$ be a reduced word in $\langle \Gamma_1, \Gamma_2 \rangle$. By Theorem 1.2 (ii), for a uniform constant $c_1 > 0$ we also have the inequality

$$\sigma_1(\gamma_1 \cdots \gamma_n) \geqslant c_1^n \sigma_1(\gamma_1) \cdots \sigma_1(\gamma_n). \tag{11}$$

Without loss of generality we can assume that $c_1 < 1$, so $\log c_1 < 0$.

Let $g = \gamma_1 \cdots \gamma_n$, and let $\ell = |g|/n$. We consider two cases.

Case 1: $\ell \geqslant \frac{-2}{L} \log c_1$**:** In this case (10) and (11) give us

$$\begin{aligned}\log(\sigma_1(g)) &\geqslant n \log(c_1) + \sum_{i=1}^{n} L|\gamma_i| \\ &= \frac{|g|}{\ell} \log(c_1) + L|g| \\ &= |g| \left(\frac{1}{\ell} \log(c_1) + L \right) \\ &\geqslant |g| \frac{L}{2}.\end{aligned}$$

Case 2: $\ell \leqslant \frac{-2}{L} \log c_1$**:** In this case, setting $L' = \frac{L}{-2 \log c_1} > 0$, we have $1/\ell \geqslant L'$ and thus

$$n = |g|/\ell \geqslant L'|g|.$$

Then Theorem 1.2 (i) implies that for some uniform $\lambda > 1$ and $c_4 > 0$ we have

$$\log \frac{\sigma_1}{\sigma_2}(g) \geqslant \log(\lambda) L' c_4 |g|.$$

□

Proof of Corollary 1.4. By assumption, there is $V_0 \in \mathsf{Gr}_{d-1}(\mathbb{K}^d)$ with $\Lambda_1(\Gamma) \subset \mathbb{P}(\mathbb{K}^d) \smallsetminus \mathbb{P}(V_0)$. Since the limit set of the dual group Γ^* is contained in an affine chart of $\mathbb{P}(\mathbb{K}^d)$, the complement of $\bigcup_{W \in \Lambda_{d-1}(\Gamma)} \mathbb{P}(W)$ is a non-empty closed subset of $\mathbb{P}(\mathbb{K}^d)$. In particular, we may choose $[v_0] \in \mathbb{P}(\mathbb{K}^d) \smallsetminus \mathbb{P}(V_0)$ and $\varepsilon > 0$ with

$$\mathrm{dist}\,([v_0], \mathbb{P}(W)) \geqslant 6\varepsilon, \ \forall\ W \in \Lambda_{d-1}(\Gamma).$$

Now let $g \in \mathsf{GL}_d(\mathbb{K})$ be a 1-proximal matrix with attracting fixed point in $[v_0] \in \mathbb{P}(\mathbb{K}^d)$ and repelling hyperplane $V_0 \in \mathsf{Gr}_{d-1}(\mathbb{K})$. By the previous choices, we may pass to a finite-index subgroup $\Gamma' < \Gamma$ and choose $r > 0$ large enough such that for every $n \in \mathbb{N}$,

$$\left(\Gamma' \smallsetminus \{I_d\}\right) B_\varepsilon([v_0]) \subset N_\varepsilon(\Lambda_1(\Gamma)),\ g^{rn} N_\varepsilon\left(\Lambda_1(\Gamma)\right) \subset B_\varepsilon([v_0])$$
$$\left(\Gamma' \smallsetminus \{I_d\}\right) N_\varepsilon(\mathbb{P}(V_0)) \subset N_\varepsilon(\Lambda_{d-1}(\Gamma)),\ g^{-rn} N_\varepsilon(\Lambda_{d-1}(\Gamma)) \subset N_\varepsilon(\mathbb{P}(V_0)).$$

By the previous inclusions we deduce that the semigroups $\Gamma' < \Gamma$ and $\{g^{rm} : m \geqslant 0\}$ are in ping-pong position in the sense of Definition 1.1; moreover, the relevant ping-pong sets are also disjoint by construction. Then by Proposition 3.4 the semigroup $\langle \Gamma', g^r \rangle < \mathsf{GL}_d(\mathbb{K})$ is isomorphic to the free product of Γ' and $H = \{g^{rm} : m \geqslant 0\}$. Then passing to a further finite-index subgroup $\Gamma'' < \Gamma'$, and increasing r if necessary, we can use Theorem 1.2 (iii) to conclude that the semigroup is 1-Anosov (alternatively, one may also apply Theorem 1.2 (i) as in the proof of Corollary 1.8 to see that the free product of Γ' and H is already 1-Anosov). ☐

3.3.1. *A 1-Anosov semigroup which does not generate a 1-Anosov subgroup.* Let $\Gamma < \mathsf{GL}_d(\mathbb{R})$ be a discrete Gromov-hyperbolic group which acts properly discontinuously and cocompactly on a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ (see Section 4.1 below). Then the inclusion $\Gamma \hookrightarrow \mathsf{GL}_d(\mathbb{R})$ is 1-Anosov by [9, Prop. 6.1]. By Corollary 1.4, we can replace Γ with some finite-index subgroup of itself and find some $w \in \mathsf{GL}_d(\mathbb{R})$ so that the semigroup $\langle \Gamma, w \rangle \subset \mathsf{GL}_d(\mathbb{R})$ is 1-Anosov and isomorphic to the free product of Γ with the semigroup $\{w^n : n \geqslant 0\}$. However, the group generated by $\Gamma \cup \{w^{\pm 1}\}$ fails to be 1-Anosov, since Γ cannot be embedded as an infinite index subgroup of any 1-Anosov subgroup of $\mathsf{GL}_d(\mathbb{R})$ (see [2, Thm. 1.5 & 1.7]).

4. Free products of quasi-isometrically embedded discrete groups

In this section we prove Theorem 1.6 and its consequence Corollary 1.8.

4.1. **Properly convex domains.** Our proof of Theorem 1.6 requires us to work with *properly convex domains* in real projective space. Recall that an open subset $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain if its closure is a bounded convex subset of some affine chart in $\mathbb{P}(\mathbb{R}^d)$. Equivalently, Ω is the projectivization of a *properly convex cone* $\tilde{\Omega}$ in $\mathbb{R}^d$, i.e. a convex subset of $\mathbb{R}^d$ invariant under multiplication by positive scalars whose closure does not contain any k-plane in $\mathbb{R}^d$ for $k > 0$.

If Ω is a properly convex domain, then the set of elements $g \in \mathsf{GL}_d(\mathbb{R})$ preserving Ω is called the *automorphism group* and denoted $\mathrm{Aut}(\Omega)$. When viewed as a subgroup of $\mathsf{PGL}_d(\mathbb{R})$ by projectivization, $\mathrm{Aut}(\Omega)$ acts properly on Ω.

We need the following basic result on sequences in $\mathrm{Aut}(\Omega)$:

Lemma 4.1. *Let K be a compact subset of a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$. There exists $\varepsilon > 0$ and $L > 0$ so that whenever $g \in \mathrm{Aut}(\Omega)$ satisfies $\frac{\sigma_k}{\sigma_{k+1}}(g) > L$, then*

$$\inf_{x \in K} \mathrm{dist}(x, \mathbb{P}(\Xi_k(g))) > \varepsilon.$$

Proof. Suppose the statement is false. Then there is an index k and a sequence $g_n \in \mathrm{Aut}(\Omega)$ with $\frac{\sigma_k}{\sigma_{k+1}}(g_n) \to \infty$, satisfying

$$\inf_{x \in K} \mathrm{dist}(x, \mathbb{P}(\Xi_k(g_n))) \to 0.$$

After extracting a subsequence, $(\Xi_k(g_n))_{n \in \mathbb{N}}$ converges to some k-plane V_k, with $\mathbb{P}(V_k) \cap K \neq \varnothing$. After further extraction, $(\Xi_{d-k}(g_n^{-1}))_{n \in \mathbb{N}}$ also converges to some $(d-k)$-plane V_{d-k}.

Now, if V_k' is any k-plane transverse to V_{d-k}, we must have $g_n \mathbb{P}(V_k') \to \mathbb{P}(V_k)$ with respect to Hausdorff distance on $\mathbb{P}(\mathbb{R}^d)$. Since Ω is properly convex, we can choose such a V_k' so that $\mathbb{P}(V_k') \subset \mathbb{P}(\mathbb{R}^d) \smallsetminus \overline{\Omega}$.

Let $r > 0$ be such that $N_r(K) \subset \Omega$. Since $\mathbb{P}(V_k)$ intersects K, for all sufficiently large n there is some $x_n \in \mathbb{P}(V_k')$ with $g_n x_n \in N_r(K)$, hence $g_n x_n \in \Omega$. But since $x_n \notin \Omega$ this contradicts the fact that $g_n \in \mathrm{Aut}(\Omega)$. □

4.2. **Establishing the theorem.** The proposition below is the key step in the proof of Theorem 1.6.

Proposition 4.2. *Let $\eta \in (0,1)$, let Ω be a properly convex domain in $\mathbb{P}(\mathbb{R}^d)$, and let Γ be a subgroup of $\mathrm{Aut}(\Omega) < \mathsf{GL}_d(\mathbb{R})$ satisfying $\sigma_1(g)\sigma_d(g) = 1$ for all $g \in \Gamma$. There exists:*

- *a compact neighborhood $\mathcal{K} \subset \mathsf{GL}_d(\mathbb{R})$ of the identity;*
- *a pair of inclusions $\rho_1, \rho_2 : \Gamma \hookrightarrow \mathsf{GL}_{d+1}(\mathbb{R})$, and*
- *a pair of subsets $\mathcal{C}_1, \mathcal{C}_2 \subset \mathbb{P}(\mathbb{R}^{d+1})$ with $\overline{\mathcal{C}_1} \cap \overline{\mathcal{C}_2} = \varnothing$*

such that, for any $g \in \Gamma \setminus \mathcal{K}$, the following two conditions hold:

(a) for $i = 1, 2$, we have $\rho_i(g)\mathcal{C}_{3-i} \subset \mathcal{C}_i$, and
(b) for $i = 1, 2$ and any $[v] \in \mathcal{C}_{3-i}$, we have

$$||\rho_i(g)v|| \geqslant \sigma_1(g)^{1-\eta}||v||.$$

To prove the proposition we need one more technical estimate.

Lemma 4.3. *Let $g \in \mathsf{GL}_d(\mathbb{K})$ satisfy $\sigma_d(g) = \sigma_1(g)^{-1}$, $d \geqslant 4$ and $0 < \epsilon < \frac{2}{d-1}$. There exists $k \in \{1, \dots, d-1\}$ such that*

$$\sigma_k(g) \geqslant \sigma_1(g)^{1-d\epsilon}, \quad \frac{\sigma_k}{\sigma_{k+1}}(g) \geqslant \sigma_1(g)^{\epsilon}.$$

Proof. Note that $\prod_{i=1}^{d-1} \frac{\sigma_i}{\sigma_{i+1}}(g) = \sigma_1(g)^2$. Consequently, there is at least one index $j \in \{1, \dots, d-1\}$ such that

$$\frac{\sigma_j}{\sigma_{j+1}}(g) \geqslant \sigma_1(g)^{\frac{2}{p-1}} \geqslant \sigma_1(g)^{\epsilon}.$$

Let k be a minimal such j. If $k = 1$, the lemma follows. If $k > 1$, by the minimality of k, for $1 \leqslant i < k$ we have $\frac{\sigma_i}{\sigma_{i+1}}(g) \leqslant \sigma_1(g)^{\epsilon}$ and therefore $\sigma_i(g) \geqslant \sigma_1(g)^{1-(i-1)\epsilon}$. The lemma follows. □

Proof of Proposition 4.2. Identify $\mathbb{R}^d$ with a subspace $V \subset \mathbb{R}^{d+1}$, and let U be an orthogonal complement to V in $\mathbb{R}^{d+1}$. Then $\mathsf{GL}_d(\mathbb{R})$ is identified with $\mathsf{GL}(V)$. We implicitly extend the action of $\mathsf{GL}(V)$ to an action on $\mathbb{R}^{d+1}$ by letting $\mathsf{GL}(V)$ act by the identity on U, and define the representation ρ_1 to be the restriction of this action to $\Gamma < \mathsf{GL}(V)$. We define the set $\mathcal{C}_1$ by taking it to be an open neighborhood of Ω in $\mathbb{P}(\mathbb{R}^{d+1})$, chosen to be small enough so that the closure of $\mathcal{C}_1$ is contained in an affine chart, and does not contain the point $\mathbb{P}(U)$.

The representation ρ_2 will be a conjugate of ρ_1 by some fixed element $w \in \mathsf{GL}_{d+1}(\mathbb{R})$. To find w, first let $\tilde{\Omega} \subset V$ be a convex cone lifting Ω. Observe that the sum $\tilde{\Omega} + U$ is an open (non-properly) convex cone $\tilde{C} \subset \mathbb{R}^{d+1}$. The projectivization C of this cone contains a copy of Ω, viewed as a subset of $\mathbb{P}(V) \subset \mathbb{P}(\mathbb{R}^{d+1})$.

Note that since $\overline{\mathcal{C}_1}$ does not contain $\mathbb{P}(U)$, the set $C \setminus \overline{\mathcal{C}_1}$ is nonempty and open. So, we can let $\mathcal{C}_2$ be an open ball in $\mathbb{P}(\mathbb{R}^{d+1})$ such that $\overline{\mathcal{C}}_2 \subset C \setminus \overline{\mathcal{C}_1}$ Then, we can choose a $\{1, d\}$-proximal element $w \in \mathsf{GL}_{d+1}(\mathbb{R})$ whose attracting and repelling fixed points both lie in $\mathcal{C}_2$, and whose attracting and repelling hyperplanes both avoid $\overline{\mathcal{C}_1}$. After replacing w with a sufficiently large power w^n, we can ensure that

$$w\overline{\mathcal{C}_1} \cup w^{-1}\overline{\mathcal{C}_1} \subset \mathcal{C}_2. \tag{12}$$

With this choice of w, we define $\rho_2(g) := w\rho_1(g)w^{-1}$.

It remains to show that we can choose a compact neighborhood $\mathcal{K}$ of the identity so that conditions (a) and (b) above hold for all $g \in \Gamma \setminus \mathcal{K}$. To show (a), we employ a strategy of

Danciger-Guéritaud-Kassel: by [3, Prop. 4.3], there is a family $\{\Omega_t : t \geqslant 0\}$ of $\rho_1(\Gamma)$-invariant properly convex open subsets of $\mathbb{P}(\mathbb{R}^{d+1})$ such that $\Omega_t \subset \Omega_{t'}$ for all $t < t'$ and

$$\bigcup_{t\geqslant 0} \Omega_t = C, \quad \bigcap_{t\geqslant 0} \Omega_t = \Omega.$$

Since $\overline{\mathcal{C}_2} \subset C$, we must have $\overline{\mathcal{C}_2} \subset \Omega_t$ for some $t < \infty$. Moreover, [3, Prop. 4.2] implies that for every $t \geqslant 0$, the *full orbital limit set* of $\rho_1(\Gamma)$ in Ω_t does not depend on t, and therefore is a subset of $\overline{\Omega}$. (When a group G acts on a properly convex domain Ω, the full orbital limit set of G in Ω is the set of all accumulation points in $\partial\Omega$ of G-orbits in Ω.)

Since Ω_t is properly convex, the ρ_1-action of Γ on Ω_t is proper. So, there is a compact neighborhood $\mathcal{K}$ of the identity so that for every $g \in \Gamma \smallsetminus \mathcal{K}$, we have

$$\rho_1(g)\mathcal{C}_2 \subset \mathcal{C}_1.$$

This implies that (a) holds when $i = 1$. The corresponding inclusion when $i = 2$ then follows from (12) and the definition of ρ_2.

Finally we turn to the proof that (b) holds for some choice of $\mathcal{K}$. Note that for any $g \in \Gamma$, if $\rho_i(g)$ has a gap of index k, and we write in the Cartan decomposition

$$\rho_i(g) = k_{g,i}\mathrm{diag}\big(\sigma_1(\rho_1(g)), \dots, \sigma_{d+1}(\rho_i(g))\big)k'_{g,i}, \ \ k_{g,i}, k'_{g,i} \in \mathsf{K}_d$$

then for any $v \in \mathbb{R}^{d+1}$, we have

$$\begin{aligned} \big|\big|\rho_i(g)v\big|\big| = \big|\big|\mathrm{diag}\big(\sigma_1(\rho_i(g)), \dots, \sigma_{d+1}(\rho_i(g))\big)k'_{g,i}v\big|\big| &\geqslant \left(\sum_{j=1}^{k} \sigma_j(\rho_j(g))^2 \big|\langle k'_{g,i}v, e_j\rangle\big|^2\right)^{1/2} \\ &\geqslant \sigma_k(\rho_i(g)) \left(\sum_{j=1}^{k} \big|\langle v, (k'_{g,i})^{-1}e_j\rangle\big|^2\right)^{1/2} \\ &= \sigma_k(\rho_i(g))\mathrm{dist}\big([v], \mathbb{P}(\Xi_{d+1-k}(g^{-1}))\big) \cdot ||v||. \end{aligned} \tag{13}$$

Observe that, since we assume $\sigma_1(g)\sigma_d(g) = 1$ for any $g \in \Gamma$, the diagonal embedding $\rho_1 : \Gamma \to \mathsf{GL}_{d+1}(\mathbb{R})$ satisfies $\sigma_1(g) = \sigma_1(\rho_1(g))$ and $\sigma_d(g) = \sigma_{d+1}(\rho_1(g))$. So, by applying Lemma 4.3 with $\epsilon = \eta/2d$, for any $g \in \Gamma$, we can find some index k (depending on g) so that

$$\sigma_k(\rho_1(g)) \geqslant \sigma_1(g)^{1-\frac{\eta}{2}}, \ \ \frac{\sigma_k}{\sigma_{k+1}}(\rho_1(g)) \geqslant \sigma_1(g)^{\frac{\eta}{2d}}. \tag{14}$$

We also know that for every $g \in \Gamma$, $\rho_1(g)$ preserves the properly convex domain Ω_t containing the compact set $\overline{\mathcal{C}_2}$. So, by using Lemma 4.1 together with the second inequality in (14), we can see that for some $\varepsilon > 0$ and some compact neighborhood $\mathcal{K}$ of the identity, any $g \in \Gamma \smallsetminus \mathcal{K}$ has a gap of some index k, and

$$\inf_{x\in\mathcal{C}_2} \mathrm{dist}\big(x, \Xi_{d+1-k}(g^{-1})\big) \geqslant \varepsilon.$$

Combining this with (13) and the first inequality in (14), we find that for any $[v] \in \mathcal{C}_2$ and $g \in \Gamma \smallsetminus \mathcal{K}$, we have

$$||\rho_1(g)v|| \geqslant \varepsilon \cdot \sigma_1(g)^{1-\frac{\eta}{2}}||v||.$$

By further increasing $\mathcal{K}$ we can also ensure that $\varepsilon > \sigma_1(g)^{-\eta/2}$ for any $g \in \Gamma \smallsetminus \mathcal{K}$, which completes the proof of part (b) when $i = 1$. The case $i = 2$ also follows: since ρ_2 is a conjugate of ρ_1 by some element w, and $w^{-1}\mathcal{C}_1 \subset \mathcal{C}_2$, we can increase $\mathcal{K}$ further and use the $i = 1$ case to see that for any $[v] \in \mathcal{C}_1$ and any $g \in \Gamma \smallsetminus \mathcal{K}$ we have

$$||w^{-1}\rho_2(g)w(w^{-1}v)|| \geqslant \sigma_1(g)^{1-\frac{\eta}{2}}||w^{-1}v||.$$

This implies that for some constant $A > 0$ depending only on w, we have

$$||\rho_2(g)v|| \geqslant A \cdot \sigma_1(g)^{1-\frac{\eta}{2}}||v||.$$

Then by increasing $\mathcal{K}$ even further we get the desired bound in this case as well. ☐

Now we can prove the main result of this section.

Proof of Theorem 1.6. Consider the representations

$$\mathrm{Sym}_d : \mathsf{SL}_d(\mathbb{R}) \to \mathsf{GL}(V),\ \mathrm{Sym}_d(g)X = g^t X g\ X \in V,$$
$$\psi_d : \mathsf{GL}(V) \to \mathsf{GL}(V \otimes V),\ \psi_d(h) = h \otimes h^*\ h \in \mathsf{GL}(V)$$

where $V = \mathrm{Sym}_d(\mathbb{R})$ is the vector space of symmetric $d \times d$ matrices. Note that both the image of Sym_d and its dual preserve the properly convex domain $D \subset \mathbb{P}(V)$ of positive definite symmetric matrices. The composition $\phi_d = \psi_d \circ \mathrm{Sym}_d$ preserves the properly convex domain $D' := \{[X \otimes Y] : [X], [Y] \in D\}$ of $\mathbb{P}(V \otimes V)$. Also, for any $g \in \mathsf{SL}_d(\mathbb{R})$, we have

$$\sigma_1(\phi_d(g^{\pm 1})) = \frac{\sigma_1(g)^2}{\sigma_d(g)^2}.$$

We are now in a position to apply Proposition 4.2. Set $\eta = \epsilon/2$, and let $\rho : \mathsf{SL}_d(\mathbb{R}) * \mathsf{SL}_d(\mathbb{R}) \to \mathsf{GL}_m(\mathbb{R})$ be the representation obtained by composing ϕ_d with ρ_1 on the first $\mathsf{SL}_d(\mathbb{R})$ factor and ρ_2 on the second factor. It is immediate from part (a) of the proposition that if $H_1, H_2 < \mathsf{SL}_d(\mathbb{R})$ are discrete subgroups with $\phi_d(H_i) \cap \mathcal{K} = \{\mathrm{id}\}$, then the restriction $\rho : H_1 * H_2 \to \mathsf{GL}_m(\mathbb{R})$ is discrete and faithful. Moreover, if $\gamma_1 \cdots \gamma_n \in H_1 * H_2$ is a reduced word, we can inductively apply part (b) of the proposition to see that

$$\frac{\sigma_1}{\sigma_m}\left(\rho(\gamma_1 \cdots \gamma_n)\right) \geqslant \prod_{i=1}^{n} \sigma_1(\phi_d(\gamma_i))^{1-\eta} = \prod_{i=1}^{n} \left(\frac{\sigma_1}{\sigma_d}(\gamma_i)\right)^{2-\epsilon}.$$

This gives us the desired estimate. □

Finally we turn to the proof of Corollary 1.8. First, we reproduce an argument from [6] to prove the following:

Lemma 4.4 (See [6, Prop. 4.1])**.** *Let Γ_1, Γ_2 be finitely-generated groups, each containing an element with infinite order, and fix finite-index subgroups $\Gamma_i' < \Gamma_i$ for $i = 1, 2$. Then a finite-index subgroup of the free product $\Gamma_1 * \Gamma_2$ is isomorphic to an undistorted subgroup of the free product $\Gamma_1' * \Gamma_2'$.*

Proof. We may replace each Γ_i' with a further finite-index subgroup which is normal in Γ_i. Now let Γ_0 be the intersection of the kernels of the compositions $\Gamma_1 * \Gamma_2 \to \Gamma_i \to \Gamma_i/\Gamma_i'$. This intersection has finite index in $\Gamma_1 * \Gamma_2$. Moreover, it is isomorphic to a free product of the form

$$\Gamma_1' * \cdots * \Gamma_1' * \Gamma_2' * \cdots * \Gamma_2' * \mathbb{Z} * \cdots * \mathbb{Z}. \tag{15}$$

To see this, let X be a graph of spaces for the free product $\Gamma_1 * \Gamma_2$, consisting of a pair of spaces X_1, X_2 with $\pi_1 X_i \simeq \Gamma_i$, attached along an edge. Let X_1', X_2' be the covers of X_1, X_2 corresponding to the subgroups Γ_1', Γ_2' respectively. Observe that the finite cover X_0 of X corresponding to Γ_0 is itself a finite graph of spaces, and that each vertex space of X_0 is homeomorphic to one of X_1', X_2'; it follows that $\pi_1 X_0$ is isomorphic to a free product of the form in (15).

However, any free product as in (15) itself embeds as an undistorted subgroup of the free product $\Gamma_1' * \Gamma_2'$. To see this, let $\gamma_i \in \Gamma_i'$ have infinite order; then the subgroup of $\Gamma_1' * \Gamma_2'$ given by

$$\left\langle \gamma_2 \Gamma_1' \gamma_2^{-1}, \gamma_2^2 \Gamma_1' \gamma_2^{-2}, \ldots \gamma_2^r \Gamma_1' \gamma_2^{-r}, \gamma_1 \Gamma_2' \gamma_1^{-1}, \gamma_1^s \Gamma_2' \gamma_1^{-s}, \gamma_1^{s+1} \gamma_2 \gamma_1^{-s-1}, \ldots, \gamma_1^{s+q} \gamma_2 \gamma_1^{-s-q} \right\rangle$$

is naturally isomorphic to the free product of r copies of Γ_1', s copies of Γ_2', and q copies of $\mathbb{Z}$. □

Proof of Corollary 1.8. Suppose that Γ_1, Γ_2 are finitely-generated discrete subgroups of $\mathsf{SL}_d(\mathbb{R})$, each of which is quasi-isometrically embedded. Theorem 1.6 implies that, for finite-index subgroups $H_1 < \Gamma_1, H_2 < \Gamma_2$, we may realize the free product $H_1 * H_2$ as a quasi-isometrically embedded subgroup of $\mathsf{SL}_n(\mathbb{R})$. Then, by the previous lemma, a finite-index subgroup Γ_0 of the free product $\Gamma_1 * \Gamma_2$ is quasi-isometrically embedded in $\mathsf{SL}_n(\mathbb{R})$.

Now, if $\Gamma < \Gamma'$ is a subgroup of index m, and $\rho : \Gamma \to \mathsf{SL}_n(\mathbb{R})$ is a faithful representation, then one may always construct a faithful representation $\rho' : \Gamma' \to \mathsf{SL}_{nm}(\mathbb{R})$ so that the restriction $\rho'|_\Gamma$ has a subrepresentation isomorphic to ρ. In particular, if ρ is a quasi-isometric embedding, then so is ρ'. Applying this to $\Gamma_0 < \Gamma_1 * \Gamma_2$ completes the proof. □

References

[1] J. Bochi, R. Potrie, and A. Sambarino, *Anosov representations and dominated splittings*, J. Eur. Math. Soc. (JEMS), 21 (2019), pp. 3343–3414.

[2] R. Canary and K. Tsouvalas, *Topological restrictions on anosov representations*, Journal of Topology, 13 (2020), pp. 1497–1520.

[3] J. Danciger, F. Guéritaud, and F. Kassel, *Combination theorems in convex projective geometry*, arXiv e-prints, (2024), p. arXiv:2407.09439.

[4] S. Dey and M. Kapovich, *Klein-Maskit combination theorem for Anosov subgroups: Amalgams*, arXiv e-prints, (2023), p. arXiv:2301.02354.

[5] S. Dey and M. Kapovich, *Klein-Maskit combination theorem for Anosov subgroups: free products*, Math. Z., 305 (2023), pp. Paper No. 35, 25.

[6] S. Douba and K. Tsouvalas, *Anosov groups that are indiscrete in rank one*, International Mathematics Research Notices, (2023), p. rnad258.

[7] ———, *Anosov groups that are indiscrete in rank one*, Int. Math. Res. Not. IMRN, (2024), pp. 8377–8383.

[8] F. Guéritaud, O. Guichard, F. Kassel, and A. Wienhard, *Anosov representations and proper actions*, Geom. Topol., 21 (2017), pp. 485–584.

[9] O. Guichard and A. Wienhard, *Anosov representations: domains of discontinuity and applications.*, Invent. Math., 190 (2012), pp. 357–438.

[10] M. Kapovich, B. Leeb, and J. Porti, *A Morse lemma for quasigeodesics in symmetric spaces and Euclidean buildings*, Geom. Topol., 22 (2018), pp. 3827–3923.

[11] F. Kassel and R. Potrie, *Eigenvalue gaps for hyperbolic groups and semigroups*, J. Mod. Dyn., 18 (2022), pp. 161–208.

[12] F. Klein, *Neue Beiträge zur Riemann'schen Functionentheorie*, Math. Ann., 21 (1883), pp. 141–218.

[13] L. Li, K. Ohshika, and X. Wang, *On Klein-Maskit combination theorem in space. I*, Osaka J. Math., 46 (2009), pp. 1097–1141.

[14] E. Martínez-Pedroza and A. Sisto, *Virtual amalgamation of relatively quasiconvex subgroups.*, Algebr. Geom. Topol., 12 (2012), pp. 1993–2002.

[15] ———, *Virtual amalgamation of relatively quasiconvex subgroups.*, Algebr. Geom. Topol., 12 (2012), pp. 1993–2002.

[16] B. Maskit, *Kleinian groups*, vol. 287 of Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer-Verlag, Berlin, 1988.

[17] A. Traaseth and T. Weisman, *Combination theorems for geometrically finite convergence groups*, (2023).

[18] T. Weisman, *An extended definition of Anosov representation for relatively hyperbolic groups*, arXiv e-prints, (2022), p. arXiv:2205.07183.

[19] A. M. Zimmer, *Proper quasi-homogeneous domains in flag manifolds and geometric structures*, Ann. Inst. Fourier (Grenoble), 68 (2018), pp. 2635–2662.

Max Planck Institute for Mathematics in the Sciences, Inselstrasse 22, 04103 Leipzig, Germany
Email address: konstantinos.tsouvalas@mis.mpg.de

Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA
Email address: tjwei@umich.edu