MORSE PROPERTIES IN CONVEX PROJECTIVE GEOMETRY

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Abstract. We study properties of "hyperbolic directions" in groups acting cocompactly on properly convex domains in real projective space, from three different perspectives simultaneously: the (coarse) metric geometry of the Hilbert metric, the projective geometry of the boundary of the domain, and the singular value gaps of projective automorphisms. We describe the relationship between different definitions of "Morse" and "regular" quasi-geodesics arising in these three different contexts. This generalizes several results of Benoist and Guichard to the non-Gromov hyperbolic setting.

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1. Introduction

Group actions on $\mathbb{H}^n$ have classically played a pivotal role in the study of discrete subgroups of Lie groups, geometric topology, and geometric group theory. Hyperbolic geometry provides a strong link between these fields, since hyperbolic manifolds (whose holonomy representations have discrete images lying in $\mathrm{PO}(n,1) \simeq \mathrm{Isom}(\mathbb{H}^n)$) give some of the most important examples of geometric structures on manifolds, and the properties of hyperbolic space are effectively coarsified via the theory of Gromov-hyperbolic groups.

It is thus natural to try and find a substitute for hyperbolic geometry which extends this connection beyond the negative curvature setting. In particular, one would like to find a model geometry which facilitates the study of discrete subgroups of higher-rank Lie groups, such as $\mathrm{SL}(d,\mathbb{R})$ for $d \geq 3$. One reasonable possibility to consider is the non-positively curved Riemannian symmetric space $\mathrm{SL}(d,\mathbb{R})/\mathrm{SO}(d)$, but actions on the symmetric space are often unsatisfyingly rigid. For instance,

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when $d \geq 3$, any discrete Zariski-dense subgroup of $\mathrm{SL}(d, \mathbb{R})$ which acts cocompactly on a convex subset of $\mathrm{SL}(d, \mathbb{R})/\mathrm{SO}(d)$ is a uniform lattice in $\mathrm{SL}(d, \mathbb{R})$ [Qui05, KL06].

Convex projective geometry fills this gap by providing examples of natural spaces—*properly convex domains*—for discrete subgroups of $\mathrm{SL}(d, \mathbb{R})$ to act on. A properly convex domain Ω is an open subset of real projective space $\mathbb{P}(\mathbb{R}^d)$, which is bounded in some affine chart. Such a domain can be equipped with its *Hilbert metric* d_Ω, and the group $\mathrm{Aut}(\Omega) \subset \mathrm{PGL}(d, \mathbb{R})$ of projective transformations preserving Ω acts by isometries of this metric. When Ω is a round ball in $\mathbb{P}(\mathbb{R}^{d+1})$, then $\mathrm{Aut}(\Omega) \simeq \mathrm{PO}(d, 1)$ and the space $(\Omega, \mathrm{d}_\Omega)$ is precisely the projective Beltrami-Klein model of the real hyperbolic space $\mathbb{H}^d$. By virtue of this example, convex projective geometry can be viewed as a far-reaching generalization of real hyperbolic geometry. This viewpoint has been of much interest lately, and consequently convex projective geometry has developed close connections with higher Teichmüller theory (see e.g. [Gol90, CG93, GW08, Wie18]) and the theory of Anosov representations [DGK17, Zim21].

Outside of coarse negative curvature, however, convex projective geometry can also be used to model examples which have a mixture of "negatively curved" and "flat" behavior. This allows for the study of discrete subgroups of Lie groups which have markedly *different* behavior from those in rank one. For instance, consider a closed 3-manifold M with a geometric decomposition along a nonempty collection of tori, into pieces whose interiors admit finite-volume complete hyperbolic structures. Benoist [Ben06a] constructed examples of such 3-manifolds M which are diffeomorphic to a quotient Ω/Γ, where Ω is a properly convex domain and Γ is a discrete subgroup of $\mathrm{Aut}(\Omega)$ In this case, $\Gamma \simeq \pi_1 M$, is relatively hyperbolic relative to 2-flats and the domain Ω is quasi-isometric to $\pi_1(M)$.

In this above example, the projective structure on the manifold Ω/Γ is not "non-positively curved," in the sense that Ω equipped with its Hilbert metric d_Ω is not a $\mathrm{CAT}(0)$ metric space. In fact, a classical theorem states that a Hilbert geometry $(\Omega, \mathrm{d}_\Omega)$ is $\mathrm{CAT}(0)$ if any only if Ω is equivalent to the projective model of hyperbolic space [KS58]. Despite this, convex projective domains share some striking similarities with $\mathrm{CAT}(0)$ geometry. Lately, there has been much activity in understanding these similarities [IZ23, Isl, Wei23, Bob21, IZ21, Bla21]. A key upshot of these recent developments is the realization that irreducible properly convex domains with a cocompact action essentially come in two flavors: rank one and higher rank [Isl, Zim23]. The higher-rank domains are special, and have a complete classification. On the other hand, many different examples of rank one domains have been constructed [Ben06b, Ben06a, Kap07, CLM16, BDL18, BV23], and their classification seems difficult. These domains are characterized by the presence of abundant "negatively curved behavior" (see Section 1.4.2).

Summary of results. Motivated by this, we initiate in this paper a study of "hyperbolic" geodesic directions in $(\Omega, \mathrm{d}_\Omega)$. We are mainly interested in the case where there is a discrete subgroup $\Gamma \subset \mathrm{Aut}(\Omega)$ acting cocompactly on Ω; in this situation, following Benzécri [Ben60] and Benoist [Ben08], we say that Γ *divides* Ω and that Ω is *divisible*. We consider *projective* geodesic rays in a divisible domain Ω, i.e. geodesic rays $c : [0, \infty) \to \Omega$ whose image is a projective line segment, and sequences $\{\gamma_n\}$ in Γ whose orbits give a coarse approximation of c. In this paper we understand "hyperbolicity" of these geodesic rays from three different perspectives:

(I) **The coarse metric geometry of the space** (Ω, d_Ω)**.** In this context, there are two notions of "hyperbolic geodesic" which are relevant for this paper: *Morse geodesics*, which are geodesics which satisfy the same "Morse" or "quasi-geodesic stability" property as geodesics in hyperbolic spaces, and *contracting geodesics*, whose nearest-point projection maps have a similar "contracting" property as hyperbolic geodesics. In CAT(0) spaces, Morse and contracting geodesics coincide; we prove that the same is true for Hilbert geometries (Theorem 1.16).

(II) **The linear algebraic behavior of the sequence** $\{\gamma_n\}$ **in** $\mathrm{SL}(d, \mathbb{R})$**.** Here, our understanding of "hyperbolicity" of a geodesic comes from results of Benoist [Ben04], Bochi-Potrie-Sambarino [BPS19], and Kapovich-Leeb-Porti [KLP17], implying that a discrete group $\Gamma \subseteq \mathrm{Aut}(\Omega)$ acting cocompactly on Ω is Gromov-hyperbolic if and only if the singular values of Γ satisfy a *uniform exponential gap* condition along all geodesics in Γ. Thus we understand "hyperbolic directions" as geodesics in Γ whose singular value gaps satisfy a similar exponential growth condition. This perspective is closely tied to the notion of a k*-Morse* quasi-geodesic in the Riemannian symmetric space $\mathrm{SL}(d, \mathbb{R})/\mathrm{SO}(d)$, introduced by Kapovich-Leeb-Porti [KLP18].

(III) **The projective geometry of the boundary of the domain** Ω**.** Our motivation for this perspective comes from results of Benoist [Ben04] and [Gui05], which imply that, if Γ is a discrete hyperbolic group dividing a domain Ω, then the boundary of Ω is a C^α hypersurface in projective space for some $\alpha > 1$. From this perspective, the "hyperbolicity" of a geodesic ray c in a general divisible domain Ω is captured by the local regularity of the hypersurface $\partial\Omega$ at the endpoint of c.

Our main aim in this paper is to establish relationships between geodesics satisfying various versions of these notions of "hyperbolicity." Many of these relationships (in the case of a projective geodesic c in a convex divisible domain with *exposed boundary*, see Definition 2.2) are summarized in Figure 1 below.

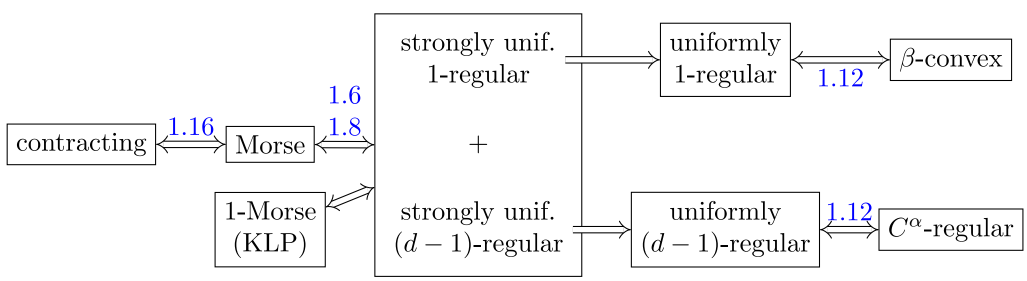

Figure 1. Relationships between various notions of "hyperbolicity" for a projective geodesic in a convex divisible domain with exposed boundary.

Before giving precise theorem statements in the next section, we briefly outline the main results expressed by this diagram. Theorem 1.6 and Theorem 1.8 relate perspectives (I) and (II) above. These theorems show that, if c is a projective geodesic in a divisible domain tracked by a sequence $\{\gamma_n\}$, then Morseness of c (in the sense of (I)) is characterized by the behavior of singular value gaps of the sequence

$\{\gamma_n\}$. In particular, this implies that for projective geodesics, the Kapovich-Leeb-Porti notion of "1-Morseness" for quasi-geodesics in symmetric spaces coincides with the coarse metric notion of Morseness in (I) (Corollary 1.10).

Theorem 1.12 in the diagram directly relates perspectives (II) and (III). The theorem concerns projective geodesics c whose endpoint $c(\infty)$ in $\partial\Omega$ satisfies a certain regularity property; roughly, this property asserts that if $\partial\Omega$ is locally the graph of a convex function $f(x)$ near $c(\infty)$, then f is sandwiched between $C_1 \|x\|^\alpha$ and $C_2 \|x\|^\beta$ for some $\alpha > 1$ and $\beta < \infty$. We prove that this property is characterized by the behavior of the singular values of the sequence $\{\gamma_n\}$, and give a formula for the optimal constants α and β in terms of these singular values. Via the results alluded to in the previous paragraph, this also relates perspectives (I) and (III), and shows that every Morse quasi-geodesic in the sense of (I) satisfies the regularity property mentioned above. However, the converse to this statement turns out to be false (see Theorem 1.9). Effectively, this theorem says that the reverse of the implications "strong uniform regularity" $\implies$ "uniform regularity" in Figure 1 do not always hold.

Statement of the main results. We now provide a more detailed and precise account of the main results in the paper.

1.1. M-Morse geodesics and uniform regularity.

Definition 1.1. Let (X, d) be a proper geodesic metric space and $M : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be any function. A geodesic (segment, ray, line) ℓ is *M-Morse* if any (λ, a)-quasi-geodesic ℓ' with endpoints $x, y \in \ell$ lies in the $M(\lambda, a)$-neighborhood of ℓ.

The function M is called a *Morse gauge* for the geodesic ℓ. At times, we will skip explicit mention of the Morse gauge and only say that ℓ is *Morse*, instead of ℓ is M-Morse.

Geodesic rays in $\mathbb{H}^2$ are all M_0-Morse for a fixed Morse gauge M_0. On the other hand, flat spaces like $\mathbb{R}^2$ and higher rank CAT(0) spaces like $\mathbb{H}^2 \times \mathbb{H}^2$ or $\mathrm{SL}(3, \mathbb{R})/\mathrm{SO}(3)$ do not contain any Morse geodesics. The results in [Isl] and [Zim23] indicate that a "generic" divisible domain Ω has many Morse geodesics that project to closed geodesics in the quotient Ω/Γ (see Section 1.4.2). It is also straightforward to check (see Section 3) that any M-Morse geodesic ray in a convex projective domain Ω is uniformly close to a projective geodesic ray.

We would like to understand the Morseness of a projective geodesic ray by studying the sequence of automorphisms that approximates the ray via an orbit map. To make this precise, we use the following terminology throughout this paper.

Definition 1.2 (Tracking sequences)**.** Let Ω be a properly convex domain, $c : [0, \infty) \to \Omega$ be a projective geodesic ray, $x_0 \in \Omega$, and $R > 0$. We say that a sequence $\{\gamma_n\}_{n \in \mathbb{N}}$ in $\mathrm{Aut}(\Omega)$ *R-tracks c with respect to x_0* if $\mathrm{d}_\Omega(x_0, \gamma_n^{-1} c(n)) < R$ for every $n \in \mathbb{N}$.

Remark 1.3. While discussing tracking sequences, unless necessary we will omit the specific constant R and the basepoint x_0, and simply say that $\{\gamma_n\}$ *tracks* c. Note that if Ω/Γ is compact, then every geodesic is R-tracked by some sequence in Γ for $R = \mathrm{diam}(\Omega/\Gamma)$. Also, by definition, $\{\gamma_n\}$ tracks c along a sequence of equally-spaced points $\{c(n)\}$. One can consider other kinds of sequences, but we do not pursue this here.

Now, for any $g \in \mathrm{GL}(d, \mathbb{R})$, let $\sigma_1(g) \geq \ldots \geq \sigma_d(g)$ be the singular values of g, and for any $1 \leq i < j \leq d$, let $\mu_{i,j}(g) := \log \frac{\sigma_i(g)}{\sigma_j(g)}$. Note that $\mu_{i,j}$ descends to a well-defined map on $\mathrm{PGL}(d, \mathbb{R})$.

Definition 1.4. For $1 \leq k < d$, we say that a sequence $\{g_n\}$ in $\mathrm{GL}(d, \mathbb{R})$ is *uniformly k-regular* if it is divergent (it leaves every compact set in $\mathrm{GL}(d, \mathbb{R})$) and

$$\liminf_{n\to\infty} \frac{\mu_{k,k+1}(g_n)}{\mu_{1,d}(g_n)} > 0.$$

We say that the sequence $\{g_n\}$ is *strongly uniformly k-regular* if it is divergent and there are constants $C, N > 0$ such that for all $n, m \in \mathbb{N}$ with $m > N$, we have

$$\frac{\mu_{k,k+1}(g_n^{-1} g_{n+m})}{\mu_{1,d}(g_n^{-1} g_{n+m})} > C.$$

Remark 1.5. It is immediate that a strongly uniformly k-regular sequence is also uniformly k-regular. In general, the converse does not hold; the construction in Section 7 of this paper provides a counterexample.

Note that our definition of "uniform regularity" is slightly different from definitions appearing in the work of Kapovich-Leeb-Porti [KLP17, KLP18, KL18]. This is unavoidable as the definitions of uniform regularity appearing in those papers are not mutually consistent. Our "strongly uniformly regular" sequences coincide with the "coarsely uniformly regular" sequences defined in [KLP18].

We prove the following:

Theorem 1.6 (Section 5)**.** *Suppose Ω is a properly convex domain, $c : [0, \infty) \to \Omega$ is a geodesic ray, and $\{\gamma_n\}$ R-tracks c with respect to $x_0 \in \Omega$. If c is M-Morse, then $\{\gamma_n\}$ is strongly uniformly k-regular for both $k = 1$ and $k = d - 1$.*

Moreover, the constants C, N in the definition of strong uniform regularity depend only on M, R, and the basepoint $x_0 \in \Omega$.

This theorem implies in particular that Morse geodesics give rise to uniformly regular sequences. We express this via the corollary below.

Corollary 1.7. *Suppose that Ω is a properly convex domain, c is a projective geodesic ray, and $\{\gamma_n\}$ R-tracks c with respect to $x_0 \in \Omega$. For any Morse gauge M, there exists $\xi = \xi(M, R, x_0) > 1$ so that*

$$\liminf_{n\to\infty} \frac{\mu_{1,d}(\gamma_n)}{\mu_{1,d-1}(\gamma_n)} > 1 + \frac{1}{\xi - 1} \quad \text{and} \quad \limsup_{n\to\infty} \frac{\mu_{1,d}(\gamma_n)}{\mu_{1,2}(\gamma_n)} < \xi.$$

If we impose additional assumptions on the domain Ω, then a partial converse to Theorem 1.6 also holds. Specifically, we have the following:

Theorem 1.8 (Section 5)**.** *Let Ω be a convex divisible domain with exposed boundary and let c be a projective geodesic ray in Ω. Suppose $\{\gamma_n\}$ R-tracks c with respect to $x_0 \in \Omega$. If $\{\gamma_n\}$ is strongly uniformly k-regular for $k = 1$ and $k = d - 1$, then c is M-Morse for some Morse gauge M.*

Moreover, M can be chosen to depend only on x_0, R, and the constants in the definition of strong uniform k-regularity.

We provide the precise definition of a convex projective domain with *exposed boundary* in Definition 2.2. The additional assumptions on Ω in Theorem 1.8 are necessary, as is the assumption that c is a *projective* geodesic; see Example 5.5 and Example 5.6.

Together, Theorem 1.6 and Theorem 1.8 show that, when Ω is a convex divisible domain with exposed boundary, it is possible to completely characterize (projective) Morse geodesics in terms of the singular values of tracking sequences. We also show that the weaker uniform regularity condition in Corollary 1.7 does *not* imply Morseness:

Theorem 1.9 (Section 7)**.** *There exists a convex divisible domain Ω with exposed boundary, a projective geodesic ray c, and a sequence $\{\gamma_n\}$ tracking c so that $\{\gamma_n\}$ is both uniformly 1-regular and uniformly $(d-1)$-regular, but* not *strongly uniformly 1-regular. In particular, due to Theorem 1.6, c is* not *M-Morse for any Morse gauge M.*

1.1.1. *k-Morseness in symmetric spaces.* In [KLP18], Kapovich-Leeb-Porti developed a notion of "Morseness" for quasi-geodesics in certain symmetric spaces. If X is the Riemannian symmetric space $\mathrm{PGL}(d,\mathbb{R})/\mathrm{PO}(d)$, then a quasi-geodesic ray $q : [0,\infty) \to X$ is *k-Morse in the sense of Kapovich-Leeb-Porti* if it satisfies a "higher-rank Morse property" with respect to the Grassmannian of k-planes $\mathrm{Gr}(k,d)$, viewed as a space of simplices in the visual boundary of X. This property says that q lies in a bounded neighborhood of a Euclidean Weyl sector asymptotic to a k-plane in $\mathrm{Gr}(k,d)$.

In the same paper, Kapovich-Leeb-Porti proved a *higher-rank Morse lemma*, characterizing Morse quasi-geodesics in terms of their uniform regularity. Applying this result with Theorem 1.6 and Theorem 1.8, we obtain the following:

Corollary 1.10. *Let Ω be a convex divisible domain with exposed boundary, let $c : [0,\infty) \to \Omega$ be a projective geodesic, and let $\{\gamma_n\}$ be a sequence in $\mathrm{Aut}(\Omega)$ which tracks c. Then c is a Morse geodesic if and only if the mapping $\mathbb{N} \to \mathrm{PGL}(d,\mathbb{R})/\mathrm{PO}(d)$ given by $n \mapsto \gamma_n \mathrm{PO}(d)$ is a 1-Morse quasi-geodesic in the sense of Kapovich-Leeb-Porti.*

1.2. **Uniform regularity and boundary regularity.** We now relate the singular value gap conditions appearing in Theorem 1.6 and Corollary 1.7 to the smoothness or regularity of the boundary $\partial\Omega$ at the endpoint of a projective geodesic. The boundary $\partial\Omega$ is a convex hypersurface in $\mathbb{P}(\mathbb{R}^d)$, meaning it is locally the graph of a convex function $f : \mathbb{R}^{d-2} \to \mathbb{R}$. Typically, this hypersurface is nowhere C^1, but we can still make sense of local regularity using convexity.

We say that a point $z \in \partial\Omega$ is a C^1 *point* if there is a unique supporting hyperplane of Ω at z, i.e. a hyperplane containing z, but not intersecting Ω. At a C^1 point z, we further have a local notion of α-Hölder regularity. We say that z is a C^α *point* if there exist Euclidean coordinates on an affine chart in $\mathbb{P}(\mathbb{R}^d)$ such that $\partial\Omega$ is the graph of a convex function f, $(z, f(z))$ is the origin, and $f(x) \leq C_1||x||^\alpha$ for some $C_1 > 0$ and all x sufficiently close to z. Dually, we say that z is a *β-convex* point if $f(x) \geq C_2||x||^\beta$ for some $C_2 > 0$ and all x sufficiently close to z.

Definition 1.11. Let Ω be a properly convex domain and $x \in \partial\Omega$ be a C^1 point. Set

$$\alpha(x,\Omega) := \sup\{\alpha > 1 : \partial\Omega \text{ is } C^\alpha \text{ at } x\}$$

and

$$\beta(x, \Omega) := \inf\{\beta < \infty : \partial\Omega \text{ is } \beta\text{-convex at } x\}.$$

If $\partial\Omega$ is not C^α at x for any $\alpha > 1$, we define $\alpha(x, \Omega) = 1$. Similarly if $\partial\Omega$ is not β-convex at x for any $\beta < \infty$, we define $\beta(x, \Omega) = \infty$.

We show that for a divisible domain Ω, the functions $\alpha(x, \Omega)$ and $\beta(x, \Omega)$ defined above are determined by singular values of tracking sequences. To state our result, we require x to be an exposed boundary point; see Definition 2.2 and Fig. 5.

Theorem 1.12 (Section 6)**.** *Let Ω be a properly convex domain, let $\{\gamma_n\}$ track a projective geodesic ray $c : [0, \infty) \to \Omega$, and suppose that $c(\infty) = x$ is an exposed C^1 extreme point in $\partial\Omega$. Define*

$$\alpha_0 := \liminf_{n\to\infty} \frac{\mu_{1,d}(\gamma_n)}{\mu_{1,d-1}(\gamma_n)} \quad \text{and} \quad \beta_0 := \limsup_{n\to\infty} \frac{\mu_{1,d}(\gamma_n)}{\mu_{1,2}(\gamma_n)}.$$

Then $\alpha_0 = \alpha(x, \Omega)$ and $\beta_0 = \beta(x, \Omega)$.

In particular, $c(\infty)$ is a C^α point for some $\alpha > 1$ if and only if $\{\gamma_n\}$ is uniformly $(d-1)$-regular, and $c(\infty)$ is β-convex for $\beta < \infty$ if and only $\{\gamma_n\}$ is uniformly 1-regular.

An immediate consequence of Corollary 1.7 and Theorem 1.12 is the following, which is our link between perspectives (I) and (III) in this paper:

Corollary 1.13. *Suppose Ω is a properly convex domain and $\{\gamma_n\}$ R-tracks a projective geodesic ray c with respect to $x_0 \in \Omega$. If c is M-Morse, then $c(\infty)$ is C^α and β-convex for some $\alpha > 1, \beta < \infty$, depending only on M, R, and x_0. Moreover*

$$\alpha(c(\infty), \Omega) = \liminf_{n\to\infty} \frac{\mu_{1,d}(\gamma_n)}{\mu_{1,d-1}(\gamma_n)} > 1 \quad \text{and} \quad \beta(c(\infty), \Omega) = \limsup_{n\to\infty} \frac{\mu_{1,d}(\gamma_n)}{\mu_{1,2}(\gamma_n)} < \infty.$$

By applying Theorem 1.9 and Theorem 1.12, we can also see that the converse to Corollary 1.13 does not hold:

Corollary 1.14. *There exists a convex divisible domain Ω with exposed boundary and a projective geodesic ray c so that c is* not *M-Morse for any Morse gauge M, but $c(\infty)$ is both a C^α point for some $\alpha > 1$ and a β-convex point for some $\beta < \infty$.*

1.3. **D-contracting geodesics and Morse local-to-global spaces.** We now mention a few additional results regarding notions of "coarsely negatively curved" geodesic directions in Ω. Recall that geodesics in hyperbolic metric spaces always satisfy a *contracting* property, which motivates the following definition:

Definition 1.15. Let (X, d) be a proper metric space, ℓ a geodesic (ray, segment, line), and let $\pi_\ell : X \to 2^\ell$ denote the closest-point projection on ℓ, i.e.

$$\pi_\ell(x) = \{y \in \ell : d(x, y) = d(x, \ell)\}.$$

Then ℓ is *D-contracting* for $D > 0$ if, for any metric ball $B_r(x)$ disjoint from ℓ,

$$\operatorname{diam}(\pi_\ell(B_r(x))) \leq D.$$

If ℓ is D-contracting for some $D > 0$, then we simply say that ℓ is *contracting.*

A result of Charney-Sultan [CS15] implies that, if X is a CAT(0) metric space, then contracting geodesics are exactly the same as Morse geodesics. We prove:

Theorem 1.16 (Section 3)**.** *Let Ω be a properly convex domain, and let c be a geodesic in Ω. Then c is Morse if and only if c is contracting.*

Remark 1.17. It is well-known that in general metric spaces, every D-contracting geodesic is M-Morse for some Morse gauge M depending only on D. So, our main contribution in Theorem 1.16 is proving the converse, i.e. that Morse geodesics are always contracting. Our proof for this direction relies on specific features of the projective geometry of Ω.

When we prove this direction, we do *not* in general obtain uniform control over the contraction constant D in terms of the Morse gauge M. However, we do have uniform control if we additionally assume that Ω has a cocompact action by automorphisms; see Corollary 3.28.

1.3.1. *Morse local-to-global property.* One may apply Theorem 1.16 to prove that divisible convex domains have the so-called *Morse local-to-global property* [RST22]. As the name suggests, this property means the following. Suppose c is a path in a metric space X, such that any sufficiently long finite sub-segment of c is an M-Morse quasi-geodesic. Then the entire path c is an M'-Morse quasi-geodesic, for some Morse gauge M'. Metric spaces that have Morse local-to-global property were studied extensively in [RST22]. This property holds for a large class of spaces, e.g. hyperbolic spaces, CAT(0) spaces, and mapping class groups of most finite-type surfaces. In Section 3.13 we observe:

Theorem 1.18. *If Ω is any convex divisible domain, the metric space $(\Omega, \mathrm{d}_\Omega)$ is Morse local-to-global.*

1.4. **Comparison to previous results.**

1.4.1. *The Gromov-hyperbolic case.* Several of the results in this paper are inspired by previous work of Benoist and Guichard on convex divisible domains. In particular, we are motivated by the following theorem:

Theorem 1.19 ([Ben04])**.** *Let Γ be a group dividing a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$. Then the following are equivalent:*

- *(a)* Γ *is Gromov-hyperbolic;*
- *(b) The inclusion* $\Gamma \hookrightarrow \mathrm{PGL}(d, \mathbb{R})$ *is a* 1-Anosov representation*.*
- *(c) The domain Ω is strictly convex, i.e. its boundary $\partial\Omega$ contains no nontrivial projective segments;*
- *(d) The boundary $\partial\Omega$ is a C^1 hypersurface.*

We can interpret this theorem as giving a link between our perspectives (I), (II) (III) in the hyperbolic setting. In particular, if Γ is a Gromov-hyperbolic group, then every geodesic in Γ is M_0-Morse for some uniform Morse gauge M_0, so every geodesic direction in Γ is "hyperbolic" in the sense of our perspective (I). Part (b) of the theorem connects to perspective (II) via work of Kapovich-Leeb-Porti [KLP17] and Bochi-Potrie-Sambarino [BPS19], who poved that a representation $\Gamma \to \mathrm{PGL}(d, \mathbb{R})$ is 1-Anosov if and only if it is a quasi-isometric embedding and every geodesic is strongly uniformly 1-regular. From this viewpoint, our Theorem 1.6, Theorem 1.8, and Corollary 1.13 give a generalization of Theorem 1.19 to the situation where Γ is not necessarily a Gromov-hyperbolic group. Effectively, we prove that the equivalences in Theorem 1.19 still hold locally, "along hyperbolic directions."

Remark 1.20. There are explicit constructions for divisible domains in $\mathbb{P}(\mathbb{R}^d)$ that are *not* strictly convex (so none of the conditions in Theorem 1.19 hold), but still contain Morse geodesics (so that our main results apply); for examples, see [Ben06a, CLM16, BDL18] when $4 \leq d \leq 7$, and [BV23] for any $d \geq 4$.

In [Gui05], Guichard also investigated the relationship between regularity of the boundary of a strictly convex divisible domain Ω, and the linear algebraic properties of the dividing group Γ. In particular, Guichard showed that, assuming Γ is a hyperbolic group, the *global* Hölder regularity of $\partial\Omega$ can be computed in terms of the asymptotic behavior of the eigenvalues of sequences in Γ; this provides another link between our perspectives (II) and (III), again in the case where Γ is assumed to be a hyperbolic group. We mention also related work of Crampon [Cra09], which shows that for *strictly* convex divisible domains, the regularity of the boundary Ω is related to the Lyapunov exponent of the geodesic flow. Our Theorem 1.12 can be thought of as a localized version of Guichard's result, which applies to geodesic directions in any (not necessarily strictly convex) divisible domain.

1.4.2. *Closed geodesics in rank-one convex projective manifolds.* In [Isl], the first author introduced a notion of rank one properly convex domains, a family that encompasses the Gromov hyperbolic ones. An infinite order element $\gamma \in \mathrm{Aut}(\Omega)$ is called a *rank one automorphism* if γ acts by a translation along a projective geodesic $\ell_\gamma \subset \Omega$, and ℓ_γ is not contained in any half triangle (see Definition 3.7). The results in [Isl] show that the axis ℓ_γ of a rank one automorphism is always a Morse geodesic, and also characterize rank-one automorphisms in terms of their eigenvalues. By combining this with results in the present paper, we obtain the following more complete description of the relationship between rank one automorphisms and Morseness.

Proposition 1.21. *Suppose an infinite order element $\gamma \in \mathrm{Aut}(\Omega)$ acts by a translation along a projective geodesic $\ell_\gamma \subset \Omega$. Then the following are equivalent:*

1. *γ is a rank one automorphism.*
2. *The geodesic ℓ_γ is Morse (equivalently, ℓ_γ is contracting).*

If, in addition, there is a discrete group $\Gamma \subseteq \mathrm{Aut}(\Omega)$ such that Γ divides Ω and $\gamma \in \Gamma$, then either of the above conditions is equivalent to:

3. *γ is biproximal, i.e. the matrix representing γ has unique eigenvalues of maximum and minimum modulus.*

Proof. (1) $\Longrightarrow$ (2) is [Isl, Proposition 1.12]. (2) $\Longrightarrow$ (1) follows from the results in Section 3 of this paper: since ℓ_γ is Morse, Corollary 3.25 implies that the endpoints of ℓ_γ cannot lie in the closure of a non-trivial projective line segment in $\partial\Omega$. Thus ℓ_γ is not contained in any half triangle.

Finally, (1) $\iff$ (3) is [Isl, Proposition 6.8]. □

A main result of [Isl] is that, if Γ divides Ω and Γ contains a rank one automorphism, then Γ in fact has many rank one automorphisms. In particular, Γ is an acylindrically hyperbolic group. That is to say that Γ has a 'nice' action on some (possibly non-proper) Gromov hyperbolic metric space, although $(\Omega, \mathrm{d}_\Omega)$ itself may not be Gromov hyperbolic.

On the other hand, in [Zim23], Zimmer proved a rank rigidity theorem for properly convex domains. This result implies that if Γ does not contain any rank one automorphisms, then either Ω is reducible (meaning a cone over Ω splits as a product of cones) or else Ω and Γ are very restricted: in particular, Ω is a projective model for the Riemannian symmetric space G/K for a simple Lie group G, and Γ is isomorphic to a uniform lattice in G. Taken together, the results in [Isl] and [Zim23] indicate that a "generic" divisible domain contains many projective geodesics that point in "hyperbolic" directions.

1.5. **Comments on the proofs.** The proof of Theorem 1.6 (our first main theorem) relies on two key ingredients. The first is a "straightness" lemma (Lemma 4.10) that does not rely on Morseness at all—it holds for any sequence $\{\gamma_n\}$ tracking a projective geodesic. The lemma says that for any three elements $\gamma_i, \gamma_j, \gamma_k$ in the tracking sequence, with $i < j < k$, the gap $\mu_{1,2}(\gamma_i^{-1}\gamma_k)$ is coarsely bounded below by the sum $\mu_{1,2}(\gamma_i\gamma_j^{-1}) + \mu_{1,2}(\gamma_j^{-1}\gamma_k)$. In particular, this implies that $\mu_{1,2}(\gamma_n)$ is coarsely nondecreasing as a function of n, which is *not* a property satisfied by arbitrary quasi-geodesic sequences in $\mathrm{PGL}(d, \mathbb{R})$. We also remark that this "straightness" property does *not* require any assumption on the regularity of the sequence $\{\gamma_n\}$, which is critical for a later application in the proof of Theorem 1.9.

The second ingredient in the proof of Theorem 1.6 relies crucially on M-Morseness, see Lemma 5.1. This lemma shows that Morseness forces growth in $\mu_{1,2}$ as one shadows a M-Morse geodesic for a sufficiently long time. The proof of Theorem 1.6 then follows by a telescoping argument splitting up the Morse geodesic into pieces with sufficiently large $\mu_{1,2}$ growth; see Proposition 5.3.

1.6. **Outline of the paper.** The first part of this paper focuses mainly on the relationship between the coarse metric geometry and projective geometry of a convex projective domain. We provide some background about convex projective geometry in Section 2. In Section 3, we give several projective geometric characterizations of Morse geodesics in Hilbert geometry, prove Theorem 1.16, and sketch the proof of Theorem 1.18. This section also introduces the notion of *conically related* pairs of points in the boundary of a pair of convex projective domains, which is an important ingredient in the proof of Theorem 1.8.

The next part of the paper focuses more on the linear algebraic viewpoint. In Section 4, we prove singular value estimates along sequences $\{\gamma_n\}$ in $\mathrm{PGL}(d, \mathbb{R})$ that tracks a projective geodesic; in particular, we prove the "straightness" Lemma 4.10 alluded to previously. Then in Section 5, we use results from Section 4 (and also Section 3) to prove the relationship between Morse geodesics and strongly uniformly regular sequences, as described by Theorem 1.6 and Theorem 1.8.

In Section 6 we consider C^α regularity and β-convexity of the boundary of a convex projective domain, and prove Theorem 1.12. Finally, we construct the counterexample described by Theorem 1.9 in Section 7.

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2. Preliminaries

2.1. **Notation.** We standardize some notation for the entire paper. If (X, d) is a metric space, $A \subseteq X$, and $r > 0$, then we denote the (open) metric r-neighborhood of A by

$$N_r^X(A) := \{x \in X : d(x, A) < r\}.$$

If X is clear from context, we will simply write $N_r^X(A) = N_r(A)$. If $A = \{x\}$, then we will use the notation $B_r(x)$ to denote the metric r-ball $N_r(\{x\})$.

2.1.1. *Projective space.* When V is a real vector space, we let $\mathbb{P}(V)$ denote the projectivization of V, i.e. the space of 1-dimensional vector subspaces of V. If v is a nonzero vector in V then $[v]$ denotes the point in $\mathbb{P}(V)$ given by the span of v.

If $U \subseteq V$ is a subset, then $\mathbb{P}(U)$ denotes the image of $U - \{0\}$ under the projectivization map $v \mapsto [v]$. If U is a vector subspace of V, this notation identifies the projective space $\mathbb{P}(U)$ as a subset of $\mathbb{P}(V)$ (a *projective subspace*). We will *never* implicitly identify a vector subspace $W \subseteq V$ with the corresponding projective subspace $\mathbb{P}(W)$. If P is a projective subspace, equal to $\mathbb{P}(W)$ for $W \subseteq V$, we write $W = \widetilde{P}$.

If $U \subseteq V$ then the *projective span* of U is the projective subspace $\operatorname{span}_{\mathbb{P}}\{U\} := \mathbb{P}(\operatorname{Span}(U))$. Similarly if $P \subset \mathbb{P}(V)$, the projective span of P is $\operatorname{span}_{\mathbb{P}}\{P\} := \mathbb{P}(\operatorname{Span}(\widetilde{P}))$, where $\widetilde{P}$ is a lift of P in V.

We let $\mathbb{P}^*(V)$ denote the space of codimension-1 subspaces of V. If $W \in \mathbb{P}^*(V)$, then the projective subspace $\mathbb{P}(W) \subset \mathbb{P}(V)$ is a *projective hyperplane*.

When $V = \mathbb{R}^d$, we have *projective coordinates* on projective space $\mathbb{P}(\mathbb{R}^d)$ defined in terms of the standard basis: the notation $[x_1 : \ldots : x_d]$ denotes the projectivization of the vector $(x_1, \ldots x_d)$.

2.2. **Properly convex domains.** In this section, we give some reminders about convex projective geometry. For a set $X \subset \mathbb{P}(\mathbb{R}^d)$, we denote by $\overline{X}$ the closure of X in the subspace topology induced from $\mathbb{P}(\mathbb{R}^d)$.

Definition 2.1. A subset $\tilde{\Omega} \subset \mathbb{R}^d$ is a *convex cone* if it is convex, nonempty, and closed under multiplication by positive scalars. If $\tilde{\Omega} \subset \mathbb{R}^d$ is a convex cone, we say that its projectivization $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a *properly convex domain* if Ω is open and $\overline{\Omega}$ does not contain any projective line in $\mathbb{P}(\mathbb{R}^d)$ (equivalently, if $\overline{\Omega}$ is a bounded convex subset of some affine chart in $\mathbb{P}(\mathbb{R}^d)$).

The *boundary* of a properly convex domain Ω is its topological boundary $\partial\Omega := \overline{\Omega} - \Omega$. Note that $\overline{\Omega}$ is topologically a closed ball and $\partial\Omega$ is homeomorphic to the boundary of this ball. A *supporting hyperplane* of a convex projective domain Ω is a projective hyperplane in $\mathbb{P}(\mathbb{R}^d)$ which intersects $\overline{\Omega}$, but not Ω.

If $x, y \in \mathbb{P}(\mathbb{R}^d)$ is a pair of distinct points, then $\operatorname{span}_{\mathbb{P}}\{x, y\}$ is a projective line that contains both of them. However, there does not exist a canonical notion of a projective line segment joining x and y in general. But in the presence of a properly convex domain Ω such that $x, y \in \overline{\Omega}$, we can make a canonical choice.

For $x, y \in \Omega$, we say that the *open projective line segment* joining x and y is the unique connected component of $\operatorname{span}_{\mathbb{P}}\{x, y\} - \{x, y\}$ that is contained entirely in $\overline{\Omega}$. We denote this by (x, y). The *projective line segment* joining x and y, denoted by $[x, y]$, is the closure of (x, y) in Ω. We will use the notation $[x, y) := [x, y] - \{y\}$ and $(x, y] := [x, y] - \{x\}$. Finally, if $x = y$, we define $[x, y] = \{x\}$ while $(x, y) = \emptyset$. Often, we will also refer to projective line segments as projective geodesics, as we explain below in Section 2.3.

A *face* of Ω is an equivalence class in $\partial\Omega$ of the relation $\sim$, where $x \sim y$ if there is an *open* projective segment in $\partial\Omega$ containing x and y.

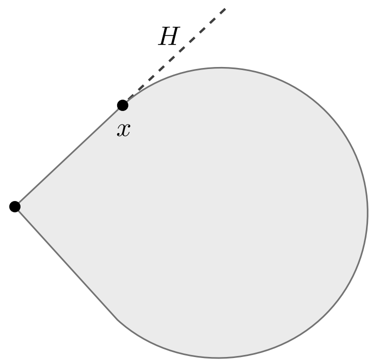

FIGURE 2. The point x is a C^1 extreme point and H is the unique supporting hyperplane at x. Here $F_\Omega(x) = \{x\}$ is a face, but not an exposed face.

Definition 2.2 (Exposed boundary)**.** We say that a face $F \subset \partial\Omega$ is *exposed* if there is a supporting hyperplane H of Ω whose intersection with $\partial\Omega$ is precisely $\overline{F}$; see Fig. 2. A point $x \in \partial\Omega$ is *exposed* if it lies in an exposed face. We say that Ω has *exposed boundary* if every point in $\partial\Omega$ is exposed.

Note that every known example of a convex divisible domain has exposed boundary. However, it is still unknown whether this property holds for every convex divisible domain.

2.3. **The Hilbert metric.** If Ω is properly convex, the *automorphism group* $\mathrm{Aut}(\Omega) \subset \mathrm{PGL}(d, \mathbb{R})$ is the group of projective transformations preserving Ω. One can always define an $\mathrm{Aut}(\Omega)$-invariant metric d_Ω on Ω, called the *Hilbert metric*, as follows. Fix a projective cross-ratio $[\cdot, \cdot; \cdot, \cdot]$ on $\mathbb{R}\mathbb{P}^1$. We choose the cross-ratio given by

$$[a, b; x, y] = \frac{|a-y| \cdot |b-x|}{|a-x| \cdot |b-y|},$$

where $|u - v|$ is the distance measured using any Euclidean metric on an affine chart of $\mathbb{R}\mathbb{P}^1$ containing u, v; the choice of chart and metric does not matter.

Definition 2.3. Let Ω be a properly convex domain. The distance between $x, y \in \Omega$ in the *Hilbert metric* is defined as

$$\mathrm{d}_\Omega(x, y) = \frac{1}{2} \log[a, b; x, y].$$

The pair $(\Omega, \mathrm{d}_\Omega)$ is always a proper geodesic metric space, on which $\mathrm{Aut}(\Omega)$ acts by isometries. This ensures that the action of $\mathrm{Aut}(\Omega)$ on Ω is always proper. When Ω is an ellipsoid, then this metric space is isometric to $(d-1)$-dimensional hyperbolic space; thus Hilbert geometry is a strict generalization of hyperbolic geometry.

A projective line segment $[x, y]$ that lies in Ω (instead of lying entirely in $\partial\Omega$) is a geodesic for the metric d_Ω. Hence, we will call $[x, y]$ the *projective geodesic segment* joining x and y. In the same vein, if $x, y \in \overline{\Omega}$, we call (x, y) a *projective geodesic.* Note that a projective geodesic may be infinite or bi-infinite and, wherever necessary, we will emphasize this by using specific terminology. If $x \in \Omega$ and $y \in \partial\Omega$, then we will call $[x, y)$ (also (x, y)) a *projective geodesic ray.* If $x, y \in \partial\Omega$

but $(x, y) \subset \Omega$, we will call (x, y) a *bi-infinite projective geodesic* (or a *projective geodesic line*).

A projective geodesic segment, however, is often not the only geodesic joining points $x, y \in \Omega$. One can easily verify the following:

Fact 2.4 (Characterizing geodesics)**.** *For pairwise distinct points $w_1, w_2, w_3 \in \Omega$, we have*

$$\mathrm{d}_\Omega(w_1, w_2) = \mathrm{d}_\Omega(w_1, w_3) + \mathrm{d}_\Omega(w_3, w_2)$$

if and only if there are segments $[y, y']$ and $[z, z']$ in $\partial\Omega$ such that y, w_1, w_3, z and y', w_3, w_2, z' are aligned in that order.

Fact 2.4 implies that if Ω is a *strictly* convex domain (i.e. if there are no nontrivial projective segments in $\partial\Omega$), then every geodesic in Ω is projective.

2.4. **Finer metric properties of** d_Ω**.** As mentioned in the introduction to this paper, the metric space $(\Omega, \mathrm{d}_\Omega)$ is typically not a CAT(0) space, and in fact the Hilbert metric often fails to satisfy some of the strong convexity properties enjoyed by general CAT(0) metrics. However, the Hilbert metric does satisfy a weak convexity property called the *maximum principle.*

Lemma 2.5 (Maximum principle; see [CLT15, Corollary 1.9])**.** *If C is a closed convex set in a properly convex domain Ω, then for every compact subset $K \subset \Omega$, the function $K \to \mathbb{R}_{\geq 0}$ given by*

$$x \mapsto \mathrm{d}_\Omega(x, C)$$

attains its maximum at an extreme point of K.

It is also true that when C is a convex subset of a convex projective domain Ω, the nearest-point projection map $\Omega \to C$ is not always well-defined. One can still define a *set-valued* nearest-point projection map $\pi_C : \Omega \to 2^C$, but this map is not necessarily continuous with respect to Hausdorff distance on 2^C. However, using the maximum principle, one can see that the nearest-point projection map onto a projective geodesic in Ω always maps convex sets to connected sets:

Lemma 2.6. *Let ℓ be a projective geodesic in a properly convex domain Ω, and let $\pi_\ell : \Omega \to 2^\ell$ denote the set-valued nearest-point projection map, i.e. the map*

$$\pi_\ell(x) = \{y \in \ell : \mathrm{d}_\Omega(x, y) = \mathrm{d}_\Omega(x, \ell)\}.$$

If $A \subset \Omega$ is convex, then $\pi_\ell(A)$ is connected.

Proof. Fix points $x', y' \in \ell$, so that $x' \in \pi_\ell(x), y' \in \pi_\ell(y)$ for $x, y \in A$, and let z' be a point on the open segment $(x', y') \subset \ell$. We wish to show that $z' \in \pi_\ell(A)$. The point z' separates ℓ into two components, so let ℓ_- be the closure of the component containing x', and let ℓ_+ be the closure of the component containing y'.

Since A is convex, it contains the projective geodesic $[x, y]$. Consider the continuous function $f : [x, y] \to \mathbb{R}$ given by

$$f(u) = \mathrm{d}_\Omega(u, \ell_-) - \mathrm{d}_\Omega(u, \ell_+).$$

For any $u \in [x, y]$, we know that $\mathrm{d}_\Omega(u, \ell) = \min\{\mathrm{d}_\Omega(u, \ell_-), \mathrm{d}_\Omega(u, \ell_+)\}$. So, since $\mathrm{d}_\Omega(x, \ell) = \mathrm{d}_\Omega(x, x') \geq \mathrm{d}_\Omega(x, \ell_-)$, it follows that $f(x)$ is nonpositive. Similarly, $f(y)$ is nonnegative, so there is some $z \in [x, y]$ with $f(z) = 0$, i.e. z satisfying $\mathrm{d}_\Omega(z, \ell_-) = \mathrm{d}_\Omega(z, \ell_+)$. Thus there are points $z_\pm \in \ell_\pm$ which satisfy

$$\mathrm{d}_\Omega(z, z_+) = \mathrm{d}_\Omega(z, z_-) = \mathrm{d}_\Omega(z, \ell_\pm) = \mathrm{d}_\Omega(z, \ell).$$

Then Lemma 2.5 implies that every point w in $[z_-, z_+]$ satisfies $\mathrm{d}_\Omega(z, w) \le \mathrm{d}_\Omega(z, \ell)$, so in fact each such w satisfies $\mathrm{d}_\Omega(z, w) = \mathrm{d}_\Omega(z, \ell)$, i.e. $w \in \pi_\ell(z)$. As $z \in A$ and the geodesic $[z_-, z_+]$ contains the previously chosen point z', this proves the claim. □

2.5. **Space of properly convex domains.** Suppose (X, d) is a metric space. This induces a notion of Hausdorff distance between closed subsets $A, B \subset X$ defined by:

$$d^{\mathrm{Haus}}(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b) \right\}.$$

Fixing a metric on the projective space $\mathbb{P}(V)$, compatible with the standard topology on $\mathbb{P}(V)$, defines a notion of Hausdorff distance between subsets of $\mathbb{P}(V)$ (or more precisely, their closures).

Definition 2.7. Let V be a real vector space. We denote by $\mathcal{C}(V)$ the space of properly convex projective domains in $\mathbb{P}(V)$. The topology on $\mathcal{C}(V)$ is the topology induced by Hausdorff distance, with respect to any metrization of $\mathbb{P}(V)$.

Note that the topology on $\mathcal{C}(V)$ is independent of the metrization on $\mathbb{P}(V)$.

2.6. **The Benzécri cocompactness theorem.**

Definition 2.8. Let $\mathcal{C}_*(V)$ denote the space of *pointed* domains in $\mathbb{P}(V)$, i.e. the space

$$\mathcal{C}_*(V) := \{(\Omega, x) \in \mathcal{C}(V) \times \mathbb{P}(V) : x \in \Omega\}.$$

The topology on $\mathcal{C}_*(V)$ is the product topology that it inherits from $\mathcal{C}(V) \times \mathbb{P}(V)$. The group $\mathrm{PGL}(V)$ acts pointwise on $\mathcal{C}(V)$, and diagonally on $\mathcal{C}_*(V)$. We have the following important result:

Theorem 2.9 (Benzécri cocompactness [Ben60]). *The action of* $\mathrm{PGL}(V)$ *on* $\mathcal{C}_*(V)$ *is both proper and cocompact.*

Theorem 2.9 turns out to be very useful when we consider the case of a *non-cococompact* group action on a properly convex domain. Although divisible domains Ω are often technically easier to work with than general domains, they require the automorphism group $\mathrm{Aut}(\Omega)$ to be 'large'. In this paper, we will often be interested in studying general properly convex domains, not necessarily divisible. In such cases, the Benzécri cocompactness theorem becomes a powerful tool that we can use to import techniques for divisible domains to the non-divisible case.

Typically, we apply the theorem to a sequence of points x_n in some domain Ω which leaves every compact subset of Ω, to find a sequence of "approximate automorphisms" taking x_n back to some fixed basepoint. The properness part of the theorem ensures that any choice of "approximate automorphisms" differ by elements in a compact set, which we can often use to obtain information about a given sequence of divergent elements in $\mathrm{Aut}(\Omega)$.

2.7. **Properties of faces in properly convex domains.** Every face F of a properly convex domain is itself a properly convex domain in its own projective span. Consequently, F can be endowed with its own Hilbert metric d_F. This Hilbert metric is related to the Hilbert metric on the larger domain Ω, and gives a way to characterize faces in terms of metric (rather than projective) geometry. This is expressed via Lemma 2.10 below.

We state this lemma in a fairly general form. In particular, we allow the domain Ω to vary continuously in the space of all properly convex domains $\mathcal{C}(\mathbb{R}^d)$ (see Definition 2.7).

Lemma 2.10. *Let $\{\Omega_n\}$ be a sequence of properly convex domains in $\mathbb{P}(\mathbb{R}^d)$, converging in $\mathcal{C}(\mathbb{R}^d)$ to a properly convex domain Ω_∞. Suppose that points $x_n, y_n \in \Omega_n$ converge to $x, y \in \overline{\Omega_\infty}$. If*

$$\liminf_{n\to\infty} d_{\Omega_n}(x_n, y_n) < \infty,$$

then x and y lie in the same face F of Ω_∞, and

$$\mathrm{d}_F(x, y) \leq \liminf_{n\to\infty} d_{\Omega_n}(x_n, y_n).$$

Since this version of the lemma is slightly more general than versions that typically appear in the literature, we provide a proof.

Proof of Lemma 2.10. Note that there is nothing to prove if $x = y$, so assume that $x \neq y$. Let $[a_n, b_n] := \overline{\Omega_n} \cap \mathrm{span}_{\mathbb{P}}\{x_n, y_n\}$ where the labels a_n, b_n are assigned in such a way that the four points a_n, x_n, y_n, b_n appear in this order along $\mathrm{span}_{\mathbb{P}}\{x_n, y_n\}$. Up to passing to a subsequence, we can assume that $a_n \to a$ and $b_n \to b$ in $\mathbb{P}(\mathbb{R}^d)$. Since $\Omega_n \to \Omega_\infty$, $a, b \in \overline{\Omega_\infty}$ and $[a_n, b_n] \to [a, b]$. Thus $x, y \in [a, b]$ which implies that $a \neq b$, since otherwise x will be equal to y. By the ordering of the labels a_n, b_n, we know that the points a, x, y, b appear in this order along $\mathrm{span}_{\mathbb{P}}\{a, b\}$. If either $a = x$ or $b = y$, then the cross-ratio $[a_n, b_n; x_n, y_n] \to \infty$ and contradicts $\liminf_{n\to\infty}(x_n, y_n) < \infty$. Thus, the four points $a, x, y, b \in \overline{\Omega_\infty}$ are pairwise distinct. Hence $x, y \in (a, b)$.

Now observe that (a, b), which is an open projective line segment in $\overline{\Omega_\infty}$, is either disjoint from $\partial\Omega_\infty$ or is entirely contained in it. Since $x, y \in (a, b) \cap \partial\Omega_\infty$, $(a, b) \subset \partial\Omega_\infty$. This implies that x, y belong to a face F in Ω_∞. Moreover, by continuity of cross-ratios,

$$d_F(x, y) \leq d_{(a,b)}(x, y) \leq \liminf_{n\to\infty} d_{(a_n,b_n)}(x_n, y_n) = \liminf_{n\to\infty} d_{\Omega_n}(x_n, y_n). \qquad \square$$

When the domain Ω is fixed, we can use Lemma 2.10 together with the maximum principle (Lemma 2.5) to obtain a related estimate for the Hausdorff distance between a pair of projective geodesics. For this lemma, we follow the convention that $F_\Omega(x) = \Omega$ if $x \in \Omega$, while $F_\Omega(x)$ is the face containing x if $x \in \partial\Omega$.

Lemma 2.11. *Suppose Ω is a properly convex domain, $x_\pm \in \Omega$, and $y_\pm \in F_\Omega(x_\pm)$. If $(x_+, x_-) \subset \Omega$, then $(y_-, y_+) \subset \Omega$ and*

$$\mathrm{d}_\Omega{}^{\mathrm{Haus}}((y_+, y_-), (x_+, x_-)) \leq \max\left\{d_{F_\Omega(x_\pm)}(x_\pm, y_\pm)\right\}.$$

2.8. **Properly embedded simplices.** For any $k \geq 0$, a *standard projective k-simplex* in $\mathbb{P}(\mathbb{R}^d)$ is

$$S_k := \{[x_1 : x_2 : \cdots : x_{k+1} : 0 : \cdots : 0] \mid x_1, \ldots, x_{k+1} > 0\}.$$

We say that S_k is the simplex spanned by $[e_1], \ldots, [e_{k+1}]$. A *projective k-simplex* is any set in $\mathbb{P}(\mathbb{R}^d)$ that is projectively equivalent to a standard projective k-simplex.

Definition 2.12. Suppose Ω is a properly convex domain and $A \subset \Omega$ is a convex subset. Then we say that:

(1) A is a *properly embedded* subset if $A \hookrightarrow \Omega$ is a proper map, or equivalently if $\partial A \subset \partial\Omega$.
(2) A is a *properly embedded k-simplex* if A is properly embedded in Ω and a projective k-simplex.

Properly embedded simplices are projective analogs of totally geodesic flats in CAT(0) spaces. Consider, for example, a properly embedded triangle, or 2-simplex. Suppose the vertices of such a triangle Δ are represented by the standard basis vectors in $\mathbb{R}^3$. Then the group

$$\left\{ \begin{pmatrix} 2^a & & \\ & 2^b & \\ & & 2^c \end{pmatrix} : a, b, c \in \mathbb{Z}, a+b+c = 0 \right\}$$

acts properly discontinuously and cocompactly on Δ. So (Δ, d_Δ) equipped with its Hilbert metric is quasi-isometric to a 2-flat. Hence properly embedded simplices serve as analogs of isometrically embedded flats in CAT(0) spaces.

2.9. **Singular values and the Cartan projection.** In this section we briefly recall the definitions and basic properties of the Cartan projection $\mathrm{GL}(d, \mathbb{R}) \to \mathbb{R}^d$. We will always equip $\mathbb{R}^d$ with its standard Euclidean inner product.

Definition 2.13. For any $g \in \mathrm{GL}(d, \mathbb{R})$, we let $\sigma_1(g) \geq \sigma_2(g) \geq \ldots \geq \sigma_d(g) > 0$ denote the *singular values* of g, counted with multiplicity. We let $\mu : \mathrm{GL}(d, \mathbb{R}) \to \mathbb{R}^d$ denote the *Cartan projection*, given by $\mu_i(g) = \log \sigma_i(g)$. The Cartan projection can be also be defined via the *Cartan decomposition* of a group element $g \in \mathrm{GL}(d, \mathbb{R})$: $\mu(g)$ is the unique vector in $\mathbb{R}^d$ with nonincreasing entries such that

$$g = k \cdot \exp(\mathrm{diag}(\mu_1(g), \ldots \mu_d(g))) \cdot \ell,$$

for some $k, \ell \in \mathrm{O}(d)$. For $1 \leq i \leq j \leq d$, we let $\mu_{i,j}(g)$ denote the nonnegative quantity $\mu_i(g) - \mu_j(g)$.

Remark 2.14. Although the Cartan projection μ is only defined on $\mathrm{GL}(d, \mathbb{R})$, the values of $\mu_{i,j}$ are well-defined on the quotient $\mathrm{PGL}(d, \mathbb{R})$.

The singular values of any $g \in \mathrm{GL}(V)$ have an interpretation in terms of the norm and the conorm of g. Recall that if $g \in \mathrm{GL}(V)$, the operator norm is

$$||g|| = \sup_{v \in \mathbb{R}^d - \{0\}} \frac{||gv||}{||v||},$$

while the conorm is

$$\mathbf{m}(g) = ||g^{-1}||^{-1}.$$

The largest singular value is given by $\sigma_1(g) = \|g\|$ while the smallest singular value is given by $\sigma_d(g) = \mathbf{m}(g)$. More generally, for any $1 \leq k \leq d$, we let $\mathrm{Gr}(k, d)$ denote the Grassmannian of k-dimensional subspaces of $\mathbb{R}^d$. Then one has the "minimax" formula:

$$\sigma_k(g) = \max_{W \in \mathrm{Gr}(i,d)} \mathbf{m}(g|_W). \tag{1}$$

Note that if $g \in \mathrm{SL}(d, \mathbb{R})$, we have $\prod \sigma_i(g) = 1$ and thus $\sum \mu_i(g) = 0$. Using this, we see that for any $g \in \mathrm{SL}(d, \mathbb{R})$, we have

$$\mu_{1,d}(g) = \log(||g||) + \log(||g^{-1}||).$$

Lemma 2.15 (Additivity of Cartan projection, see [GGKW17, Fact 2.18])**.** *There is a constant $K_0 > 0$ so that for any $g, h_1, h_2 \in \mathrm{GL}(d, \mathbb{R})$, we have*

$$||\mu(h_1 g h_2) - \mu(g)|| \leq K_0(||\mu(h_1)|| + ||\mu(h_2)||). \tag{2}$$

In particular, for any $1 \leq i < j \leq d$, there is a constant $K > 0$ such that

$$|\mu_{i,j}(h_1 g h_2) - \mu_{i,j}(g)| \leq K(||\mu(h_1)|| + ||\mu(h_3)||). \tag{3}$$

Remark 2.16. For an appropriate choice of norm on $\mathbb{R}^d$ (which is typically not the standard norm), the inequality (2) can be strengthened to

$$||\mu(h_1 g h_2) - \mu(g)|| \leq ||\mu(h_1)|| + ||\mu(h_2)||.$$

This immediately implies the version of the inequality we have stated above.

Lemma 2.17. *Suppose $g \in \mathrm{GL}(d, \mathbb{R})$ and there exist $C > 0$ and $1 \leq i \leq j \leq d$ such that*

$$|\mu_{i,j}(g) - \mu_{1,d}(g)| \leq C.$$

Then:

1. *$\mu_{1,k}(g) \leq C$ for $k \in \{1, \ldots, i\}$,*
2. *$\mu_{k,d}(g) \leq C$ for $k \in \{j, \ldots, d\}$, and*
3. *$\mu_{k,k+1}(g) \leq C$ for $k \in \{1, \ldots, i-1\} \cup \{j, \ldots, d-1\}$.*

Proof. Since the values of $\mu_k(g)$ are non-increasing, $\mu_{i',j'}(g) \geq 0$ for any $1 \leq i' \leq j' \leq d$. But $\mu_{1,d}(g)$ is equal to the sum $\mu_{1,i}(g) + \mu_{i,j}(g) + \mu_{j,d}(g)$. Thus $\mu_{1,i}(g) \leq C$ and $\mu_{j,d}(g) \leq C$. The first two inequalities are then immediate as $\mu_{1,k}(g) \leq \mu_{1,i}(g)$ whenever $k \in \{1, \ldots, i-1\}$, and $\mu_{k,d}(g) \leq \mu_{j,d}(g)$ for any $k \in \{j, \ldots, d\}$. The third inequality follows from the first two and the fact that $\mu_{k,k+1}(g)$ is bounded by either $\mu_{1,i}(g)$ or $\mu_{j,d}(g)$ whenever $k \in \{1, \ldots, i-1\} \cup \{j, \ldots, d-1\}$. □

Let $\angle$ be the standard angle in $\mathbb{R}^d$ induced by the standard Euclidean inner product. Note that $\angle$ also defines a Riemannian metric $\mathrm{d}_{\mathbb{P}}$ on $\mathbb{P}(\mathbb{R}^d)$, by setting $\mathrm{d}_{\mathbb{P}}(u, v) = \angle(u, v)$ for any $u, v \in \mathbb{P}(\mathbb{R}^d)$. There is an analogous notion of angles between subspaces.

Definition 2.18. If U, W are two transverse subspaces of $\mathbb{R}^d$, we define the angle $\angle(U, W)$ by

$$\angle(U, W) = \inf_{\substack{u \in U - \{0\} \\ w \in W - \{0\}}} \angle(u, w).$$

Lemma 2.19. *For any $\varepsilon > 0$, there exists $C \equiv C(\varepsilon)$ satisfying the following. Suppose we have two decompositions*

$$\mathbb{R}^d = U_1 \oplus \ldots \oplus U_k,$$
$$\mathbb{R}^d = W_1 \oplus \ldots \oplus W_k,$$

such that $\dim(U_i) = \dim(W_i)$ for all i, and $\angle(U_i, U_j) \geq \varepsilon$ and $\angle(W_i, W_j) \geq \varepsilon$ for all $i \neq j$. Then there is some $k \in \mathrm{GL}(d, \mathbb{R})$ such that $k(U_i) = W_i$ for all i and $\mu_{1,d}(k) \leq C$.

Proof. By choosing orthogonal bases for each U_i and each W_i, we can reduce to the case where the subspaces U_i and W_i are all one-dimensional. Then, using Lemma 2.15, we can further reduce to the case where the subspaces U_i give the decomposition of $\mathbb{R}^d$ into the lines spanned by the standard basis vectors $e_1, \ldots, e_d$.

We can pick unit vectors $w_1, \ldots, w_d$ spanning each W_i, and consider the matrix k whose columns are $w_1, \ldots, w_d$. Then k takes U_i to W_i, and lies in the compact subset $K(\varepsilon)$ of $\mathrm{GL}(d, \mathbb{R})$ consisting of matrices whose columns are unit vectors having pairwise angles at least ε. By compactness there is a uniform upper bound C on $\mu_{1,d}(K(\varepsilon))$, and the result follows. □

3. Morse geodesics are contracting

Our main goal in this section is to prove Theorem 1.16, which says that Morse geodesics (Definition 1.1) in a convex projective domain Ω are equivalent to contracting geodesics (Definition 1.15). As part of the proof, we also introduce the framework of *conically related* pairs of points in boundaries of convex projective domains, and use this to provide a number of other characterizations of Morse geodesics in Ω. These ideas will reappear later in Section 5, when we use them to study the linear algebraic behavior of Morse geodesics.

Our proof of the equivalence between Morse and contracting geodesics goes through a δ-*slimness* property for geodesic triangles, which is reminiscent of a similar property that also characterizes Morseness in CAT(0) spaces. We define this property below. Note that the definition does not apply in general metric spaces, since it relies on the existence of a preferred geodesic between every pair of points (in this case, a projective geodesic).

Definition 3.1. Let ℓ be a projective geodesic in a properly convex domain Ω and $\delta \geq 0$. We say that ℓ is *projectively δ-slim* if, any projective geodesic triangle $[x, y] \cup [y, z] \cup [z, x]$ with $x, y, z \in \Omega$ and $[x, y] \subset \ell$ is δ-slim, i.e., for $\{a, b, c\} = \{x, y, z\}$, we have

$$[a, c] \subset N_\delta([a, b]) \cup N_\delta([b, c]).$$

Remark 3.2. [IZ23, Lemma 13.8] implies that for a projective geodesic triangle to be δ-slim, it suffices that one of its edges is contained in the $\frac{\delta}{2}$ neighborhood of its other two edges. More precisely, $[x, y] \cup [y, z] \cup [z, x] \subset \Omega$ is δ-slim if $[x, y] \subset N_r([x, z]) \cup N_r([z, y])$ with $r := \frac{\delta}{2}$.

Our main result in this section is the following:

Proposition 3.3. *Let Ω be a properly convex domain and let ℓ be a projective geodesic in Ω. The following are equivalent:*

(1) ℓ is Morse.
(2) ℓ is projectively δ-slim.
(3) ℓ is contracting.

In Proposition 3.3, the implication (3) $\implies$ (1) follows from a well-known general result, stated below. The proof is standard; see e.g. [Sul14, Lemma 3.3].

Proposition 3.4. *Let X be a proper geodesic metric space and let $D > 0$. There exists a Morse gauge M, depending only on D and X, so that any D-contracting geodesic in X is M-Morse.*

The implication (2) $\implies$ (3) in Proposition 3.3 is also straightforward, and we provide a quick proof below. Most of the rest of this section is then devoted to the proof of the implication (1) $\implies$ (2).

3.1. **Projectively δ-slim implies contracting.** This is the implication (2) $\implies$ (3) in Proposition 3.3. For this result, we mostly imitate the proof, due to Charney-Sultan, of an analogous statement for CAT(0) spaces (see Theorem 2.14 in [CS15]). It turns out that in most situations, the Charney-Sultan proof does not use the full strength of the CAT(0) condition, but only the weaker *maximum principle* (see Lemma 2.5).

The Charney-Sultan proof does also appeal to the CAT(0) condition in ways not covered by the maximum principle. However, it is not difficult to modify the proof to avoid this, at the cost of increasing some of the constants appearing in the proof. The first step is the following lemma.

Lemma 3.5. *Let Ω be a properly convex domain and let $\ell \subset \Omega$ be a projective geodesic. Suppose that ℓ is projectively δ-slim. Then, for any $x \in \Omega$, $y \in \ell$, and $z \in \pi_\ell(x)$, we have $\mathrm{d}_\Omega(z, [x, y]) < 4\delta$.*

Proof. If $\mathrm{d}_\Omega(y, z) \le 2\delta$ we are done, so assume that $\mathrm{d}_\Omega(y, z) > 2\delta$, and then choose a point $w \in [y, z]$ so that $2\delta < \mathrm{d}_\Omega(w, z) < 3\delta$. Then let u be a point on $[x, z]$ so that

$$\mathrm{d}_\Omega(w, u) = \mathrm{d}_\Omega(w, [x, z]).$$

From the triangle inequality we have

$$\mathrm{d}_\Omega(x, u) + \mathrm{d}_\Omega(u, w) \ge \mathrm{d}_\Omega(x, w),$$

and since $z \in \pi_\ell(x)$ and $w \in \ell$ we know that

$$\mathrm{d}_\Omega(x, w) \ge \mathrm{d}_\Omega(x, z) = \mathrm{d}_\Omega(x, u) + \mathrm{d}_\Omega(u, z).$$

Putting the previous two lines together we see that $\mathrm{d}_\Omega(u, z) \le \mathrm{d}_\Omega(u, w)$. But then

$$2\delta < \mathrm{d}_\Omega(w, z) \le \mathrm{d}_\Omega(w, u) + \mathrm{d}_\Omega(u, z) \le 2\, \mathrm{d}_\Omega(u, w),$$

which implies that $\mathrm{d}_\Omega(w, [x, z]) = \mathrm{d}_\Omega(w, u) > \delta$.

Now, as ℓ is projectively δ-slim, the projective geodesic triangle $[x, y] \cup [y, z] \cup [z, x]$ is δ-slim. Since $\mathrm{d}_\Omega(w, [x, z]) > \delta$, we have $\mathrm{d}_\Omega(w, [x, y]) < \delta$. Thus

$$\mathrm{d}_\Omega(z, [x, y]) \le \mathrm{d}_\Omega(z, w) + \mathrm{d}_\Omega(w, [x, y]) < 4\delta.$$

□

The following completes the proof that (2) $\implies$ (3) in Proposition 3.3.

Proposition 3.6. *Let Ω be a properly convex domain and let ℓ be a projective geodesic in Ω. If ℓ is projectively δ-slim, then ℓ is 24δ-contracting.*

Proof. Let $B = B(x, r)$ be a ball not intersecting ℓ. Let $y \in B$ and let $x' \in \pi_\ell(x), y' \in \pi_\ell(y)$.

By Lemma 3.5, there is a point $u \in [y, x']$ such that $d(y', u) < 4\delta$. The maximum principle (Lemma 2.5) implies that

$$\mathrm{d}_\Omega(x, u) \le \max\{\mathrm{d}_\Omega(x, y), \mathrm{d}_\Omega(x, x')\} = \mathrm{d}_\Omega(x, x'),$$

so

$$\mathrm{d}_\Omega(x, y') \le \mathrm{d}_\Omega(x, x') + 4\delta.$$

Then we apply Lemma 3.5 again to see that there is a point $w \in [y', x]$ so that $\mathrm{d}_\Omega(x', w) < 4\delta$. Then

$$\begin{aligned}\mathrm{d}_\Omega(x, y') &= \mathrm{d}_\Omega(x, w) + \mathrm{d}_\Omega(w, y') \\ &\geq \mathrm{d}_\Omega(x, x') - \mathrm{d}_\Omega(x', w) + \mathrm{d}_\Omega(y', x') - \mathrm{d}_\Omega(x', w) \\ &\geq \mathrm{d}_\Omega(x, x') + \mathrm{d}_\Omega(y', x') - 8\delta.\end{aligned}$$

That is, we have

$$\mathrm{d}_\Omega(x, x') + \mathrm{d}_\Omega(y', x') - 8\delta \leq \mathrm{d}_\Omega(x, y') \leq \mathrm{d}_\Omega(x, x') + 4\delta,$$

which implies $\mathrm{d}_\Omega(y', x') < 12\delta$. Thus the diameter of the nearest-point projection of B onto ℓ is at most 24δ. □

Having proved that (2) $\implies$ (3) $\implies$ (1) in Proposition 3.3, we now turn to the implication (1) $\implies$ (2). Our proof of this implication relies much more heavily on the convex projective geometry of the domain Ω. In particular, we develop a notion of *conically related* pairs of points in the boundary of certain pairs of properly convex domains, and show that Morseness is preserved between conically related points. This allows us to develop several other more technical characterizations of Morse geodesics in a convex projective domain Ω, which we ultimately use to establish the desired implication in Proposition 3.3.

We state all of our different equivalences below in Proposition 3.10. First, however, we need a few more definitions.

3.2. **Half triangles.** Half-triangles in convex projective domains extend the analogy between properly embedded triangles and flats in CAT(0) spaces (see Section 2.8) to isometrically embedded *half-flats* in CAT(0) spaces.

Definition 3.7. Let Ω be a properly convex domain. A *half-triangle* in Ω consists of three points $x, y, z \in \partial\Omega$, such that exactly two of the segments $[x, y], [y, z], [z, x]$ are contained in $\partial\Omega$.

Note that, as a subspace of Ω with its restricted Hilbert metric, a half-triangle is *not* necessarily isometric to a half-space (i.e. a subspace bounded by a geodesic) in a properly embedded triangle. Nevertheless half-triangles still bear some resemblance to half-flats, since segments in the boundary of a properly convex domain correspond roughly to "flat directions" (see e.g. Lemma 3.22).

3.3. **Conically related points.** The idea behind our next definition (that of *conically related points*) is to consider what a properly convex domain Ω "looks like" from the perspective of a sequence of points traveling along a projective geodesic ray c towards the ideal endpoint $c(\infty)$ in $\partial\Omega$. If Ω has a cocompact action by projective automorphisms, we can consider a sequence of points $\{x_n\}$ limiting to an ideal endpoint z of c, and a sequence of group elements $\{\gamma_n\}$ in $\mathrm{Aut}(\Omega)$ taking x_n back to some fixed compact subset of Ω. The projective geometry of the accumulation points of the sequence $\{\gamma_n z\}$ in $\partial\Omega$ should inform the metric geometry of the geodesic c.

More generally, when $\mathrm{Aut}(\Omega)$ does *not* act cocompactly on Ω, we can use the Benzécri cocompactness theorem (Theorem 2.9) to find elements g_n in $\mathrm{PGL}(V)$ which "translate" points in the sequence $\{x_n\}$ into a fixed compact subset of some limiting domain Ω_∞. Again, we can understand the metric geometry of the geodesic c by looking at accumulation points of $\{g_n z\}$ in $\partial\Omega_\infty$.

In [Ben03], Benoist used essentially this approach to investigate the global hyperbolicity of arbitrary convex projective domains. The definition below gives one way to formalize this idea. (For another, see e.g. [Wei23, Section 5]).

Definition 3.8. Let Ω_1, Ω_2 be properly convex domains, let $z_1 \in \partial\Omega_1$, and let z_2^+, z_2^- be points in $\partial\Omega_2$ such that $(z_2^+, z_2^-) \subset \Omega_2$. Suppose that:

(1) there is a sequence of points $\{x_n\}$ in the projective geodesic ray $[x, z_1) \subset \Omega_1$ such that x_n converges to z_1, and
(2) there is a divergent sequence of group elements $\{g_n\}$ in $\mathrm{PGL}(V)$ (i.e. a sequence $\{g_n\}$ which leaves every compact set in $\mathrm{PGL}(V)$) such that $g_n(z_1, x_n)$ converges to $(z_2^+, z_2^-) \subset \Omega_2$ and $g_n\Omega_1$ converges to Ω_2.

Then we say that (z_1, Ω_1) is *forward conically related to* (z_2^+, Ω_2) by the sequence $\{g_n\}$, and *backward conically related to* (z_2^-, Ω_2) by the sequence $\{g_n\}$.

If the domains Ω_1, Ω_2 are understood from context, we will sometimes just say that z_1 is (forward or backward) conically related to z_2^+ or z_2^-.

Observe that if (z_1, Ω_1) is (forward or backward) conically related to (z_2, Ω_2), it follows immediately that for any $g_1, g_2 \in \mathrm{PGL}(V)$, $(g_1 z_1, g_1\Omega_1)$ is also (forward or backward) conically related to $(g_2 z_2, g_2\Omega_2)$. That is:

Proposition 3.9. *The relation "(z_1, Ω_1) is conically related to (z_2, Ω_2)" is well-defined when we regard (z_i, Ω_i) as elements in the quotient set*

$$\{(x, \Omega) \in \mathbb{P}(V) \times \mathcal{C}(V) : x \in \partial\Omega\} / \mathrm{PGL}(V),$$

where $\mathrm{PGL}(V)$ acts diagonally on $\mathbb{P}(V) \times \mathcal{C}(V)$.

Later we will prove a number of other straightforward but useful properties of conically related points. In particular, we will show that Morseness is preserved between geodesics with conically related endpoints (see Lemma 3.21).

3.4. **Characterizations of Morseness.** We can now state our full characterization of Morse projective geodesics, giving a more general version of Proposition 3.3:

Proposition 3.10. *Suppose ℓ is a projective geodesic in a properly convex domain Ω. Then the following are equivalent:*

(M) The geodesic ℓ is Morse for some Morse gauge M.
(HT) For every endpoint z_1 of ℓ in $\partial\Omega$, if z_1 is forward conically related to a point $z_2 \in \partial\Omega_2$, then z_2 does not lie in the boundary of a half-triangle in Ω_2.
(HT−) For every endpoint z_1 of ℓ in $\partial\Omega$, if z_1 is backward conically related to a point $z_2 \in \partial\Omega_2$, then z_2 does not lie in the boundary of a half-triangle in Ω_2.
(SC) For every endpoint z_1 of ℓ in $\partial\Omega$, if z_1 is forward conically related to a point $z_2 \in \partial\Omega_2$, then $(z_2, w) \subset \Omega_2$ for every $w \in \partial\Omega_2 - \{z_2\}$.
(SC−) For every endpoint z_1 of ℓ in $\partial\Omega$, if z_1 is backward conically related to a point $z_2 \in \partial\Omega_2$, then $(z_2, w) \subset \Omega_2$ for every $w \in \partial\Omega_2 - \{z_2\}$.
(δ) The geodesic ℓ is projectively δ-slim for some $\delta > 0$.
(C) The geodesic ℓ is D-contracting for some D.

Remark 3.11. We allow the projective geodesic ℓ in the statement of Proposition 3.10 to have zero, one, or two endpoints in the boundary of the properly convex domain Ω. In the case where ℓ has zero ideal endpoints (meaning it is a compact segment in Ω), then the conditions *(HT)*, *(HT−)*, *(SC)*, and *(SC−)*

are vacuous. In this case, conditions (M) and (C) hold trivially since ℓ has finite diameter, and condition (δ) follows from Lemma 2.11.

The proof of Proposition 3.10 follows the scheme given in Figure 3 below. Each implication is labeled with the number of the intermediate result(s) that provide its proof.

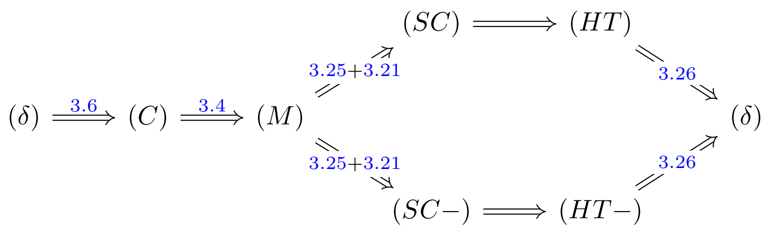

FIGURE 3. Proof outline for Proposition 3.10

Note that we have already shown the implications $(\delta) \implies (C) \implies (M)$. There are no labels on the implications $(SC) \implies (HT)$ and $(SC-) \implies (HT-)$ as they are immediate. Indeed, if $z_2 \in \partial\Omega_2$ is in the boundary of a half-triangle in Ω_2, then there exists $w \in \partial\Omega_2 - \{z_2\}$ such that $[z_2, w] \subset \partial\Omega_2$.

3.5. **Projective geodesics in triangles and half-triangles.** The first step towards proving the remaining implications in Proposition 3.10 is to observe that Morse geodesics cannot have endpoints lying in the boundary of triangles or half-triangles. This should be unsurprising if we accept that triangles and half-triangles are analogs of flats and half-flats.

Lemma 3.12. *Suppose that $y \in \partial\Omega$ lies in the boundary of a properly embedded triangle in Ω. Then for any $x \in \Omega$, the projective geodesic $[x, y)$ is not Morse.*

Proof. Since Morseness does not depend on the choice of basepoint, we can assume that x lies in the interior of the properly embedded triangle Δ whose boundary contains y. Then the projective geodesic $[x, y)$ is also contained in Δ. But Δ is quasi-isometric to a 2-flat, and 2-flats contain no Morse quasi-geodesics, so $[x, y)$ cannot be Morse. ☐

Lemma 3.13. *Let x, y, z be the vertices of a half-triangle in $\partial\Omega$ with $(y, z) \subset \Omega$, and suppose that $[x, y]$ is a maximal segment in $\partial\Omega$. Then for any $w \in \Omega$, the projective geodesic $[w, y)$ is not Morse.*

The proof of Lemma 3.13 is somewhat more complicated than the proof of Lemma 3.12, because half-triangles in a properly convex domain are not necessarily quasi-isometric to half-flats. Our proof instead takes advantage of the following result of Cordes:

Lemma 3.14 ([Cor17, Key Lemma])**.** *Let X be a geodesic metric space. For any Morse gauge M, there exists a constant δ_M so that, if $\alpha : [0, \infty) \to X$ is an M-Morse geodesic ray, and $\beta : [0, \infty) \to X$ is a geodesic ray such that $\beta(0) = \alpha(0)$ and the images of α, β have finite Hausdorff distance, then for all $t \in [0, \infty)$ we have $d_X(\alpha(t), \beta(t)) < \delta_M$.*

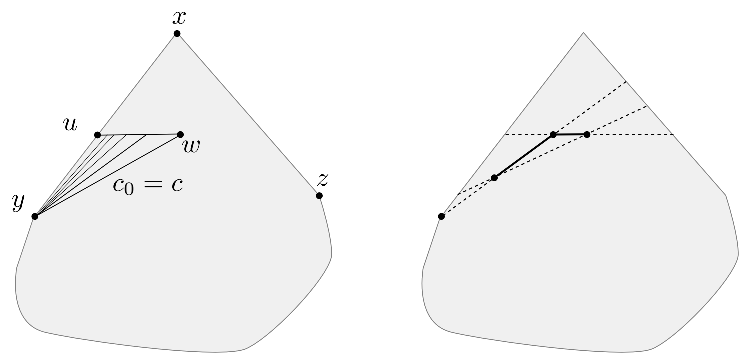

FIGURE 4. Left: the sequence of "broken geodesics" c_n. Right: verifying that each c_n is actually a geodesic, using Fact 2.4.

Proof of Lemma 3.13. Let x, y, z, w be as in the statement of the lemma. Since Morseness is basepoint-independent, we may assume that w actually lies in the convex hull of x, y, z. Consider the projective geodesic $[w, y)$, and fix a point $u \in (y, x)$ so that the projective line spanned by u, w has its other ideal endpoint in the interval (z, x). Let $c : [0, \infty) \to \Omega$ be a unit-speed parameterization of the geodesic ray $[w, y)$, and let $s : [0, \infty) \to \Omega$ be a unit-speed parameterization of $[w, u)$.

For each $n \in \mathbb{N}$, let $r_n : [0, \infty) \to \Omega$ be a unit-speed parameterization of the projective geodesic $[s(n), y)$. Consider the sequence of "broken geodesics" $c_n : [0, \infty) \to \Omega$ given by

$$c_n(t) = \begin{cases} s(t), & t \leq n \\ r_n(t - n), & t > n \end{cases}.$$

Fact 2.4 implies that each c_n is actually a geodesic in Ω, with endpoint y (see Figure 4). Moreover, by Lemma 2.11, the Hausdorff distance between $c_n([0, \infty))$ and (w, y) is bounded by $\mathrm{d}_\Omega(c_n(n), (w, y))$.

Now suppose that (w, y) is a M-Morse geodesic for some M. Then Lemma 3.14 tells us that $\mathrm{d}_\Omega(c_n(n), c(n))$ is bounded above by a constant that depends only on M. As $n \to \infty$, the sequence $c_n(n)$ approaches u, and $c(n)$ approaches y. Then Lemma 2.10 implies that $u \in F_\Omega(y)$, which contradicts the maximality of the line segment $[x, y] \in \partial\Omega$. Thus (w, y) cannot be Morse. □

3.6. **Properties of conically related points.** The previous two lemmas show that, for a projective geodesic ℓ in Ω, having an endpoint in a triangle or half-triangle is an obstruction to Morseness for ℓ. For the proof of Proposition 3.10, we want to show that having an endpoint which is *conically related* to an endpoint in a triangle or half-triangle also obstructs Morseness; this will follow from Lemma 3.21 below. Before we state and prove this lemma, however, we make a brief digression to develop the theory of conically related points a little further.

First, observe that "(z_1, Ω_1) is conically related to (z_2, Ω_2)" is *not* an equivalence relation, since it is not in general symmetric. In addition, the relation is not even necessarily reflexive, since we require the sequence of group elements g_n appearing in the definition to be divergent. However, the relation does satisfy the following:

Lemma 3.15. *Suppose Ω is a properly convex domain and $x \in \partial\Omega$. Then there exists a properly convex domain Ω' and $x'_\pm \in \partial\Omega'$ such that (x, Ω) is forward conically related to (x'_+, Ω') and backward conically related to (x'_-, Ω').*

Proof. Fix a basepoint $x_0 \in \Omega$ and pick a sequence $\{p_n\}$ in $[x_0, x)$ such that $p_n \to x$. Pick another sequence $\{q_n\}$ such that $q_n \in [p_n, x)$ and $\mathrm{d}_\Omega(p_n, q_n) = n$. By Theorem 2.9, there exists a sequence $\{g_n\}$ in $\mathrm{PGL}(V)$ such that $g_n(\Omega, q_n)$ converges to $(\Omega_\infty, q_\infty) \in \mathcal{C}_*(V)$. Up to passing to a subsequence, we can assume that $g_n p_n \to p_\infty$ and $g_n x \to x_\infty$.

Since $x \in \partial\Omega$, we have $x_\infty \in \partial\Omega_\infty$. We also know that $p_\infty \in \partial\Omega_\infty$, because $\mathrm{d}_\Omega(q_n, p_n) \to \infty$ and $q_\infty \in \Omega_\infty$. But $(p_\infty, x_\infty) \subset \Omega_\infty$ as $q_\infty \in (p_\infty, x_\infty) \cap \Omega_\infty$. Thus $g_n(p_n, x) \to (p_\infty, x_\infty)$. Hence $g_n(x_0, x) \to (p_\infty, x_\infty)$.

As the sequence $q_n \in (x_0, x)$ converges to $q_\infty \in \Omega_\infty$ under g_n, we see that (x, Ω) is forward (resp. backward) conically related to $(x_\infty, \Omega_\infty)$ (resp. $(p_\infty, \Omega_\infty)$). □

Remark 3.16. It turns out that the conical relation is also transitive, in the sense that, if (z_1, Ω_1) is forward conically to (z_2, Ω_2), and (z_2, Ω_2) is forward conically related to (z_3, Ω_3), then (z_1, Ω_1) is forward conically related to (z_3, Ω_3). The proof of this fact is straightforward; we omit it as we have no need for it in this paper.

3.6.1. *Conically related points along k-sectors.* It is often useful to consider the behavior of projective automorphisms on a lower-dimensional "projective slice" of a convex projective domain. Following Benoist and Benzécri, we consider the following:

Definition 3.17 (k-sectors)**.** Let Ω be a properly convex domain in $\mathbb{P}(V)$. A *k-sector* ω of Ω is a nonempty intersection $\mathbb{P}(W) \cap \Omega$, where $\mathbb{P}(W)$ is a projective subspace of dimension k.

Fix a k-dimensional projective space $\mathbb{P}(W_0)$ of $\mathbb{P}(V)$. Then the space of k-dimensional projective subspaces of $\mathbb{P}(V)$ is a $\mathrm{PGL}(V)$ orbit of $\mathbb{P}(W_0)$. Thus, any k-sector in Ω can be canonically identified with a projective equivalence class of properly convex domains in W_0. So, owing to Proposition 3.9, if Ω_1 and Ω_2 are properly convex domains in $\mathbb{P}(V)$ and $x_i \in \partial\omega_i$ for k-sectors ω_i of Ω_i ($i = 1, 2$), it makes sense to say that (x_1, ω_1) is (forward or backward) conically related to (x_2, ω_2), as elements in $W_0 \times \mathcal{C}(W_0)$. The lemma below essentially follows from [Ben03, Lemma 2.8]:

Lemma 3.18. *Let Ω_1, Ω_2 be properly convex domains in $\mathbb{P}(V)$, and fix $1 \leq k < d$. Then (x_1, Ω_1) is (forward or backward) conically related to (x_2, Ω_2) if and only if there are k-sectors ω_1, ω_2 so that $x_i \in \partial\omega_i$ for $i = 1, 2$, and (x_1, ω_1) is (forward or backward) conically related to (x_2, ω_2).*

3.6.2. *Uniqueness for conically related points.* In general, a pair (x_1, Ω_1) may be conically related to many different pairs (x_2, Ω_2), even up to projective equivalence. However, as a basic application of Theorem 2.9, we can recover some uniqueness given additional information about the sequence $\{g_n\}$ realizing the conical relation.

Definition 3.19. If (x_1, Ω_1) is (forward or backward) conically related to (x_2, Ω_2) by some $g_n \in \mathrm{PGL}(V)$, and there is some sequence $\{p_n\}$ in Ω_1 so that $g_n(\Omega_1, p_n)$ converges in $\mathcal{C}_*(V)$, then we say that (x_1, Ω_1) is conically related to (x_2, Ω_2) *along the sequence* $\{p_n\}$.

Lemma 3.20. *Let x_1 be a point in the boundary of a properly convex domain Ω_1, and suppose that (x_1, Ω_1) is forward (resp. backward) conically related to both (x_2, Ω_2) and (x_2', Ω_2') along the same sequence $\{p_n\}$ in Ω. Then there is some $h \in \mathrm{PGL}(V)$ such that $(hx_2, h\Omega) = (x_2', \Omega_2')$.*

Proof. Consider sequences $\{g_n\}, \{g_n'\}$ in $\mathrm{PGL}(V)$ so that (x_1, Ω_1) is conically related to (x_2, Ω_2) by g_n, (x_1, Ω_1) is conically related to (x_2', Ω_2') by $\{g_n'\}$, and the sequences $g_n(\Omega_1, p_n)$ and $g_n'(\Omega_1, p_n)$ both converge in the space $\mathcal{C}_*(V)$ of pointed domains in $\mathbb{P}(V)$.

This means that we can find compact subsets $\mathcal{K}, \mathcal{K}'$ in $\mathcal{C}_*(V)$ so that the intersection $g_n' g_n^{-1} \mathcal{K} \cap \mathcal{K}' \neq \emptyset$. From Theorem 2.9, it then follows that $g_n' = k_n g_n$ for k_n in a fixed compact subset of $\mathrm{PGL}(V)$. Any subsequence of k_n has a further subsequence which converges to some $h \in \mathrm{PGL}(V)$; it follows that h takes the limit of $g_n(x_1, \Omega_1)$ to the limit of $g_n'(x_1, \Omega_1)$, i.e. $h(x_2, \Omega_2) = (x_2', \Omega_2')$. □

3.7. **Points conically related to Morse points.** We now return to our main task of proving Proposition 3.10. The next lemma is a key tool we need for several of the equivalences in the proposition. It says that Morseness is preserved (in one direction) along a conical relation.

Lemma 3.21. *Let $y_1 \in \partial\Omega_1$, and suppose that the projective geodesic $[x_1, y_1)$ is Morse for some (any) $x_1 \in \Omega_1$. If y_1 is forward or backward conically related to $y_2 \in \partial\Omega_2$, then for some (any) $x_2 \in \Omega_2$, the projective geodesic $[x_2, y_2)$ is Morse.*

Proof. We first remark that the choice of x_1 and x_2 in the statement of the lemma is not significant, since the Morseness of a a geodesic ray is independent of its basepoint. So, fix any x_1 in Ω_1 and $y_1 \in \partial\Omega_1$. We will prove the contrapositive of the desired statement, and show that if $y_1 \in \partial\Omega_1$ is forward or backward conically related to $y_2 \in \partial\Omega_2$, and $[x_2, y_2)$ is non-Morse for some $x_2 \in \Omega_2$, then $[x_1, y_1)$ is also non-Morse.

Let (z_1, y_1) be the bi-infinite projective geodesic in Ω_1 that contains $[x_1, y_1)$. As y_1 is conically related to y_2, there is a sequence $\{g_n\}$ in $\mathrm{PGL}(V)$ so that $g_n\Omega_1 \to \Omega_2$ and y_2 is the limit of either $g_n y_1$ or $g_n z_1$ (depending on whether y_1 is forward or backward conically related to y_2). By definition of the conical relation, there exists $(z_2, y_2) \subset \Omega_2$ such that $g_n(x_1, y_1) \to (z_2, y_2)$. Fix a point $x_2 \in (z_2, y_2)$.

Assume that the projective geodesic ray $[x_2, y_2)$ is not Morse. This means that there exist quasi-geodesic constants $K \geq 1, C \geq 0$ such that for every $m \in \mathbb{N}$, there is a (K, C)-quasi-geodesic $q_m : [0, T_m] \to \Omega_2$ with endpoints in $[x_2, y_2)$ such that the image of q_m does not lie in the m-neighborhood of $[x_2, y_2)$.

We now claim that there exist constants K', C' such that: for any $m \in \mathbb{N}$, there exists a (K', C')-quasi-geodesic $q_m' : [0, T_m] \to \Omega_1$ with endpoints on $[x_1, y_1)$, but not contained in the $(m-1)$-neighborhood of $[x_1, y_1)$ in the metric d_{Ω_1}. This claim essentially follows from the fact that the convergence of $g_n\Omega_1$ to Ω_2 in $\mathcal{C}(V)$ is uniform on compact subsets of $\mathbb{P}(V)$ that intersect Ω_2.

Fix any $m \in \mathbb{N}$, and pick a compact convex set $D_m \subset \Omega_2$ large enough to contain the m-neighborhood of the set $q_m([0, T_m])$. Then for sufficiently large n (depending on m), the subset D_m is contained in $g_n\Omega_1$. Moreover, we have

$$d_{g_n\Omega_1}|_{D_m \times D_m} \to d_{\Omega_2}|_{D_m \times D_m}$$

uniformly as $n \to \infty$.

As $q_m(0), q_m(T_m) \in (z_2, y_2)$, the projective geodesic (z_2, y_2) intersects D_m in a finite length projective geodesic segment. As n tends to infinity, we have $g_n(x_1, y_1) \cap D_m \to (z_2, y_2) \cap D_m$. Hence, the endpoints $q_m(0), q_m(T_m)$ lie at a distance at most 1 from $g_n[x_1, y_1)$, with respect to the Hilbert metric on Ω_2. So, for each sufficiently large n, we can define a map $q_{m,n} : [0, T_m] \to \Omega_2$, agreeing with q_m on the open interval $(0, T_m)$, and whose endpoints lie on the ray $g_n[x_1, y_1)$ at a distance at most 1 from the endpoints of $q_m([0, T_m])$. The image of each $q_{m,n}$ lies in the set D_m. Since q_m is a (K, C)-quasi-geodesic with respect to the Hilbert metric on Ω_2, $q_{m,n}$ must be a $(K, C+1)$-quasi-geodesic with respect to the same metric.

Now, we know that the Hilbert on $g_n\Omega_1$ converges to the Hilbert distance on Ω_2 uniformly on D_m. So, if we fix $K' = K + 1$ and $C' = C + 2$, then for n large enough, the map $q_{m,n} : [0, T_m) \to \Omega_2$ must also be a (K', C')-quasi-geodesic with respect to the Hilbert metric on $g_n\Omega_1$.

By construction of q_m, we also know that there is some $t_m \in (0, T_m)$ so that the $(m-1)$-ball B_2 about $q_m(t_m)$ (with respect to the Hilbert metric on Ω_2) does not intersect the geodesic (z_2, y_2). Letting $B_{1,n}$ be the $(m-1)$-ball about $q_m(t_m)$ with respect to the Hilbert metric on $g_n\Omega_1$, the uniform convergence of Hilbert metrics on D_m implies that $B_{1,n} \subset D_m$ for large enough n and that $B_{1,n} \to B_2$ as $n \to \infty$. Then, as $g_n(x_1, y_1) \cap D_m$ converges to $(z_2, y_2) \cap D_m$, for large enough n we see that $B_{1,n}$ cannot intersect the projective geodesic $g_n(x_1, y_1)$.

This implies that, for all sufficiently large n, the quasi-geodesic $q_{m,n}$ is not contained in the $(m-1)$-neighborhood of $g_n(x_1, y_1)$ with respect to the Hilbert metric on $g_n\Omega_1$. But then $g_n^{-1}q_{m,n}$ is a (K', C')-quasi-geodesic with endpoints on $[x_1, y_1)$, whose image does not lie in the $(m-1)$-neighborhood of $[x_1, y_1)$ with respect to the Hilbert metric on Ω_1. Since m was arbitrary, and K', C' are independent of m, this proves that $[x_1, y_1)$ cannot be Morse. $\square$

Combining the above lemma with Lemma 3.12 and Lemma 3.13, we obtain a direct proof of the implications $(M) \implies (HT)$ and $(M) \implies (HT-)$ in Proposition 3.10. However, we need to do some more work to prove the implications $(M) \implies (SC)$ and $(M) \implies (SC-)$.

3.8. **Conically related points in triangles and half-triangles.** The lemma below is well-known to experts, and a similar proof already appears in [Ben60]. This result expresses the idea that, in any domain Ω, segments (or non-C^1 points) in the boundary correspond to "flat directions:" as we follow a projective geodesic towards a segment or corner in $\partial\Omega$, the domain "looks more like" a domain containing a properly embedded triangle, with the original segment or corner in its boundary. We give a statement which uses the language of conically related points, and provide a proof for convenience.

Lemma 3.22. *Suppose that $x_1 \in \partial\Omega_1$ is forward conically related to $x_2^+ \in \partial\Omega_2$, and backward conically related to $x_2^- \in \partial\Omega_2$.*

1. *If x_1 lies in the interior of a nontrivial segment in $\partial\Omega_1$, then there is a properly embedded triangle Δ in Ω_2 so that x_2^+ lies in the interior of an edge of Δ, and x_2^- is a vertex of Δ.*
2. *If x_1 is not a C^1 point, then there is a properly embedded triangle Δ in Ω_2 so that x_2^+ is a vertex of Δ and x_2^- is on the interior of an edge of Δ.*

Proof. Via Lemma 3.18, it suffices to consider the case where Ω_1, Ω_2 are 2-dimensional. The definition of conically related points implies that there exists a projective geodesic ray $[a, x_1) \subset \Omega_1$, a sequence in $[a, x_1)$, and a sequence $\{g_n\}$ in $\mathrm{PGL}(d, \mathbb{R})$ such that $g_n[a, x_1) \to (x_2^-, x_2^+) \subset \Omega_2$. Let x_1^- be a point in $\partial\Omega_1$ so that $[a, x_1) \subset (x_1^-, x_1)$, and hence, $(x_1^-, x_1^+) \subset \Omega_1$. Then $g_n(x_1^-, x_1)$ converges to (x_2^-, x_2^+). Let $\{p_n\}$ be a sequence in $[a, x_1)$ so that $g_n p_n$ converges to some point in the interior of (x_2^-, x_2^+).

(1) By assumption, there exists a maximal nontrivial projective line segment $[b, c] \subset \partial\Omega_1$ with $x_1 \in (b, c)$. Consider a sequence of projective transformations h_n, defined (with respect to the projective basis $\{b, c, x_1^-\}$) by

$$h_n := \begin{pmatrix} \lambda_n^{-1} & & \\ & \lambda_n^{-1} & \\ & & \lambda_n^2 \end{pmatrix},$$

where λ_n is chosen so that $h_n p_n$ converges to a point in the interior of the line segment x_1, x_1^-. Then $h_n\Omega_1$ converges to the triangle with vertices at b, c, x_1^-. The result then follows from Lemma 3.20.

(2) This case is similar. Fix a supporting line L_- for Ω_1 at the point x_1^-. Since x_1 is not a C^1-point, we can choose two distinct supporting hyperplanes of Ω_1 at x_1 that we label H_b and H_c. Let $b = H_b \cap L_-$ and $c = H_c \cap L_-$. Here we consider the sequence of projective transformations (defined with respect to the projective basis $\{x_1, b, c\}$) by

$$h_n := \begin{pmatrix} \lambda_n^{-2} & & \\ & \lambda_n & \\ & & \lambda_n \end{pmatrix},$$

where λ_n is again chosen so that $h_n p_n$ converges to a point in the interior of (x_1, x_1^-). This time, since Ω_1 is not C^1 at x_1, the sequence of domains converges to a triangle with a vertex at x_1, and an edge containing x_1^- in its interior and again we are done by Lemma 3.20. ☐

The next lemma does not appear to be well-known. It says that, just as a point z in the interior of a boundary segment in a properly convex domain Ω can be thought of as a "flat direction," a point z in the *closure* of a segment can be thought of as a "half-flat" direction: as we approach z along a projective geodesic, the domain "looks more" like it contains a properly embedded half-triangle.

Lemma 3.23. *Suppose that $x_1 \in \partial\Omega$ is forward conically related to $x_2^+ \in \partial\Omega_2$ and backward conically related to $x_2^- \in \partial\Omega_2$. If x_1 lies in the closure of a nontrivial segment in $\partial\Omega$, then both x_2^+ and x_2^- lie in the boundary of a half-triangle in Ω_2.*

Proof. After applying Lemma 3.18 we may assume that Ω and Ω_2 are both two-dimensional, and using Lemma 3.22, we can further reduce to the case where x_1 is the endpoint of a maximal nontrivial segment in $\partial\Omega$. Let z be the other endpoint of this segment, and let L_+ be the projective span of x_1 and z, so that L is a supporting line of Ω at x_1.

Fix a sequence $\{g_n\}$ realizing the conical relations between x_1 and $x_2^\pm$, so that $g_n x_1 \to x_2^+$ and for some $x_1^- \in \partial\Omega$, we have $(x_1^-, x_1) \subset \Omega$ and $g_n x_1^- \to x_2^-$. Let L_- be a supporting line of Ω at x_1^-, let $x_0 = L_- \cap L_+$, and let $p_n \in (x_1, x_1^-)$ be a sequence converging to x_1 so that $g_n p_n$ converges to a point $p_0 \in \Omega_2$.

We fix lifts v_1, v_0, v_1^- for x_1, x_0, x_1^- respectively, so that $\{v_1, v_0, v_1^-\}$ is a basis for $\mathbb{R}^3$ and the projectivization of the convex hull of v_1, v_0, v_1^- lies in Ω. We consider a sequence of linear maps $\{h_n\}$, defined with respect to this basis by

$$h_n := \begin{pmatrix} \lambda_n^{-1} & & \\ & \lambda_n^{-1} & \\ & & \lambda_n^2 \end{pmatrix}.$$

Here $\lambda_n > 0$ is chosen so that $h_n p_n$ converges to a point $p_0' \in (x_1, x_1^-)$. The sequence of domains $h_n\Omega$ converges to a triangle with vertices x_1, x_1^-, z (so this triangle does *not* contain p_0' in its interior).

Our chosen basis $\{v_1, v_0, v_1^-\}$ also determines projective coordinates $[x : y : z]$ for projective space $\mathbb{P}(\mathbb{R}^3)$. Consider the affine chart

$$\{[x : y : 1 - x] : x, y \in \mathbb{R}\} \simeq \mathbb{R}^2,$$

which has affine coordinates given by (x, y). In these coordinates, the projective line spanned by x_1, x_1^- corresponds to the horizontal axis $y = 0$, so without loss of generality the triangle limited to by $h_n\Omega$ is a bounded convex subset of the upper half-plane. Therefore, since $h_n p_n$ lies in the interior of $h_n\Omega$, the intersection of $h_n\Omega$ with the lower-half plane is nonempty, and contained in an open subset of the form $I \times (0, -\varepsilon_n)$, where I is a fixed interval and ε_n is a positive constant tending to zero.

We can then compose h_n with a projective transformation f_n given by a "vertical rescaling" (i.e. a transformation which preserves vertical lines, and acts on them by homotheties centered at zero) so that the intersection of $f_n h_n \Omega$ with the lower half-plane converges to a bounded nonempty convex set; explicitly, in our chosen projective basis, each f_n has the form

$$\begin{pmatrix} \eta_n^{-1} & & \\ & \eta_n^2 & \\ & & \eta_n^{-1} \end{pmatrix}.$$

Since $\varepsilon_n \to 0$, the vertical scaling factor of each f_n must tend to infinity, which means that the intersection of $f_n h_n \Omega$ with the upper half-plane limits to a subset of the form $I \times (0, \infty)$. But this subset is projectively equivalent to a half-triangle in the limit of $f_n h_n \Omega$.

Altogether, we have seen that the sequence of pointed domains $f_n h_n(\Omega, p_n)$ converge to a pointed domain (Ω_2', p_0'), so that $f_n h_n x_1 = x_1$ and $x_1^- = f_n h_n x_1^-$ both lie in the boundary of a half-triangle in Ω_2'. We can then apply Lemma 3.20 to complete the proof. □

3.9. **Projective geodesics with endpoints in segments.** We can combine our previous results regarding Morse geodesics and conically related points to prove some more facts about the endpoints of Morse geodesics.

Corollary 3.24. *Suppose that $y \in \partial\Omega$ is the endpoint of a M-Morse geodesic ray. Then y is a C^1 extreme point of $\partial\Omega$.*

Proof. Fix a projective geodesic ray $[y_0, y)$ that is M-Morse. By Lemma 3.15, (y, Ω) is forward conically related to (y', Ω'). Now suppose that y is not an extreme point, meaning that y is contained in the interior of a projective line segment in $\partial\Omega$. So Lemma 3.22 part (1) implies that there is a properly embedded triangle $\Delta' \subset \Omega'$ whose boundary contains y'. Let $p' \in \Delta'$. By Lemma 3.21, $[p', y')$ is a Morse geodesic ray. This contradicts Lemma 3.12, so y is an extreme point. That

y is a C^1 point follows from similar reasoning, applying part (2) of Lemma 3.22 instead of part (1). ☐

We can use this result to obtain:

Corollary 3.25. *Suppose that y lies in the closure of a nontrivial segment in $\partial\Omega$. Then for any $x \in \Omega$, the projective geodesic $[x, y)$ is not Morse.*

Proof. Suppose, for a contradiction, that $[x, y)$ is M-Morse for some $x \in \Omega$. In this case Corollary 3.24 implies that y is a C^1 extreme point. So, we may assume that y is the endpoint of a nontrivial segment in $\partial\Omega$.

By Lemma 3.15, (y, Ω) is forward conically related to (y_+, Ω_∞) and backward conically related to (y_-, Ω_∞) for some properly convex domain Ω_∞. Fix a point $p \in \Omega_\infty$. By Lemma 3.21, $[p, y_\infty)$ is also Morse, and by Lemma 3.23, y_∞ lies in the boundary of a half-triangle in $\partial\Omega_\infty$.

Now, Corollary 3.24 again implies that y_∞ cannot lie in the interior of a segment in $\partial\Omega_\infty$. But in this case Lemma 3.13 implies that $[p, y_\infty)$ cannot be Morse and we get a contradiction. ☐

Combining the above with Lemma 3.21 immediately yields the implications $(M) \implies (SC)$ and $(M) \implies (SC-)$ in Proposition 3.10.

3.10. **δ-slimness.** To finish the proof of Proposition 3.10, we need to prove the final two implications $(HT) \implies (\delta)$ and $(HT-) \implies (\delta)$ (again see Figure 3). These both follow from the lemma below.

Lemma 3.26. *Let ℓ be a projective geodesic in Ω. If ℓ is not projectively δ-slim for any $\delta > 0$, then there is an endpoint y of ℓ in $\partial\Omega$ and points z_+, z_- lying in the boundary of a half-triangle in some domain Ω_2 so that y is forward (resp. backward) conically related to z_+ (resp. z_-).*

Proof. The argument is essentially identical to the proof of Proposition 2.5 in [Ben04]; we reproduce it here for convenience. Fix a projective geodesic ℓ which is not projectively δ-slim for any δ. We choose a sequence of triples $\{(a_n, b_n, c_n)\}$ in Ω, with $a_n, b_n \in \ell$, such that the projective geodesic triangle with vertices a_n, b_n, c_n is not $2n$-slim. Then, by Remark 3.2, the segment $[a_n, b_n]$ cannot be contained in the union of metric n-neighborhoods

$$N_n([a_n, c_n]) \cup N_n([b_n, c_n]).$$

Since the projective geodesic segment $[a_n, b_n]$ is connected, there is a point $x_n \in [a_n, b_n]$ so that $d_\Omega(x_n, [a_n, c_n]) \geq n$ and $d_\Omega(x_n, [b_n, c_n]) \geq n$. Applying Theorem 2.9, we can choose elements $g_n \in \mathrm{PGL}(V)$ and extract a subsequence so that the pointed domains $g_n(\Omega, x_n)$ converge to some limiting pointed domain $(\Omega_\infty, x_\infty)$, and the points $g_n a_n, g_n b_n, g_n c_n$ converge to points a, b, c in $\partial\Omega_\infty$.

Since $g_n x_n$ converges to $x_\infty \in \Omega_\infty$, the distances $d_{g_n\Omega}(g_n x_n, g_n[a_n, c_n])$ and $d_{g_n\Omega}(g_n x_n, g_n[b_n, c_n])$ must tend to infinity, which means the segments $[a, c]$ and $[b, c]$ must converge to subsets of $\partial\Omega_\infty$. However, since the limit of $g_n x_n$ lies in the interior of $(a, b) \cap \Omega_\infty$, the segments $[a, c]$ and $[b, c]$ must also be nontrivial and distinct. As (a, b) contains the limit of $g_n x_n \in \Omega_\infty$, the points a, b, c are the vertices of a half-triangle in Ω_∞.

If $\{g_n\}$ is a divergent sequence in $\mathrm{PGL}(V)$, then the properness condition in Theorem 2.9 implies that x_n must tend towards an endpoint of ℓ in $\partial\Omega$. This endpoint is forward conically related to one of the limiting endpoints a, b of $g_n[a_n, b_n]$, and backward conically related to the other. In this case, we have proved the lemma.

On the other hand, if $\{g_n\}$ is *not* divergent in $\mathrm{PGL}(V)$, then $(\Omega_\infty, [a, b]) = g(\Omega, \bar{\ell})$ for some $g \in \mathrm{PGL}(V)$. Thus both endpoints of ℓ already lie in a half-triangle in Ω. If the conical relation were reflexive, this would finish the proof. But since conical relation satisfies only a weak form of reflexivity, we must appeal to Lemma 3.15 followed by Lemma 3.23. This leads to the conclusion that the endpoints of ℓ are both forward and backward conically related to points in a half-triangle Δ in some properly convex domain Ω', as required. □

This concludes the proof of Proposition 3.10, hence of Proposition 3.3.

3.11. **Uniformity.** Proposition 3.4 gives us a stronger version of the implication (*C*) $\implies$ (*M*) in Proposition 3.10: it says that any D-contracting projective geodesic in a properly convex domain Ω is M-Morse for a Morse gauge M determined solely by D and Ω. In the case where Ω is divisible, we can strengthen the opposite implication in a similar manner.

Proposition 3.27. *Let Ω be a properly convex divisible domain. For every Morse gauge M, there exists a constant $\delta > 0$ (depending only on M and Ω) so that any M-Morse geodesic in Ω is projectively δ-slim.*

Observe that, by applying this proposition together with Proposition 3.6, we obtain the following uniform version of (*M*) $\implies$ (*C*):

Corollary 3.28. *Let Ω be a properly convex divisible domain, M be a Morse gauge, and δ be the constant (determined solely by M and Ω) from Proposition 3.27. Then any M-Morse geodesic in Ω is 24δ-contracting.*

Proof of Proposition 3.27. Fix a Morse gauge M, and suppose for a contradiction that there is an infinite sequence of M-Morse geodesics $\{\ell_n\}$ in X so that ℓ_n fails to be projectively n-slim. Then for each n there is a projective triangle $[a_n, b_n] \cup [b_n, c_n] \cup [c_n, a_n]$ in Ω with $[b_n, c_n] \subset \ell_n$ which is not n-slim. By Remark 3.2, this implies that there is a point $u_n \in [b_n, c_n]$ such that

$$\mathrm{d}_\Omega(u_n, [a_n, b_n] \cup [a_n, c_n]) \geq n.$$

As Ω is divisible, there exists a discrete subgroup $\Gamma \subseteq \mathrm{Aut}(\Omega)$ and a compact set $D \subset \Omega$ such that $\Gamma \cdot D = \Omega$. Then, we can find γ_n in Γ such that $\gamma_n u_n \in D$. Up to passing to a subsequence, we can assume that the points $\gamma_n a_n, \gamma_n b_n, \gamma_n c_n, \gamma_n u_n, \gamma_n \ell_n$ converge to a, b, c, u, ℓ in $\overline{\Omega}$ respectively. By construction $u \in D$. Since $u_n \in \ell_n$, this implies that $u \in \ell$ and hence ℓ is a bi-infinite projective geodesic in Ω. Moreover, $[a, b] \cup [a, c] \subset \partial\Omega$, because

$$\begin{aligned}\liminf_{n\to\infty} \mathrm{d}_\Omega(u, [a_n, b_n] \cup [a_n, c_n]) &\geq \Big(\liminf_{n\to\infty} \mathrm{d}_\Omega(u_n, [a_n, b_n] \cup [a_n, c_n])\Big) - \lim_{n\to\infty} \mathrm{d}_\Omega(u, u_n) \\ &= \infty.\end{aligned}$$

Then $\ell = (a, b)$ and the points a, b, c lie in the boundary of a half triangle in Ω.

Now, as ℓ_n is a sequence of M-Morse geodesics converging uniformly to a geodesic ℓ on compact sets, it follows from [Cor17, Lemma 2.10] that ℓ is M-Morse. But then this contradicts Lemma 3.13. □

3.12. **Morseness, C^1 points, and extreme points.** We record a few more consequences of Proposition 3.10. These results will be relevant later in the paper, when we consider the behavior of Morse geodesics as subsets of the automorphism group $\mathrm{Aut}(\Omega) \subset \mathrm{PGL}(V)$.

Proposition 3.29. *Suppose that $y \in \partial\Omega_1$ is M-Morse and (y, Ω_1) is forward conically related to (x, Ω_2). Then x is an extreme point and a C^1 point in $\partial\Omega_2$.*

Proof. Follows immediately from Lemma 3.21 and Corollary 3.24. □

Below, we provide a partial converse to Proposition 3.29. Recall that $\mathcal{C}(V)$ denotes the space of properly convex domains in $\mathbb{P}(V)$.

Definition 3.30. Let Ω be a convex projective domain in $\mathbb{P}(V)$. We let $\mathscr{O}(\Omega)$ denote the closure of the $\mathrm{PGL}(V)$-orbit of Ω in $\mathcal{C}(V)$.

Recall the notion of domains with exposed boundary from Definition 2.2.

Proposition 3.31. *Suppose Ω_1 is a properly convex domain such that every $\Omega \in \mathscr{O}(\Omega_1)$ has exposed boundary. Let $y \in \partial\Omega_1$ be such that: if (y, Ω_1) is forward conically related to (x, Ω_2), then x is a C^1 extreme point in $\partial\Omega_2$. Then y is M-Morse for some Morse gauge M.*

Proof. Fix a point $y \in \partial\Omega_1$ satisfying the two assumptions above, and suppose that y is not a Morse point in $\partial\Omega_1$. We will show that if this holds, there is a domain $\Omega \in \mathscr{O}(\Omega_1)$ which does not have exposed boundary.

The implication *(SC)* $\implies$ *(M)* in Proposition 3.10 means that y is forward conically related to a point $y_2 \in \partial\Omega_2$, lying in the closure of a nontrivial segment s in $\partial\Omega_2$. By definition Ω_2 lies in $\mathscr{O}(\Omega_1)$. By our assumptions, y_2 must be an extreme point in $\partial\Omega_2$, so y_2 lies in the boundary of s. Any hyperplane supporting Ω_2 at a point in the relative interior of s must also contain y_2. But our assumptions also imply that y_2 is a C^1 point in $\partial\Omega_2$, i.e. there is a unique supporting hyperplane H of Ω_2 at y_2. Then H must contain all of s and therefore y_2 cannot be an exposed point. Thus Ω_2 cannot have exposed boundary. □

When Ω is a *divisible* domain in $\mathbb{P}(V)$, the $\mathrm{PGL}(V)$-orbit of Ω in $\mathcal{C}(V)$ is closed, as a direct consequence of Theorem 2.9. So in this case every domain in $\mathscr{O}(\Omega)$ has exposed boundary if and only if Ω has exposed boundary, and we can combine Proposition 3.29 and Proposition 3.31 to obtain the following:

Corollary 3.32. *Let Ω be a convex divisible domain with exposed boundary, and let $y \in \partial\Omega$. Then the following are equivalent:*

(1) For some (any) $x \in \Omega$, the projective geodesic $[x, y)$ is M-Morse for some Morse gauge M.
(2) If y is forward conically related to $z \in \partial\Omega$, then z is a C^1 extreme point in $\partial\Omega$.

3.13. **Morse local-to-global.** Using the above results, one may also prove Theorem 1.18, showing that convex divisible domains satisfy a *Morse local-to-global property* defined by Russell-Spriano-Tran.

Definition 3.33. Let X be a metric space, fix constants $K \geq 1$, $L, A > 0$, and let M be a Morse gauge. We say that a path $c : [a, b] \to X$ is a $(L; M; K, A)$-*local Morse quasi-geodesic* if for every $[t_1, t_2] \subset [a, b]$ with $|t_2 - t_1| \leq L$, the restriction of c to $[t_1, t_2]$ is an M-Morse (K, A)-quasi-geodesic.

Definition 3.34. A metric space X has the *Morse local-to-global property* if, for every Morse gauge M and every $K \geq 1, A \geq 0$, there exist constants $K' \geq 1$, $L, A' \geq 0$ and a Morse gauge M' (depending only on K, A, M) so that every $(L; M; K, A)$-local Morse quasi-geodesic is an M'-Morse (K', A')-quasigeodesic.

Note that even if X is a metric space containing no Morse geodesic rays, it is still possible for X to satisfy the Morse local-to-global property. In [RST22], Russell-Spriano-Tran also proved that any CAT(0) space X satisfies the Morse local-to-global property. It turns out that their proof applies essentially verbatim to divisible Hilbert geometries, as applications of the CAT(0) condition in the proof are limited to:

(a) the maximum principle for geodesics in X,
(b) continuity of the nearest-point projection map to geodesics in X, and
(c) uniform equivalence between Morse geodesics and contracting geodesics.

One can thus follow their proof, employing:

(1) Lemma 2.5 in place of (a),
(2) Lemma 2.6 in place of (b), and
(3) Proposition 3.4 and Corollary 3.28 in place of (c)

to prove Theorem 1.18. We refer the reader to [RST22, Section 4.2] rather than reproducing the entire proof here. ☐

4. Estimating singular values using convex projective geometry

In this section, we will estimate singular values using projective geometry. Specifically, if $\{g_n\}$ is a sequence in $\mathrm{PGL}(d, \mathbb{R})$ that "almost" preserves a properly convex domain, then we obtain asymptotic estimates for various singular values of $\{g_n\}$. We will use these estimates in the next section to study the singular values of sequences that track Morse geodesic rays.

4.1. Singular value gap estimates when a domain is preserved.

We first record some known estimates on singular values of automorphisms of Ω. The first estimate relates Hilbert distances to the $\mu_{1,d}$ singular value gap.

Proposition 4.1 ([DGK17, Proposition 10.1])**.** *Let Ω be a properly convex domain in $\mathbb{P}(\mathbb{R}^d)$. For any basepoint $x_0 \in \Omega$, there exists a constant D so that for any $\gamma \in \mathrm{Aut}(\Omega)$, we have*

$$\left| \mu_{1,d}(\gamma) - \frac{1}{2} \mathrm{d}_\Omega(x_0, \gamma x_0) \right| \leq D.$$

Moreover, the constant D can be chosen to vary continuously as (x_0, Ω) varies in the space of pointed properly convex domains.

To obtain estimates for other singular value gaps, we can consider the faces in the boundary of a properly convex domain Ω. Let F be a k-dimensional face of Ω, fix a basepoint $x_0 \in \Omega$, and let $\{\gamma_n\}$ be a sequence in $\mathrm{Aut}(\Omega)$ so that $\gamma_n x_0$ accumulates on F. Lemma 2.10 tells us that, if $B(x_0, r)$ is any open ball about x_0 (with respect to d_Ω), then $\gamma_n B(x_0, r)$ also accumulates on the k-dimensional face F. This can be used to see that the sequence has a singular value gap at some index j with $j \leq k$. Precisely, we have the following.

Proposition 4.2 (See e.g. [IZ21, Proposition 5.6])**.** *Suppose $\{\gamma_n\}$ is a sequence in $\mathrm{Aut}(\Omega)$, $x_0 \in \Omega$, and $\gamma_n x_0 \to x \in \partial\Omega$. If $\dim(F_\Omega(x)) = k$, then $\mu_{1,k+2}(\gamma_n) \to \infty$.*

Proof. The proof of [IZ21, Proposition 5.6] immediately implies this (although the result is stated differently in that paper). In the notation of [IZ21], suppose $\gamma_n \to T$ in $\mathbb{P}(\mathrm{End}(\mathbb{R}^d))$. Then T is a projective linear map with $\dim(\mathrm{Im}(T)) = q$ where $q := \max\{i : \liminf_{n\to\infty} \mu_{1,i}(\gamma_n) < \infty\}$ and $\mathrm{Im}(T) \subset \mathrm{Span}\, F_\Omega(x)$. Thus $q \leq k+1$ where $k := \dim F_\Omega(x)$. Hence $\mu_{1,k+2}(\gamma_n) \geq \mu_{1,q+1}(\gamma_n) \to \infty$. □

In the above proposition, $\{\gamma_n\}$ does not need to track the projective geodesic $[x_0, x)$. But if the sequence $\{\gamma_n\}$ does actually track the projective geodesic ray $[x_0, x)$, then we get a stronger statement. In this case, it is possible to show that the balls $\gamma_n B(x_0, r)$ limit onto a relatively *open* subset of $F_\Omega(x)$, which in turn implies that the sequence γ_n does *not* have singular value gaps at an index less than k. Using this idea, one proves the following:

Proposition 4.3 (See e.g. [IZ21, Proposition 5.7]). *Let Ω be a properly convex domain, let $c : [0,\infty) \to \Omega$ be a projective geodesic ray, and let $\{\gamma_n\}$ track c. The following are equivalent:*

1. *(1) The endpoint $c(+\infty) \in \partial\Omega$ lies in a k-dimensional face in $\partial\Omega$.*
2. *(2) There exists some constant $D > 0$ such that $\mu_{k+1,k+2}(\gamma_n)$ tends to infinity as $n \to \infty$, and for any $1 \leq \ell \leq k$, we have $\mu_{\ell,\ell+1}(\gamma_n) < D$.*

4.2. **Singular value estimates when a domain is almost preserved.** The remaining estimates in this section are somewhat more technical. This is partly because we no longer restrict our attention to automorphisms of a fixed convex projective domain Ω. Rather, we consider projective transformations that "almost preserve" a domain. This idea is closely tied to the notion of conically related points from the previous section.

It will be useful to introduce the following definitions.

Definition 4.4. Suppose V is a real vector space. Recall that $\mathcal{C}(V)$ denotes the space of properly convex domains in $\mathbb{P}(V)$.

Let $\ell \subset V$ be a projective line segment with endpoints $x_\pm$, and let H be a projective subspace in $\mathbb{P}(V)$ with codimension 2. We let

$$\mathcal{C}(V; \ell, H)$$

denote the set of domains $\Omega \subset \mathbb{P}(V)$ such that ℓ is properly embedded in Ω, and the projective hyperplanes $\mathbb{P}(x_+ \oplus \widetilde{H})$ and $\mathbb{P}(x_- \oplus \widetilde{H})$ are both supporting hyperplanes of Ω. This set is equipped with the subspace topology from $\mathcal{C}(V)$.

The lemma below is one of the main technical estimates in this section.

Lemma 4.5. *Fix a projective line segment $\ell = (x_+, x_-)$ and a codimension-two projective subspace $H \subset \mathbb{P}(\mathbb{R}^d)$. Let $\mathcal{K}_1, \mathcal{K}_2$ be two compact subsets of $\mathcal{C}(\mathbb{R}^d; \ell, H)$. There exists a constant C (depending only on $\mathcal{K}_1, \mathcal{K}_2$) so that if $g \in \mathrm{GL}(d,\mathbb{R})$ preserves the decomposition $x_+ \oplus \widetilde{H} \oplus x_-$, with $||g|_{x_+}|| > ||g|_{x_-}||$, and $g\mathcal{K}_1 \cap \mathcal{K}_2 \neq \emptyset$, then:*

1. *(1) $\left|\log \left\|g|_{x_+}\right\| - \mu_1(g)\right| < C$,*
2. *(2) $\left|\log \left\|g|_{x_-}\right\| - \mu_d(g)\right| < C$,*
3. *(3) $\left|\log \left\|g|_{\widetilde{H}}\right\| - \mu_2(g)\right| < C$,*
4. *(4) $\left|\log \mathbf{m}(g|_{\widetilde{H}}) - \mu_{d-1}(g)\right| < C$.*

Proof. Let $W = \widetilde{H}$. Because of Lemma 2.19 and Lemma 2.15, we may assume that the decomposition $x_+ \oplus W \oplus x_-$ is orthogonal. In this situation, whenever g satisfies the hypotheses of the lemma, we can find indices $1 \leq i < j \leq d$ so $||g|_{x_+}|| = \sigma_i(g)$ and $||g|_{x_-}|| = \sigma_j(g)$. We first claim that:

Claim 4.5.1. *It suffices to prove only part (1).*

Proof of Claim. Suppose we have part (1). Part (2) follows immediately by applying part (1) to g^{-1} (and interchanging the roles of $\mathcal{K}_1, \mathcal{K}_2$). So we only need to see that parts (1) and (2) together imply parts (3) and (4). Parts (1) and (2) imply that $0 \leq \mu_1(g) - \mu_i(g) < C$ and $0 \leq \mu_j(g) - \mu_d(g) < C$, giving us $|\mu_{1,d}(g) - \mu_{i,j}(g)| \leq C'$ where $C' := 2C$. Then Lemma 2.17 implies that

$$\max\left\{\max_{1 \leq k \leq i} \mu_{1,k}(g), \max_{j \leq k \leq d} \mu_{k,d}(g)\right\} \leq C'. \tag{4}$$

Let i' and j' be the minimum and the maximum, respectively, of the set $(\{1, \dots, d\} - \{i, j\})$. Since $x_+ \oplus W \oplus x_-$ is an orthogonal decomposition,

$$\|g|_W\| = \sigma_1(g|_W) = \sigma_{i'}(g) \text{ and } \mathbf{m}(g|_W) = \sigma_{d-2}(g|_W) = \sigma_{j'}(g).$$

We consider several cases depending on the value of i'. If $i' = 2$, then part (3) is immediate. On the other hand, if $i' = 1$, then the definition of i' implies that $i \geq 2$. Then (4) implies that

$$|\mu_{i'}(g) - \mu_2(g)| = \mu_{1,2}(g) \leq C'$$

which again implies part (3). So we are left with the case that $i' > 2$. Note that this occurs precisely when $i = 1$ and $j = 2$. But in that case, (4) implies that

$$|\mu_{i'}(g) - \mu_2(g)| = \mu_{j,i'}(g) \leq C'$$

which again implies part (3). Thus we have shown that part (1) implies part (3).

Finally, since $\mathbf{m}(g|_W) = \left\|(g|_W)^{-1}\right\|$, we can apply part (3) to $(g|_W)^{-1}$ to prove part (4). This finishes the proof of the claim that it suffices to prove only part (1). □

We now proceed with the proof of **part** (1). Suppose, on the contrary, that part (1) fails. Then there is a sequence $\{g_n\}$ in $\mathrm{GL}(d, \mathbb{R})$ satisfying the hypotheses of the lemma, but with

$$\frac{\sigma_1(g_n)}{\sigma_i(g_n)} \to \infty \tag{5}$$

as $n \to \infty$. Here, i is an index such that $||g_n|_{x_+}|| = \sigma_i(g_n)$ for every n (after passing to a subsequence, we can ensure that the same fixed index i works for each g_n). Up to passing to a further subsequence, we may also assume that there exists $j \in \{i+1, \dots, d\}$ such that $\left\|g_n|_{x_-}\right\| = \sigma_j(g_n)$. Note that in particular, (5) implies that $i > 1$.

We can fix an orthonormal basis for W, and extend it to an orthonormal basis for $\mathbb{R}^d$ by adding unit vectors spanning x_+, x_-. With respect to this basis, g_n is block-diagonal, of the form

$$\begin{pmatrix} \sigma_i(g_n) & & \\ & g_n|_W & \\ & & \sigma_j(g_n) \end{pmatrix}.$$

The restriction $g_n|_W$ has a Cartan decomposition $k_n a_n l_n$, where a_n is a diagonal matrix with respect to our chosen basis on W, and k_n, l_n lie in the group $\mathrm{O}(W)$ of orthogonal transformations of W.

Observe that, if we pre-compose or post-compose g_n with any orthogonal matrix of $\mathbb{R}^d$ fixing $\ell = (x_+, x_-)$ pointwise and preserving W, the values of $\mu_i(g_n)$, $\|g_n|_{x_\pm}\|$, $\|g_n|_W\|$, and $\mathbf{m}(g_n|_W)$ do not change. So, after replacing g_n with the sequence

$$\begin{pmatrix} 1 & & \\ & k_n^{-1} & \\ & & 1 \end{pmatrix} \begin{pmatrix} \sigma_i(g_n) & & \\ & g_n|_W & \\ & & \sigma_j(g_n) \end{pmatrix} \begin{pmatrix} 1 & & \\ & l_n^{-1} & \\ & & 1 \end{pmatrix},$$

and replacing the sets $\mathcal{K}_1, \mathcal{K}_2$ with the sets

$$\begin{pmatrix} 1 & & \\ & \mathrm{O}(W) & \\ & & 1 \end{pmatrix} \mathcal{K}_1, \qquad \begin{pmatrix} 1 & & \\ & \mathrm{O}(W) & \\ & & 1 \end{pmatrix} \mathcal{K}_2,$$

we can assume that each g_n is a diagonal matrix with respect to a fixed orthonormal basis $e_1, \ldots, e_d$, compatible with the orthogonal decomposition $\mathbb{R}^d = x_+ \oplus W \oplus x_-$. In particular, we order our basis so that $x_+ = [e_i]$ and $x_- = [e_j]$ and $g_n e_k = \sigma_k(g_n) e_k$ for each $1 \le k \le d$.

Now fix a point $v \in \mathbb{R}^d$ so that $[v] \in \ell$. We may write $v = a e_i + b e_j$ for a, b both nonzero. Fix any $t \neq 0$. Then, using (5) (and the fact that $i < j$), we have

$$\frac{1}{\sigma_1(g_n)} g_n(te_1 + v) = te_1 + a\frac{\sigma_i(g_n)}{\sigma_1(g_n)} e_i + b\frac{\sigma_j(g_n)}{\sigma_1(g_n)} e_j \to te_1.$$

Thus $g_n[v + te_1] \to [e_1]$ for any $t \neq 0$.

Now, choose domains $\Omega_n \in \mathcal{K}_1$ so that $g_n\Omega_n \in \mathcal{K}_2$. By compactness of $\mathcal{K}_2$, we can pass to a subsequence and assume that $g_n\Omega_n$ converges to a domain Ω_∞. Since $\mathcal{K}_1$ is a compact subset of $\mathcal{C}(\mathbb{R}^d; \ell, H)$, there is some $\varepsilon > 0$ so that for each $\Omega \in \mathcal{K}_1$, the Hilbert distance in Ω between $[v + te_1]$ and $[v]$ is uniformly bounded for $t \in (-\varepsilon, \varepsilon)$. Our assumption (5) means that $\{g_n\}$ is divergent when viewed as a sequence of projective transformations, so Theorem 2.9 implies that $[g_n v]$ only accumulates on $\partial\Omega_\infty$. Since $[v]$ lies in the g_n-invariant subspace ℓ and $\|g_n|_{x_+}\| > \|g_n|_{x_-}\|$, the only possibility is that $[g_n v]$ converges to x_+.

Since $\lim_{n\to\infty} g_n[v + te_1] = [e_1]$ for any $t \neq 0$, it follows from Lemma 2.10 that $[e_1] \in F_{\Omega_\infty}(x_+)$. Moreover, by the same lemma, for any $t \in (-\varepsilon, \varepsilon) - \{0\}$, we have

$$\begin{aligned} d_{F_{\Omega_\infty}(x_+)}(x_+, [e_1]) &\le \liminf_{n\to\infty} d_{g_n\Omega_n}(g_n[v], g_n[v + te_1]) \\ &= d_{\Omega_n}([v], [v + te_1]). \end{aligned}$$

As Ω_n lies in a compact set $\mathcal{K}_1$, the Hilbert distances $d_{\Omega_n}([v], [v + te_1])$ tend to 0 uniformly in n as $t \to 0$. This means that in fact $d_{F_{\Omega_\infty}(x_+)}(x_+, [e_1]) = 0$, i.e. $x_+ = [e_1]$. But this is a contradiction since we have also arranged $x_+ = [e_i]$ for $i \neq 1$, and $\{e_1, \ldots, e_d\}$ is a basis for $\mathbb{R}^d$. $\square$

4.3. **Application to automorphisms of properly convex domains.** We now apply the previous lemma to establish estimates on singular values of projective transformations which *actually* (instead of "approximately") preserve a convex domain. First we introduce some more notation.

Definition 4.6. Let Ω be a properly convex domain in $\mathbb{P}(\mathbb{R}^d)$.

- We let $\mathcal{G}(\Omega)$ denote the space of all projective bi-infinite geodesics in Ω, with unit-speed (in d_Ω) parameterization. Let $c(\pm\infty) \in \partial\Omega$ denote the ideal endpoints of any $c \in \mathcal{G}(\Omega)$.

- We let $\mathcal{T}(\Omega)$ denote the set of triples (c, H_+, H_-), such that $c \in \mathcal{G}(\Omega)$ and $H_\pm$ are supporting hyperplanes of Ω at $c(\pm\infty)$.
- For any compact subset $K \subset \Omega$, we let $\mathcal{G}_K(\Omega)$ denote the set of geodesics $c \in \mathcal{G}(\Omega)$ such that $c(0) \in K$. Similarly, we use $\mathcal{T}_K(\Omega)$ to denote the set

$$\mathcal{T}_K(\Omega) := \{(c, H_+, H_-) \in \mathcal{T}(\Omega) : c \in \mathcal{G}_K(\Omega)\}.$$

If Ω does not have C^1 boundary, then the projection map $\mathcal{T}(\Omega) \to \mathcal{G}(\Omega)$ is not a homeomorphism. However, this map is always proper, due to the compactness of the set of supporting hyperplanes at any point in $\partial\Omega$. The map $\mathcal{G}(\Omega) \to \Omega$ given by $c \mapsto c(0)$ is also proper, as the space of projective geodesics passing through a given basepoint in Ω is also compact.

If we fix an element $(c, H_+, H_-) \in \mathcal{T}(\Omega)$, we know that H_+ cannot contain $c(-\infty)$, since otherwise H_+ would also contain $c(0)$ and would not be a supporting hyperplane of Ω. Similarly H_- cannot contain $c(+\infty)$. So, we have a direct sum decomposition

$$\mathbb{R}^d = c(+\infty) \oplus (\widetilde{H_+} \cap \widetilde{H_-}) \oplus c(-\infty).$$

For triples lying in some $\mathcal{T}_K(\Omega)$, this decomposition is actually uniformly transverse in the following sense:

Lemma 4.7. *For any compact set $K \subset \Omega$, there exists some $\varepsilon_0 > 0$ such that for any $(c, H_+, H_-) \in \mathcal{T}_K(\Omega)$, we have*

$$\min\{\angle(c(+\infty), H_+ \cap H_-), \angle(c(-\infty), H_+ \cap H_-), \angle(c(+\infty), c(-\infty))\} \geq \varepsilon_0.$$

Proof. The map $\mathcal{T}(\Omega) \to \mathbb{R}$ given by

$$(c, H_+, H_-) \mapsto \min\{\angle(c(+\infty), H_+ \cap H_-), \angle(c(-\infty), H_+ \cap H_-), \angle(c(+\infty), c(-\infty))\}$$

is continuous and positive on $\mathcal{T}(\Omega)$. The set $\mathcal{T}_K(\Omega)$ is compact since it is precisely the preimage of K under the proper map $\mathcal{T}(\Omega) \to \Omega$. So the result is immediate. $\square$

Using this observation, we can apply Lemma 4.5 to obtain the following estimate on singular values for automorphisms of a convex projective domain. In this lemma, and throughout the paper, if g is an element of $\mathrm{GL}(d, \mathbb{R})$, and $W \subseteq \mathbb{R}^d$ is a subspace (not necessarily g-invariant), then the restriction $g|_W$ is interpreted as a map $W \to \mathbb{R}^d$; since both W and $\mathbb{R}^d$ are normed spaces, both $\|g|_W\|$ and $\mathbf{m}(g|_W)$ make sense.

Proposition 4.8. *Let Ω be a properly convex domain and $K \subset \Omega$ be compact. Then there exists $D > 0$ (depending only on K, Ω) satisfying the following: if $(c, H_+, H_-) \in \mathcal{T}_K(\Omega)$, and $\gamma \in \mathrm{Aut}(\Omega)$ satisfies $\gamma^{-1}c(t) \cap K \neq \varnothing$ for some $t > 0$, then:*

(1) $\left|\log\left\|\gamma^{-1}|_{c(-\infty)}\right\| - \mu_1(\gamma^{-1})\right| < D$,
(2) $\left|\log\left\|\gamma^{-1}|_{c(+\infty)}\right\| - \mu_d(\gamma^{-1})\right| < D$,
(3) $\left|\log\left\|\gamma^{-1}|_{\widetilde{H_0}}\right\| - \mu_2(\gamma^{-1})\right| < D$,
(4) $\left|\log\mathbf{m}(\gamma^{-1}|_{\widetilde{H_0}}) - \mu_{d-1}(\gamma^{-1})\right| < D$, *where $H_0 = H_+ \cap H_-$.*

Remark 4.9. We have slightly abused notation in the statement of this proposition, since elements in $\mathrm{Aut}(\Omega)$ are *projective* transformations and so the quantities $\mu_i(\gamma)$, etc. are not well-defined. So, strictly speaking, the inequalities above apply to lifts $\widetilde{\gamma} \in \mathrm{GL}(d, \mathbb{R})$ of γ, but the validity of the inequalities is independent of the choice of lift.

Proof. This proof is mainly an application of Lemma 4.5. We first fix, once and for all, a decomposition $\mathbb{R}^d = x_+ \oplus \widetilde{H} \oplus x_-$ where $x_\pm \in \mathbb{P}(\mathbb{R}^d)$ and H is a codimension-2 projective subspace. Let ℓ be a projective line segment in $\mathbb{P}(\mathbb{R}^d)$ joining x_+ and x_-, i.e. ℓ is one of the two connected components of $\mathrm{span}_{\mathbb{P}}\{x_+, x_-\} - \{x_+, x_-\}$.

Now we need to modify γ^{-1} so that it preserves the decomposition $\mathbb{R}^d = x_+ \oplus \widetilde{H} \oplus x_-$. Applying Lemma 2.19 and Lemma 4.7 above, we see that there exists a compact set $Q \subset \mathrm{GL}(d, \mathbb{R})$ (depending only on K) so that for any $(c, H_+, H_-) \in \mathcal{T}_K(\Omega)$, with $H_0 = H_+ \cap H_-$, we can find some $k = k(c, H_+, H_-) \in Q$ taking the decomposition $\mathbb{R}^d = c(+\infty) \oplus \widetilde{H_0} \oplus c(-\infty)$ to $x_- \oplus \widetilde{H} \oplus x_+$. Moreover, we can also assume that this k takes the image of c to the projective line segment ℓ. Indeed, ℓ is one of the two connected components of $\mathrm{span}_{\mathbb{P}}\{x_+, x_-\} - \{x_+, x_-\}$. Thus, if necessary, we can compose all of the elements in Q with a fixed involution interchanging the connected components of $\mathrm{span}_{\mathbb{P}}\{x_+, x_-\} - \{x_+, x_-\}$ and ensure that k takes $c(\mathbb{R})$ to ℓ.

Possibly after replacing Q with the closure of the set

$$Q' := \{k(c, H_+, H_-) : (c, H_+, H_-) \in \mathcal{T}_K(\Omega)\},$$

we may assume that for every $q \in Q$, the projective segment $q^{-1}\ell$ is properly embedded in Ω, and $q^{-1}H$ is disjoint from Ω. This means that the set $qK \cap \ell$ has bounded diameter with respect to the Hilbert metric d_ℓ on ℓ, and that the set

$$\mathcal{K} := \{q\Omega : q \in Q\}$$

is a compact subset of $\mathcal{C}(\mathbb{R}^d; \ell, H)$. Further, since Q is compact, the diameter (with respect to the Hilbert metric d_ℓ) of the set $\left(\bigcup_{q \in Q}(\ell \cap qK)\right)$ is also bounded. Let L be an upper bound for the diameter of this set.

Now fix some $(c, H_+, H_-) \in \mathcal{T}_K(\Omega)$, and assume that $\gamma^{-1}c(t) \in K$ for $\gamma \in \mathrm{Aut}(\Omega)$ and $t > 0$. As $\mathrm{Aut}(\Omega)$ acts properly on Ω, if $t \le L$ then γ^{-1} belongs to a fixed compact subset of $\mathrm{Aut}(\Omega)$ depending only on L, and we may choose D sufficiently large so that each of the inequalities in the statement of the proposition holds for every γ in this set. So, we may assume from now on that $t > L$. Fix a lift of γ in $\mathrm{GL}(d, \mathbb{R})$; abusing notation we also denote this lift by γ (see Remark 4.9).

We let c' be the translated and reparameterized geodesic $s \mapsto \gamma^{-1}c(s + t)$, so that $c' \in \mathcal{G}_K(\Omega)$, and

$$(c', \gamma^{-1}H_+, \gamma^{-1}H_-) \in \mathcal{T}_K(\Omega).$$

By our construction of Q, we can choose $k, k' \in Q$ so that k takes the decomposition $c(\infty) \oplus \widetilde{H_0} \oplus c(-\infty)$ to $x_- \oplus \widetilde{H} \oplus x_+$, and similarly for k', c' and $\gamma^{-1}H_\pm$. Then, the group element $g \in \mathrm{GL}(d, \mathbb{R})$ defined by $g = k'\gamma^{-1}k^{-1}$ preserves the decomposition $x_+ \oplus \widetilde{H} \oplus x_-$ and the projective line ℓ, which verifies one of the hypotheses of Lemma 4.5. Moreover, recalling that $\mathcal{K} = \{k\Omega : k \in Q\}$ is a compact subset of $\mathcal{C}(\mathbb{R}^d; \ell, H)$, we see that $g\mathcal{K}$ contains $k'\gamma^{-1}k^{-1}k\Omega) = k'\Omega \in \mathcal{K}$, hence $g\mathcal{K} \cap \mathcal{K} \neq \emptyset$. This verifies another hypothesis of Lemma 4.5, when we take $\mathcal{K}_1 = \mathcal{K}_2 = \mathcal{K}$.

Finally, we need to verify that $||g_{x_+}|| \ge ||g_{x_-}||$, by considering the action of g on the projective line segment ℓ. Let $x_0 = kc(0)$. Observe that the 4-tuple $[c(-\infty), c(0), c(t), c(+\infty)]$ is arranged on the image of c in this order. Applying $k'\gamma^{-1}$ to c, we then observe that the points

$$[k'c(-\infty), k'\gamma^{-1}c(0), k'\gamma^{-1}c(t), k'c(+\infty)] = [x_+, gx_0, k'\gamma^{-1}c(t), x_-]$$

lie on the projective segment ℓ in this order. Moreover, since $\gamma^{-1}c(t) \in K$, and $c(0) \in K$, we know from the definition of L that $d_\ell(k'\gamma^{-1}c(t), x_0) \leq L$ (with respect to the Hilbert metric d_ℓ on ℓ). Since $d_\ell(k'\gamma^{-1}c(0), k'\gamma^{-1}c(t)) = t > L$, it follows that the 4-tuple of points

$$[x_+, k'\gamma^{-1}c(0), kc(0), x_-] = [x_+, gx_0, x_0, x_-]$$

are also arranged in this order on ℓ. Since g fixes the endpoints of ℓ, the eigenvalue of g on x_+ must be larger than the eigenvalue of g on x_-, or equivalently, $\left\|g|_{x_+}\right\| > \left\|g|_{x_-}\right\|$.

We have now verified that we can apply Lemma 4.5 to g. Then $\log(\left\|g|_{x_+}\right\|)$ is within bounded additive error C of $\mu_1(g)$, where the constant C depends only on K and Ω. However, since $gk = k'\gamma^{-1}$ for k, k' in the fixed compact set Q, and $kc(-\infty) = x_+$, we can apply Lemma 2.15 to get the first desired estimate for γ^{-1}. The other estimates follow similarly. □

4.4. **A "straightness" lemma.** The estimate given by Proposition 4.1 implies that, if γ_n is a sequence tracking a projective geodesic in a convex projective domain Ω, then the gap $\mu_{1,d}(\gamma_n)$ increases roughly linearly in n. The same linear estimate need not hold for other singular value gaps. In fact, [BPS19] proves that uniform linear growth in n imposes a strong restriction. Suppose Γ divides Ω and there is an index j such that $\mu_{j,j+1}(\gamma_n)$ grows uniformly linearly in n for all tracking sequences $\{\gamma_n\}$. Then Γ must be a hyperbolic group [BPS19]. Thus, in the non-hyperbolic setting, there is no way to obtain such a sharp estimate. However, for a sequence $\{\gamma_n\}$ tracking a Morse geodesic, we can prove a "coarse monotonicity" property for $\mu_{1,2}(\gamma_n)$ and $\mu_{d-1,d}(\gamma_n)$. Our main tool is the following "straightness" lemma.

Lemma 4.10. *Suppose Ω is a properly convex domain and $K \subset \Omega$ is a compact set. Then there exists a constant $D > 0$ satisfying the following: if $c \in \mathcal{G}_K(\Omega)$ and $\{\gamma_n\}$ is a sequence in $\operatorname{Aut}(\Omega)$ such that $\gamma_n^{-1}c(n) \in K$ for all $n \in \mathbb{N}$, then for any $n, m \in \mathbb{N}$, we have*

$$\mu_{i,i+1}(\gamma_n) + \mu_{i,i+1}\left(\gamma_n^{-1}\gamma_{n+m}\right) \leq \mu_{i,i+1}(\gamma_{n+m}) + D,$$

where $i \in \{1, d-1\}$.

Remark 4.11. Results of a similar flavor were also obtained by Canary-Zhang-Zimmer [CZZ22] in their work on transverse subgroups; see Section 6 in [CZZ22], especially Lemma 6.4. A crucial difference in our context is that we impose no assumption on the regularity properties of the sequence $\{\gamma_n\}$ in $\operatorname{Aut}(\Omega)$. In particular, the sequence $\{\gamma_n\}$ does *not* need to lie in a uniformly 1-regular subgroup of $\operatorname{Aut}(\Omega)$, which is a condition required by the results in [CZZ22].

Proof. Throughout the proof, we implicitly identify each γ_n in the sequence with a chosen lift in $\mathrm{GL}(d, \mathbb{R})$. As in the proof of Proposition 4.8, we start by fixing a direct sum decomposition $\mathbb{R}^d = x_+ \oplus W \oplus x_-$, a projective line ℓ joining $x_\pm$, and a compact subset $Q \subset \mathrm{GL}(d, \mathbb{R})$ so that for any $(c, H_+, H_-) \in \mathcal{T}_K(\Omega)$, we can find some $k \in Q$ taking $c(+\infty) \oplus (\widetilde{H_+} \cap \widetilde{H_-}) \oplus c(-\infty)$ to $x_- \oplus W \oplus x_+$ and the image of c to ℓ. We also fix a constant $L > 0$ as in the proof of the same proposition, so that the diameter (in the Hilbert metric d_ℓ on ℓ) of the set

$$\bigcup_{q \in Q} (\ell \cap qK)$$

is bounded by L.

Next, observe that, if $D > 0$ is chosen large enough (depending on L), then the desired inequality holds whenever $n < L$. This follows from Lemma 2.15 and the fact that $\mathrm{Aut}(\Omega)$ acts properly on Ω: if $n \leq L$, then since $\mathrm{d}_\Omega(c(n), K) \leq n$ and $\gamma_n^{-1}c(n) \in K$, the automorphism γ_n lies in compact subset of $\mathrm{Aut}(\Omega)$ depending only on L, and both quantities $\mu_{i,i+1}(\gamma_n)$ and $|\mu_{i,i+1}(\gamma_n^{-1}\gamma_{n+m}) - \mu_{i,i+1}(\gamma_{n+m})|$ are uniformly bounded by Lemma 2.15.

Similarly, since $\mathrm{d}_\Omega(\gamma_n^{-1}c(n+m), K) \leq m$ and $\gamma_{n+m}^{-1}\gamma_n \cdot \gamma_n^{-1}c(n+m) \in K$ for any m, we may also choose D so that the desired inequality holds whenever $m \leq L$. So, for the rest of the proof, we may assume that both $n > L$ and $m > L$.

For each $j \in \mathbb{N}$, since $\gamma_j^{-1}c$ passes through K, we can choose $k_j \in Q$ taking the decomposition

$$\gamma_j^{-1}c(+\infty) \oplus \gamma_j^{-1}(\widetilde{H}_+ \cap \widetilde{H}_-) \oplus \gamma_j^{-1}c(-\infty)$$

to $x_- \oplus W \oplus x_+$. Here, we assume that $\gamma_0 = \mathrm{id}$. Then, defining $g_j := k_j\gamma_j^{-1}k_0^{-1}$, we observe that g_j preserves the decomposition $x_- \oplus W \oplus x_+$. Moreover, by Proposition 4.8 and Lemma 2.15, there is a uniform constant C so that for given $n \geq 1$, we have

$$\left|\mu_{d-1,d}(\gamma_n^{-1}) - \left(\log \mathbf{m}(g_n|_W) - \log \left\|g_n|_{c(+\infty)}\right\|\right)\right| \leq 2C, \text{ and} \tag{6}$$

$$\left|\mu_{1,2}(\gamma_n^{-1}) - \left(\left\|g_n|_{c(-\infty)}\right\| - \log \|g_n|_W\|\right)\right| \leq 2C. \tag{7}$$

Next, for given $n, m \in \mathbb{N}$, we consider the group element

$$T_{n,m} := g_{n+m}g_n^{-1}.$$

By (6) we know

$$\begin{aligned}\mu_{1,2}(\gamma_n) - \mu_{1,2}(\gamma_{n+m}) &= \mu_{d-1,d}(\gamma_n^{-1}) - \mu_{d-1,d}(\gamma_{n+m}^{-1}) \\ &\leq \log \frac{\mathbf{m}(g_n|_W)}{\mathbf{m}(g_{n+m}|_W)} + \log \frac{\left\|g_{n+m}|_{x_-}\right\|}{\left\|g_n|_{x_-}\right\|} + 4C.\end{aligned}$$

Since $g_{n+m} = T_{n,m}g_n$, the inequality $\mathbf{m}(gh) \geq \mathbf{m}(g)\mathbf{m}(h)$ implies that the first term above is at most $\log \frac{1}{\mathbf{m}(T_{n,m}|_W)}$. And, since x_- is a one-dimensional eigenspace of both g_n and g_{n+m}, the second term is equal to $\log \left\|T_{n,m}|_{x_-}\right\|$. Thus

$$\mu_{1,2}(\gamma_n) - \mu_{1,2}(\gamma_{n+m}) \leq \log \frac{\left\|T_{n,m}|_{x_-}\right\|}{\mathbf{m}(T_{n,m}|_W)} + 4C. \tag{8}$$

We wish to apply Lemma 4.5 to the element $T_{n,m}$, so we let $\mathcal{K} = \{k\Omega : k \in Q\}$. Then, since

$$T_{n,m} = g_{n+m}g_n^{-1} = k_{n+m}\gamma_{n+m}^{-1}\gamma_n k_n^{-1}, \tag{9}$$

we have

$$T_{n,m}k_n\Omega = k_{n+m}\Omega \in \mathcal{K}$$

and therefore $T_{n,m}\mathcal{K} \cap \mathcal{K} \neq \varnothing$. We also need to verify the other hypothesis of Lemma 4.5, and show that $\left\|T_{n,m}|_{x_+}\right\| \geq \left\|T_{n,m}|_{x_-}\right\|$. For this, we again argue as in the proof of Proposition 4.8, and consider the 4-tuple of points

$$[c(-\infty), c(n), c(n+m), c(+\infty)]$$

arranged in this order on the image of c. Then the 4-tuple

$$k_n\gamma_n^{-1} \cdot [c(-\infty), c(n), c(n+m), c(+\infty)] \tag{10}$$

is arranged in the corresponding order on the projective segment ℓ. We let $y_0 := k_n\gamma_n^{-1}c(n)$ and $y_m := k_n\gamma_n^{-1}c(n+m)$. Then the 4-tuple in (10) is the same as

$$[x_+, y_0, y_m, x_-].$$

Since $T_{n,m}$ fixes the endpoints $x_\pm$ of ℓ and preserves ℓ, the points

$$[x_+, T_{n,m}y_0, T_{n,m}y_m, x_-]$$

are also arranged in this order on ℓ. Further, since $y_0 \in QK \cap \ell$, and $T_{n,m}y_m = k_{n+m}\gamma_{n+m}^{-1}c(n+m) \in QK \cap \ell$, we have $\mathrm{d}_\ell(T_{n,m}y_m, y_0) \le L$. But

$$\mathrm{d}_\ell(y_0, y_m) = d_c(c(n), c(n+m)) = m > L.$$

So, $T_{n,m}y_m$ must lie in the open projective segment $(x_+, y_m) \subset \ell$. Thus it follows that the points

$$[x_+, T_{n,m}y_0, y_0, x_-]$$

are arranged on ℓ in that order which implies that $\left\|T_{n,m}|_{x_+}\right\| > \left\|T_{n,m}|_{x_-}\right\|$.

We may therefore apply estimate (2) and estimate (4) from Lemma 4.5 to $T_{n,m}$. This tells us that there is a uniform constant $C' > 0$ so that

$$\log \frac{\left\|T_{n,m}|_{x_-}\right\|}{\mathbf{m}(T_{n,m}|_W)} \le \mu_d(T_{n,m}) - \mu_{d-1}(T_{n,m}) + 2C' = -\mu_{1,2}(T_{n,m}^{-1}) + 2C'.$$

Putting this together with (8) and (9), we obtain

$$\mu_{1,2}(\gamma_n) - \mu_{1,2}(\gamma_{n+m}) \le -\mu_{1,2}(k_n\gamma_n^{-1}\gamma_{n+m}k_{n+m}^{-1}) + 4C + 2C'.$$

Then an application of Lemma 2.15 proves that the desired inequality holds when $i = 1$.

The case where $i = d-1$ is similar; we apply (7) in place of (6) to see that

$$\mu_{d-1,d}(\gamma_n) - \mu_{d-1,d}(\gamma_{n+m}) \le \log \frac{\left\|T_{n,m}|_{x_+}\right\|}{\left\|T_{n,m}|_W\right\|} + 4C.$$

Then we use the other estimates from Lemma 4.5 to see that

$$\log \frac{\left\|T_{n,m}|_{x_+}\right\|}{\left\|T_{n,m}|_W\right\|} \le -\mu_{d-1,d}(T_{n,m}^{-1}) + 2C',$$

and apply Lemma 2.15 again to complete the proof. □

5. Singular values of Morse geodesics in convex projective geometry

In this section, we combine results from Sections 3 and 4 to study the behavior of singular value gaps of sequences that track Morse projective geodesic rays. The main aim of the section is to prove Theorem 1.6 and Theorem 1.8.

5.1. Morse geodesics are strongly uniformly regular.

We first address Theorem 1.6. We start with the following lemma, which we will strengthen later.

Lemma 5.1. *Let M be a Morse gauge, let $C > 0$, and let $x_0 \in \Omega$. There exists $k = k(M, C, x_0) > 0$ such that, for any $\gamma \in \mathrm{Aut}(\Omega)$, if $\mathrm{d}_\Omega(x_0, \gamma x_0) > k$ and $[x_0, \gamma x_0]$ is M-Morse, then $\mu_{1,2}(\gamma) > C$.*

Proof. Fix M, C, x_0. Suppose for a contradiction that there exists a sequence $\{\gamma_n\}$ in $\mathrm{Aut}(\Omega)$ such that $\mathrm{d}_\Omega(x_0, \gamma_n x_0) > n$ and $[x_0, \gamma_n x_0]$ M-Morse, but $\mu_{1,2}(\gamma_n) \leq C$. After passing to a subsequence, we can assume that $\gamma_n x_0$ converges to $x \in \partial\Omega$. Then $[x_0, \gamma_n x_0] \to [x_0, x)$ uniformly on compact subsets of Ω. As each $[x_0, \gamma_n x_0]$ is M-Morse, so is $[x_0, x)$.

By Corollary 3.24, x is a C^1 extreme point in $\partial\Omega$, so $\dim F_\Omega(x) = 0$. Then Proposition 4.2 implies that $\mu_{1,2}(\gamma_n) \to \infty$. This contradicts the assumption that $\mu_{1,2}(\gamma_n) \leq C$. ☐

For the next result, we slightly refine the notion of tracking sequences from Definition 1.2. If $c : [0, L] \to \Omega$ is a projective geodesic segment of length $L > 0$, then we will say that a finite sequence $\{\gamma_n\}$ in $\mathrm{Aut}(\Omega)$ *R-tracks c* if $\mathrm{d}_\Omega(\gamma_n x_0, c(n)) < R$ for all $n \in \mathbb{N} \cap [0, L]$.

Remark 5.2. If $\{\gamma_n\}$ R-tracks c (a geodesic ray or segment), then there exists a constant D' depending on x_0 such that

$$\mu_{1,d}(\gamma_i^{-1}\gamma_{i+k}) \leq 2R + D' + \frac{k}{2}.$$

This is immediate from Proposition 4.1 and the definition of a tracking sequence.

Proposition 5.3. *Fix a Morse gauge M, a positive real number R, and $x_0 \in \Omega$. There exist constants $A, B > 0$ (depending on M, x_0, and R), such that: if $\gamma \in \mathrm{Aut}(\Omega)$ for which $[x_0, \gamma x_0]$ is M-Morse, $\mathrm{d}_\Omega(x_0, \gamma x_0) > B$, and there is a finite sequence $\{\eta_n\}$ in $\mathrm{Aut}(\Omega)$ that R-tracks $[x_0, \gamma x_0]$, then*

$$\frac{\mu_{1,2}(\gamma)}{\mu_{1,d}(\gamma)} > A \quad \text{and} \quad \frac{\mu_{d-1,d}(\gamma)}{\mu_{1,d}(\gamma)} > A.$$

Proof. It suffices to only prove the first inequality. The second inequality follows from the first after replacing γ with γ^{-1}, since $\mu_{1,2}(\gamma) = \mu_{d-1,d}(\gamma^{-1})$ and $[x_0, \gamma x_0]$ is M-Morse if and only if $[x_0, \gamma^{-1} x_0]$ is M-Morse.

Fix $\gamma \in \mathrm{Aut}(\Omega)$ such that $[x_0, \gamma x_0]$ is M-Morse, and let $L := \lceil \mathrm{d}_\Omega(x_0, \gamma x_0) \rceil$. We also suppose that there is a sequence $\{\eta_n\}_{n=1}^{L}$ that R-tracks $[x_0, \gamma x_0]$. Observe that for any n, m, the geodesic segment $[\eta_n x_0, \eta_m x_0]$ is M'-Morse for a Morse gauge M' depending only on M and R.

Now we apply Lemma 4.10, taking the compact set K in the lemma to be $\overline{B_R(x_0)}$. Let D be the constant in Lemma 4.10. Then for all $n, n+m \in \{1, \dots, L\}$,

$$\mu_{1,2}(\eta_n) + \mu_{1,2}(\eta_n^{-1}\eta_{n+m}) \leq \mu_{1,2}(\eta_{n+m}) + D. \tag{11}$$

By Lemma 5.1, there exists a constant $k > 0$ so that for every $n = 1, \dots, L$,

$$\mu_{1,2}(\eta_n^{-1}\eta_{n+k}) > 3D.$$

Fix any $n \in \{k, k+1, \dots, L\}$. Let $j \in \{1, \dots, \lfloor L/k \rfloor\}$ be such that $kj \leq n < kj + k$. Then,

$$\mu_{1,2}(\eta_n) \geq \mu_{1,2}(\eta_{kj}^{-1}\eta_n) + \mu_{1,2}(\eta_{kj}) - D \geq \mu_{1,2}(\eta_{kj}) - D.$$

But the inequality (11) further implies that

$$\mu_{1,2}(\eta_{kj}) \geq -D + \mu_{1,2}(\eta_{kj-k}) + \mu_{1,2}(\eta_{kj-k}^{-1}\eta_{kj}).$$

We can then conclude (by inducting on j, and assuming $\eta_0 = \mathrm{id}$) that for all j, we have

$$\mu_{1,2}(\eta_{kj}) \geq -jD + \sum_{i=0}^{j-1} \mu_{1,2}(\eta_{ki}^{-1}\eta_{ki+k}).$$

By our choice of k this implies

$$\mu_{1,2}(\eta_{kj}) \geq -jD + j(3D) > 2Dj.$$

Thus

$$\mu_{1,2}(\eta_n) \geq \mu_{1,2}(\eta_{kj}) - D > 2jD - D \geq jD.$$

On the other hand, Remark 5.2 implies that $\mu_{1,d}(\eta_n) \leq D' + \frac{n}{2}$. Set $A := \frac{D}{2k(1+2D')}$. Then

$$\frac{\mu_{1,2}}{\mu_{1,d}}(\eta_n) \geq \frac{jD}{n+2D'} \geq \frac{D}{k(1+2D')}\frac{kj}{n} \geq 2A \cdot \frac{kj}{kj+k} \geq A.$$

The result then follows with $A := \frac{D}{2k(1+2D')}$ and $B := k$. □

Now we can prove the proposition below, which is a restatement of Theorem 1.6 from the introduction.

Proposition 5.4. *Let c be a projective geodesic in a properly convex domain Ω, and let $\{\gamma_n\}$ be a sequence which R-tracks c with respect to a basepoint $x_0 \in \Omega$. If c is M-Morse, then there are constants $C, N > 0$ (depending only on M, x_0, R) so that, for all $n \geq 1$ and $m > N$, we have*

$$\frac{\mu_{1,2}(\gamma_n^{-1}\gamma_{n+m})}{\mu_{1,d}(\gamma_n^{-1}\gamma_{n+m})} > C \text{ and } \frac{\mu_{d-1,d}(\gamma_n^{-1}\gamma_{n+m})}{\mu_{1,d}(\gamma_n^{-1}\gamma_{n+m})} > C.$$

Proof. Fix an M-Morse geodesic c and a tracking sequence $\{\gamma_n\}$ as in the statement. Since $\{\gamma_n\}$ R-tracks c, there is some Morse gauge M' (depending only on M and R) so that for any n, m, the projective geodesic segment $[\gamma_n x_0, \gamma_{n+m} x_0]$ is M'-Morse, hence so is the projective geodesic $[x_0, \gamma_n^{-1}\gamma_{n+m}x_0]$. So then Proposition 5.3 implies that there are positive constants C, N depending only on M' so that if $m > N$, then $\mu_{1,2}(\gamma_n^{-1}\gamma_{n+m})/\mu_{1,d}(\gamma_n^{-1}\gamma_{n+m}) > C$, as required. □

5.2. **The partial converse.** The examples below show that the full converse to Theorem 1.6 does not always hold.

Example 5.5. Identify the hyperbolic plane $\mathbb{H}^2$ with its projective model in $\mathbb{P}(\mathbb{R}^3)$, so that $\mathrm{PO}(2,1) \subset \mathrm{PSL}(3,\mathbb{R})$ acts by isometries. Let ℓ be a geodesic in $\mathbb{H}^2$. The two tangent lines to $\mathbb{H}^2$ at the endpoints of ℓ meet in unique dual point ℓ^* to ℓ; this point is the orthogonal complement to ℓ, with respect to the Minkowski bilinear form defining this model of $\mathbb{H}^2$.

Let Ω be the convex hull of $\mathbb{H}^2$ and ℓ^*, let $x_0 \in \ell$, and let h be a loxodromic element in $\mathrm{PO}(2,1)$ preserving ℓ, with translation length 1. Then the sequence $\{h^n x_0\}$ lies along ℓ, i.e. $\{h^n\}$ tracks a projective geodesic sub-ray of ℓ. As a subset of $\mathbb{H}^2$, the projective geodesic ℓ is Morse, since $\mathbb{H}^2$ is hyperbolic; in particular by Theorem 1.6 this means that the sequence $\{h^n\}$ is strongly uniformly k-regular for $k = 1, 2$. However, while ℓ is still a geodesic in the larger domain Ω, it cannot be a Morse geodesic in this domain, as both of its endpoints lie in the closure of nontrivial segments in $\partial\Omega$ (see Corollary 3.25).

There are two important points to observe in the previous example: first, Ω does not have exposed boundary, and second, $\{h^n : n \in \mathbb{Z}\}$ does not divide Ω. In the next example, we observe that problems can still occur even if we assume that the domain Ω is divisible.

Example 5.6. Consider the projective 2-simplex $\Delta := \{[x : y : z] \mid x, y, z > 0\}$ in $\mathbb{P}(\mathbb{R}^3)$ and fix $x_0 := [1 : 1 : 1]$. Let $\Gamma \subset \mathrm{PSL}(3, \mathbb{R})$ be the projectivization of the group of diagonal matrices whose entries are integer powers of 2. Then Γ is an abelian subgroup dividing Δ. So if $h \in \Gamma$ is the diagonal matrix $h = \mathrm{diag}(2, 1, 1/2)$, then the mapping $n \mapsto h^n x_0$ is a quasi-isometric embedding. The sequence $\{h^n\}$ is also strongly uniformly k-regular for $k = 1, 2$. However, the set of points $\{h^n x_0\}$ cannot be in a uniform neighborhood of a Morse geodesic, since Δ is quasi-isometric to the 2-dimensional Euclidean space, which contains no Morse geodesics.

Note that, although the example above fails to be irreducible, one can find irreducible divisible domains (indeed, irreducible rank-one domains) which contain an embedded copy of this example; we work closely with such an example in Section 7 of this paper. So the precise converse to Theorem 1.6 can still fail even in the case where the ambient domain Ω is divisible and rank one.

Despite the existence of the examples above, it is still possible to prove Theorem 1.8 – a partial converse to Theorem 1.6. We recall the statement of this partial converse.

Theorem 1.8 (Section 5)**.** *Let Ω be a convex divisible domain with exposed boundary and let c be a projective geodesic ray in Ω. Suppose $\{\gamma_n\}$ R-tracks c with respect to $x_0 \in \Omega$. If $\{\gamma_n\}$ is strongly uniformly k-regular for $k = 1$ and $k = d - 1$, then c is M-Morse for some Morse gauge M.*

Moreover, M can be chosen to depend only on x_0, R, and the constants in the definition of strong uniform k-regularity.

Remark 5.7.

(1) The sequence given in Example 5.5 tracks a projective geodesic, but the domain Ω in this example both fails to have exposed boundary and also fails to be divisible. We do not know if the "exposed boundary" assumption is necessary in Theorem 1.8; there are no known examples of divisible domains without exposed boundary.
(2) Theorem 1.8 tells us that the quasi-geodesic considered in Example 5.6 cannot track any projective geodesic, which can also be verified directly.

The main idea in the proof of Theorem 1.6 is to use the characterization of Morse geodesics in divisible domains with exposed boundary given at the end of Section 3. This allows us to prove a weaker version of the theorem, which does not provide uniform control over the Morse gauge; then we use a compactness argument to prove the full (uniform) result.

The non-uniform version of Theorem 1.8 is given by the proposition below.

Proposition 5.8. *Let Ω be a convex divisible domain with exposed boundary, let c be a projective geodesic ray in Ω, and let $\{\gamma_n\}$ be a sequence which tracks c. If $\{\gamma_n\}$ is both strongly uniformly 1-regular and strongly uniformly $(d-1)$-regular, then c is M-Morse.*

Proof. We will prove the contrapositive. We let $c : [0, \infty) \to \Omega$ be a projective geodesic which is *not* Morse, and let $\{\gamma_n\}$ be a sequence tracking c. Extend c (uniquely) to a bi-infinite projective geodesic $c : (-\infty, \infty) \to \Omega$ and let $y = c(-\infty)$.

By Corollary 3.32, we know that $z = c(+\infty)$ is either forward conically related to a non-extreme point in $\partial\Omega$, or else $c(+\infty)$ is forward conically related to a non-C^1 point in $\partial\Omega$. Since γ_n tracks c, the properness part of the Bénzecri cocompactness theorem tells us that we can use γ_n to realize the conical relation: there is a subsequence of γ_n so that $\gamma_n^{-1}(z, y)$ converges to a properly embedded projective segment $(z_\infty, y_\infty) \subset \Omega$, so that z_∞ is either in the interior of a segment or a non-C^1 point. In this proof, we will consider the case where z_∞ lies in the interior of a nontrivial segment; the case where z_∞ is a non-C^1 point is nearly identical.

Now we begin the proof. Let L be a projective line spanned by a nontrivial segment in $\partial\Omega$ containing z_∞, and let P be the projective 2-plane spanned by (y_∞, z_∞) and L. Fix a basis $\{v_1, v_2, v_3\}$ for $\widetilde{P}$, so that $\mathrm{span}_{\mathbb{P}}\{v_1, v_2\} = L$ and $[v_3] = y_\infty$. Then, for each $m \in \mathbb{N}$, let $\widetilde{h}_m$ be linear map on $\widetilde{P}$ defined (with respect to the chosen basis) by

$$h_m := \begin{pmatrix} e^{-2m} & & \\ & e^{-2m} & \\ & & 1 \end{pmatrix},$$

and let h_m be the corresponding projective transformation on P.

Claim 5.8.1. *For infinitely many $m \in \mathbb{N}$, there exists $g_m \in \mathrm{PGL}(d, \mathbb{R})$ and $n = n(m) \in \mathbb{N}$ so that each pair $(g_m\Omega, g_m\gamma_{n(m)}^{-1}c(n(m) + m))$ lies in a fixed compact subset of the space of pointed domains and $g_m|_P = h_m$.*

Proof of Claim. Observe that as $m \to \infty$, the sequence of domains $h_m(\Omega \cap P)$ converges (after extraction) to some fixed properly convex domain in P. So, by [Ben03, Lemma 2.8], we may extend each h_m to a linear map $g_m \in \mathrm{GL}(d, \mathbb{R})$ agreeing with h_m on $\widetilde{P}$, so that, as $m \to \infty$, a subsequence of $g_m\Omega$ converges to a properly convex domain Ω_∞ in $\mathbb{P}(\mathbb{R}^d)$, containing $\lim_{m\to\infty} h_m(\Omega \cap P)$ as a 2-sector (see Definition 3.17).

Now, as $n \to \infty$, we know that (after extracting a subsequence) the sequence $\gamma_n^{-1}c(n)$ converges to some point in the geodesic (y_∞, z_∞). We may fix a unit-speed parameterization $c_\infty : (-\infty, \infty) \to \Omega$ of this geodesic so that $c_\infty(\infty) = z_\infty$ and $\gamma_n^{-1}c(n) \to c_\infty(0)$. Then, for any fixed m, $\gamma_n^{-1}c(n + m) \to c_\infty(m)$ as $n \to \infty$.

Fix an auxiliary metric $\mathrm{d}_{\mathbb{P}}$ on $\mathbb{P}(\mathbb{R}^d)$. Since $\mathrm{GL}(d, \mathbb{R})$ acts by homeomorphisms on $\mathbb{P}(\mathbb{R}^d)$, for each fixed m we may choose some δ so that if $\mathrm{d}_{\mathbb{P}}(u, v) < \delta$, then $\mathrm{d}_{\mathbb{P}}(g_m u, g_m v) < 1/m$. In particular, for each m we can find $n(m)$ so that

$$\mathrm{d}_{\mathbb{P}}(g_m\gamma_{n(m)}^{-1}c(n(m) + m), g_m c_\infty(m)) < 1/m. \tag{12}$$

However, by construction, we know that $g_m c_\infty(m) = h_m c_\infty(m) = c_\infty(0)$, as h_m acts by a translation of Hilbert distance m along (y_∞, z_∞) in the direction of y_∞. Moreover, $c_\infty(0)$ lies in the limit of the 2-sectors $h_m(P \cap \Omega)$. Thus $c_\infty(0)$ lies in the limiting domain Ω_∞. Then (12) implies that for m large enough, $g_m\gamma_{n(m)}^{-1}c(n(m) + m)$ lies in a fixed compact subset of the domain $\Omega_\infty = \lim_{m\to\infty} g_m\Omega$. ◻

The last part of the previous claim tells us that the projective transformations g_m "approximate" the automorphisms $\gamma_{n(m)}^{-1}\gamma_{n(m)+m}$. To be precise, we have:

Claim 5.8.2. *There is a fixed compact subset Q of $\mathrm{PGL}(d,\mathbb{R})$ so that, if g_m and $n(m)$ are as in the previous claim, then $g_m\gamma_{n(m)}^{-1}\gamma_{n(m)+m} \in Q$.*

Proof of Claim. Since $\{\gamma_n\}$ tracks c, $x_m := \gamma_{n(m)+m}^{-1}c(n(m)+m)$ lies in a fixed compact subset of Ω for all m. By the previous claim, $g_m\gamma_{n(m)}^{-1}\gamma_{n(m)+m}(\Omega, x_m) = g_m(\Omega, \gamma_{n(m)}^{-1}c(n(m)+m))$ lies in a compact subset of the space of pointed domains. The claim is then immediate from the properness part of the Benzécri compactness Theorem 2.9. □

Finally, we can show:

Claim 5.8.3. *The sequence $\{\gamma_n\}$ is not strongly uniformly $(d-1)$-regular.*

Proof of Claim. Since $\{\gamma_n\}$ tracks c, Proposition 4.1 implies that the quantity $\mu_{1,d}(\gamma_{n(m)}^{-1}\gamma_{n(m)+m})$ tends to infinity as $m \to \infty$. We will show that $\mu_{d-1,d}(\gamma_{n(m)}^{-1}\gamma_{n(m)+m})$ is bounded, independent of m. Owing to the previous claim and Lemma 2.15, it suffices to show that $\mu_{d-1,d}(g_m)$ is bounded.

To prove this, fix supporting hyperplanes H_+, H_- of Ω at $c(\pm\infty)$, and let $H_0 = H_+ \cap H_-$. Using Lemma 4.7 and Lemma 2.19 (as in the proof of Proposition 4.8), we can find a fixed compact set $Q \subset \mathrm{PGL}(d,\mathbb{R})$ and elements $q_n, q_n' \in Q$ so that any lift of $q_n g_m q_n'$ preserves the decomposition

$$c(+\infty) \oplus \widetilde{H_0} \oplus c(-\infty).$$

Let $\widetilde{g}_m$ be a lift of g_n agreeing with $\widetilde{h}_m$ on $\widetilde{P}$, and let $\widetilde{q}_n$, $\widetilde{q}_n'$ be lifts of q_n, q_n' lying in a fixed compact subset of $\mathrm{GL}(d,\mathbb{R})$. Then, Lemma 4.5 and Lemma 2.15 imply that $\mu_d(\widetilde{g}_m)$ is within uniformly bounded additive error of $-2m$. In addition, since the e^{-2m}-eigenspace of $\widetilde{g}_m$ is at least 2-dimensional, it follows from the "minimax" formula (1) for singular values that $\sigma_{d-1}(\widetilde{g}_m) \le e^{-2m}$ and therefore $\mu_{d-1,d}(\widetilde{g}_m) = \mu_{d-1,d}(g_m)$ is uniformly bounded. □

This finishes the proof of Proposition 5.8 in the first case, where z_∞ is not an extreme point. In the other case (where z_∞ is not a C^1 point) we argue similarly, but we instead pick our projective 2-plane P so that z_∞ is not a C^1 point in $\Omega \cap P$. Then we pick a basis $\{v_1, v_2, v_3\}$ so that v_1 spans z_∞, and take $\widetilde{h}_m$ to be the sequence of matrices $\widetilde{h}_m = \mathrm{diag}(1, e^{2m}, e^{2m})$. Arguing as in the other case, we see that for a sequence of indices $n(m)$, the gap $\mu_{1,2}(\gamma_{n(m)}^{-1}\gamma_{n(m)+m})$ is uniformly bounded, which implies that $\{\gamma_n\}$ is not strongly uniformly 1-regular. □

5.2.1. *Proof of Theorem 1.8.* We proceed by contradiction and suppose that there is a sequence of projective geodesics $\{c_m\}$ and tracking sequences $\{\gamma_{n,m}\}_{n\in\mathbb{N}}$, so that

$$\mathrm{d}_\Omega(\gamma_{n,m}x_0, c_m(n)) \le R,$$

and each $\{\gamma_{n,m}\}_{n\in\mathbb{N}}$ is both strongly uniformly 1-regular and strongly uniformly $(d-1)$-regular (with uniform constants), but c_m eventually fails to be M-Morse for any given Morse gauge M. Applying Proposition 3.4 and Proposition 3.6, it then follows that c_m eventually fails to be projectively δ-slim, for any given $\delta > 0$. After extracting a subsequence, we can then assume that each c_m fails to be projectively m-slim.

We now argue as in the proof of Lemma 3.26: for each m, let $x_m, y_m, z_m \in \Omega$ be points such that x_m, y_m lie on the image of c_m, but $[x_m, y_m]$ is not contained in the m-neighborhood $N_m([x_m, z_m] \cup [y_m, z_m])$. Then let w_m be a point in $[x_m, y_m]$ such that $d_\Omega(w_m, [x_m, z_m] \cup [y_m, z_m]) \geq m$. Choose some n_m so that $\gamma_{n_m,m}$ satisfies $d_\Omega(\gamma_{n_m,m}^{-1} w_m, x_0) \leq R$. After extracting a further subsequence, the geodesic rays $\gamma_{n_m,m}^{-1} c_m$ converge to a bi-infinite projective geodesic c_∞ whose endpoints lie in the boundary of a half-triangle. Then by Lemma 3.13, no sub-ray of c_∞ is Morse.

For each $m \in \mathbb{N}$, define the geodesic sub-ray $c'_m : [0, \infty) \to \Omega$ of c_m by $c'_m(t) := \gamma_{n_m,m}^{-1} c_m(n_m + t)$. Note that the n_m-tail of the sequence $\{\gamma_{n_m,m}^{-1} \gamma_{n,m}\}_{n \in \mathbb{N}}$ R-tracks c'_m with respect to x_0. Moreover, as $m \to \infty$, c'_m converges to c_∞ uniformly on compact subsets of Ω. Then, we can run a diagonalization argument along the sequences $\{\gamma_{n_m,m}^{-1} \gamma_{n,m}\}_{n \in \mathbb{N}}$ to produce a sequence $\{f_n\}$ in Γ that tracks c_∞. Moreover, $\{f_n\}$ is also strongly uniformly 1-regular since the sequences $\{\gamma_{n,m}\}_{n \in \mathbb{N}}$ are all strongly uniformly 1-regular with uniform regularity constants. Thus, by Proposition 5.8, the corresponding sub-ray of c_∞ is Morse, giving a contradiction. □

6. Regularity at boundary points and singular value gaps

Our goal in this section is to prove Theorem 1.12, which connects the linear algebraic behavior of a tracking sequence in a properly convex domain Ω with the regularity of the endpoint of this geodesic in $\partial\Omega$.

6.1. **Pointwise regularity in convex hypersurfaces.** As we have alluded to previously, the boundary of a properly convex domain is often nowhere C^1, but differentiable in a dense set. We therefore wish to have a notion of "C^α-regularity" which makes sense at a single point in a convex hypersurface. Morally, x is a C^α point if the convex hypersurface $\partial\Omega$ is majorized by the graph of $y \mapsto \|y\|^\alpha$ near x.

Definition 6.1. Let Ω be a properly convex domain, $x \in \partial\Omega$, and $\alpha > 1$. Fix an Euclidean distance d on an affine chart that contains $\overline{\Omega}$. We say that x is a C^α *point* if there is a neighborhood U of x and a constant $C > 0$ so that: for any supporting hyperplane H of Ω at x and any $y \in U \cap \partial\Omega$,

$$d(y, H) \leq C d(y, x)^\alpha. \tag{13}$$

Remark 6.2. This notion of a C^α point is independent of the choice of the distance d. Indeed, changing the affine chart or the distance is a bi-Lipschitz map in a neighborhood of x and does not impact the definition. We observe further that if the inequality (13) holds for some $\alpha > 1$, $\partial\Omega$ has a *unique* supporting hyperplane at x, i.e. x is a C^1 point.

One can alternatively define C^α points in $\partial\Omega$ in the following equivalent way. Suppose that in some affine chart, the hypersurface $\partial\Omega$ is the graph of a convex function $f : \mathbb{R}^{\dim(\partial\Omega)} \to \mathbb{R}$ such that $x = (0, f(0))$ and there exists a linear map $D_f(0) : \mathbb{R}^{\dim(\partial\Omega)} \to \mathbb{R}$ such that $\ker D_f(0)$ is a supporting hyperplane at x. We say that x is a C^α point if and only if the following limit exists:

$$\lim_{y \to 0} \frac{f(y) - f(0) - D_f(0)(y)}{||y||^\alpha}.$$

Dual to the notion of a C^α point is a β-*convex point*. Just as the C^α property strengthens the condition that there is a unique supporting hyperplane of Ω at x, β-convexity strengthens the condition that $x \in \partial\Omega$ is an extreme point of $\overline{\Omega}$.

Morally, x is a β-convex point if the convex hypersurface $\partial\Omega$ majorizes the graph of $y \mapsto \|y\|^\beta$ near x.

Definition 6.3. Let Ω be a properly convex domain, let $x \in \partial\Omega$, and let $\beta < \infty$. We say that x is a *β-convex point* if there is a neighborhood U of x and a constant $C > 0$ so that for any $y \in U \cap \partial\Omega$, we have

$$d(y, H) \geq C d(y, x)^\beta.$$

As for C^α regularity, we have an alternative characterization of β-convex points. If U is a neighborhood of 0 $\mathbb{R}^n$, and $f : U \to \mathbb{R}$ is a convex function, we say that f is *β-convex at* $0 \in U$ if there is a linear map $A_f : \mathbb{R}^n \to \mathbb{R}$ so that $f(y) - f(0) > A_f(y)$ for all $y \in U$, and the limit

$$\lim_{y \to 0} \frac{||y||^\beta}{f(y) - f(0) - A_f(y)}$$

exists. Then a point x in the boundary of a properly convex domain Ω is β-convex if, in coordinates on some (any) affine chart containing x, $\partial\Omega$ is locally the graph of a function $f : \mathbb{R}^{\dim(\partial\Omega)} \to \mathbb{R}$ such that $x = (0, f(0))$ and f is β-convex at 0.

Note that the linear map A_f defining β-convexity of the function f may not be uniquely determined—so in particular a non-C^1 point in $\partial\Omega$ can be a β-convex point. However, a β-convex point in $\partial\Omega$ is always an extreme point in $\overline{\Omega}$.

Example 6.4. Consider the graph of the function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x^2$ for $x \geq 0$ and $f(x) = -x$ otherwise. Set A_f to be the constant function 0. Then $f(x)$ is β-convex at 0 with $\beta = 2 + \varepsilon$ for any $\varepsilon > 0$.

Now consider a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^3)$, whose boundary in a neighborhood of a point $x \in \partial\Omega$ is projectively equivalent to the graph of f. Then x is a β-convex point of Ω that is not C^1.

We recall Definition 1.11 from the introduction.

Definition 1.11. Let Ω be a properly convex domain and $x \in \partial\Omega$ be a C^1 point. Set

$$\alpha(x, \Omega) := \sup\{\alpha > 1 : \partial\Omega \text{ is } C^\alpha \text{ at } x\}$$

and

$$\beta(x, \Omega) := \inf\{\beta < \infty : \partial\Omega \text{ is } \beta\text{-convex at } x\}.$$

If $\partial\Omega$ is not C^α at x for any $\alpha > 1$, we define $\alpha(x, \Omega) = 1$. Similarly if $\partial\Omega$ is not β-convex at x for any $\beta < \infty$, we define $\beta(x, \Omega) = \infty$.

6.2. **Boundary regularity and uniform regularity.** We will devote the rest of this section to the proof of Theorem 1.12 whose statement we recall below.

Theorem 1.12 (Section 6)**.** *Let Ω be a properly convex domain, let $\{\gamma_n\}$ track a projective geodesic ray $c : [0, \infty) \to \Omega$, and suppose that $c(\infty) = x$ is an exposed C^1 extreme point in $\partial\Omega$. Define*

$$\alpha_0 := \liminf_{n \to \infty} \frac{\mu_{1,d}(\gamma_n)}{\mu_{1,d-1}(\gamma_n)} \quad \text{and} \quad \beta_0 := \limsup_{n \to \infty} \frac{\mu_{1,d}(\gamma_n)}{\mu_{1,2}(\gamma_n)}.$$

Then $\alpha_0 = \alpha(x, \Omega)$ and $\beta_0 = \beta(x, \Omega)$.

In particular, $c(\infty)$ is a C^α point for some $\alpha > 1$ if and only if $\{\gamma_n\}$ is uniformly $(d-1)$-regular, and $c(\infty)$ is β-convex for $\beta < \infty$ if and only $\{\gamma_n\}$ is uniformly 1-*regular.*

The proof is largely an application of the estimates we proved in Section 4, together with a computation in appropriate coordinates (Lemma 6.10).

6.2.1. *Choosing coordinates.* For the rest of this section, we will fix the following general setup. Let Ω be a properly convex domain, let $c : [0, \infty) \to \Omega$ be a projective geodesic ray, and let $\{\gamma_n\}$ be a sequence in $\mathrm{Aut}(\Omega)$ tracking c. We will also denote by $c : (-\infty, \infty) \to \Omega$ the unique bi-infinite projective geodesic that extends the geodesic ray $c([0, \infty))$. Fix supporting hyperplanes $H_\pm$ of Ω at $c(\pm\infty)$ and set $H_0 := H_+ \cap H_-$.

We fix a coordinate system on the d-dimensional affine chart $A := \mathbb{P}(\mathbb{R}^d) \setminus H_-$, chosen so that $c(\infty)$ is the origin, H_+ is the codimension one "horizontal" coordinate plane, and $(c(\infty), c(-\infty))$ is the "vertical" ray based at the origin.

More formally, let $W_0, W_+, W_- \subset \mathbb{R}^d$ be the linear subspaces such that $\mathbb{P}(W_*) = H_*$ for $* \in \{\pm, 0\}$. Fix representatives $v_\pm \in \mathbb{R}^d$ for $c(\pm\infty)$ in $\mathbb{P}(\mathbb{R}^d)$, chosen so that the image of c is the projectivization of $\{tv_+ + sv_- : s, t > 0\}$. Consider the identification $\Psi : W_- \to A$ defined by

$$\Psi(v) = [v + v_+].$$

Note that Ψ is a diffeomorphism such that $\Psi(0) = c(\infty)$, $\Psi(\mathbb{R}_{>0}\, v_-) = c(\mathbb{R})$, $\Psi(W_0) = H_+ \cap A$. So the decomposition of $W_- = W_0 \oplus [v_-]$ into "horizontal" W_0 and "vertical" $[v_-]$ corresponds to making $A \cap H_+$ "horizontal" and $A \cap \mathrm{span}_{\mathbb{P}}\{c(\infty), c(-\infty)\}$ "vertical". Note that the map Ψ^{-1} identifies open neighborhoods U of $c(+\infty)$ in H_+ with open subsets of W_0 containing the origin.

The set $\Psi^{-1}(\partial\Omega \cap A)$ is a convex hypersurface in W_- passing through the origin in W_-, with tangent hyperplane W_0. So, we can make the following definition.

Definition 6.5. Let $f : W_0 \to \mathbb{R}$ be the function such that the image of the mapping $x \mapsto \Psi(x, f(x))$ is $\partial\Omega \cap A$.

Remark 6.6. As $\partial\Omega \cap A$ is a convex hypersurface, f is a convex function. The assumption that $c(\infty)$ is a C^1 point ensures that f is differentiable at 0. The assumption that $c(\infty)$ is an exposed extreme point ensures that f is uniquely minimized at 0.

Next, we define a function h whose level sets determine annular neighborhoods of $c(+\infty)$ in the hyperplane H_+.

Definition 6.7 (see Fig. 5)**.** For each point $z \in H_+ - \{c(\infty)\}$ which is sufficiently close to $c(\infty)$, let y_z be the unique point in $\partial\Omega$ such that

$$\mathrm{span}_{\mathbb{P}}\{y_z, c(-\infty)\} = \mathrm{span}_{\mathbb{P}}\{z, c(-\infty)\}.$$

Let H_{y_z} be the projective hyperplane spanned by y_z and $H_0 = H_+ \cap H_-$. Then $H_{y_z} \cap c(\mathbb{R})$ is a singleton set $\{c(t_z)\}$ for some $t_z \in \mathbb{R}$.

Let U be a neighborhood of the origin in W_0. We define a function $h : U - \{0\} \to \mathbb{R}$ as follows: for any $x \in U - \{0\}$, define

$$h(x) = t_{\Psi^{-1}(x)}.$$

Remark 6.8.

(1) The intersection $H_{y_z} \cap c(\mathbb{R})$ is always a singleton set for $z \in H_+ - \{c(\infty)\}$. Indeed, since $y_z \in \partial\Omega - \{c(-\infty)\}$, this can only possibly fail if H_{y_z} is a supporting hyperplane of Ω at y_x. But if this is the case, then the projective

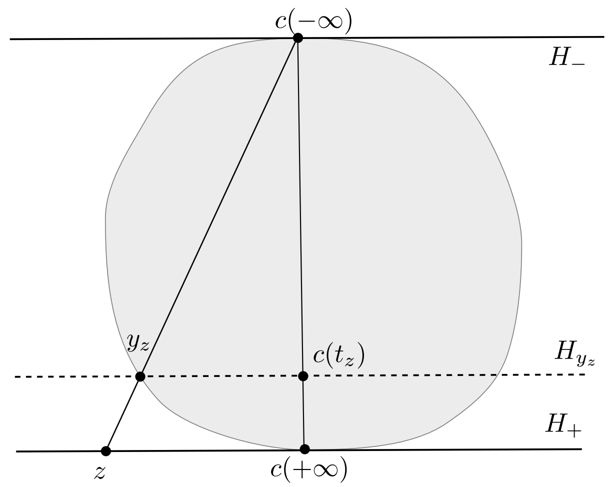

FIGURE 5. Illustration of the function $h(x) = t_{\Psi^{-1}(x)}$ in Definition 6.7. In this affine chart, the intersection $H_0 = H_+ \cap H_-$ is the point at infinity corresponding to the "horizontal direction."

segment $[y_z, c(+\infty)]$ lies in $\partial\Omega$. Since $c(+\infty)$ is extreme and exposed, this implies that $y_z = c(+\infty) = z$.

(2) We always have $h(U - \{0\}) = (a, \infty)$ for some $a \in \mathbb{R}$. So, by reparameterizing c, we can assume that the image of h is $(0, \infty)$. Further, as $x \in U - \{0\}$ tends towards 0, the function h tends to ∞.

(3) The definition of the function h does not require $c(\infty)$ to be a C^1 point – it makes sense whenever $c(\infty)$ is an extreme and exposed point in $\partial\Omega$.

Now we define annular neighborhoods of $c(\infty)$ using h.

Definition 6.9. Suppose U is a sufficiently small neighborhood of 0 in W_0 and h is as in Definition 6.7 above. We define a family $\{S_n\}_{n\in\mathbb{N}}$ of subsets of U by $S_n := h^{-1}([n-1, n])$. Note that $\cup_{n\in\mathbb{N}} S_n = U - \{0\}$.

6.2.2. *The key lemma.* The lemma below gives the key estimates we need for the proof of Theorem 1.12.

Lemma 6.10. *Suppose U is a sufficiently small neighborhood of 0. Then, there is a constant $B > 0$ satisfying the following: for any $n \in \mathbb{N}$ and any $x \in S_n$, we have*

$$-\mu_{1,d}(\gamma_n) - B \leq \log f(x) \leq -\mu_{1,d}(\gamma_n) + B, \tag{14}$$

$$-\mu_{1,d-1}(\gamma_n) - B \leq \log ||x|| \leq -\mu_{1,2}(\gamma_n) + B. \tag{15}$$

In addition, for any $n \in \mathbb{N}$, there are points $x_2(n), x_{d-1}(n)$ in S_n satisfying

$$\log ||x_{d-1}(n)|| \leq -\mu_{1,d-1}(\gamma_n) + B, \tag{16}$$

$$\log ||x_2(n)|| \geq -\mu_{1,2}(\gamma_n) - B. \tag{17}$$

Proof. Note that there are two disjoint properly convex cones in $\mathbb{R}^d$ that project to $\Omega \subset \mathbb{P}(\mathbb{R}^d)$, each of which is the negative of the other. We fix one of them, denoted by $\widetilde{\Omega}$, and call it the cone above Ω. For each $\gamma_n \in \mathrm{Aut}(\Omega)$, we fix a lift $\widetilde{\gamma_n}$ in $\mathrm{GL}(d, \mathbb{R})$ that preserves $\widetilde{\Omega}$. By definition $\mu_{i,j}(\gamma_n) = \mu_{i,j}(\widetilde{\gamma_n})$, so our estimates will not be affected by switching between γ_n and its lifts. So, by slight abuse of notation, we will henceforth denote the lifts by γ_n.

We can use Lemma 4.7 and Lemma 2.19 to find a fixed compact subset $Q \subset \mathrm{GL}(d,\mathbb{R})$ and a sequence $\{k_n\}$ in Q so that for every n, the group element $g_n := k_n\gamma_n^{-1}$ preserves the decomposition $c(\infty) \oplus \widetilde{H_0} \oplus c(-\infty)$. Then, we can apply Proposition 4.8 and Lemma 2.15 to see that there is a positive real number $D > 0$ so that for every n, we have

$$|\log(||k_n\gamma_n^{-1}|_{c(\infty)}||) - \mu_d(\gamma_n^{-1})| < D, \tag{18}$$

$$|\log(||k_n\gamma_n^{-1}|_{c(-\infty)}||) - \mu_1(\gamma_n^{-1})| < D, \tag{19}$$

$$|\log(||k_n\gamma_n^{-1}|_{\widetilde{H_0}}||) - \mu_2(\gamma_n^{-1})| < D, \tag{20}$$

$$|\log(\mathbf{m}(k_n\gamma_n^{-1}|_{\widetilde{H_0}})) - \mu_{d-1}(\gamma_n^{-1})| < D. \tag{21}$$

Let $\lambda_\pm(g_n)$ be the eigenvalues of g_n on $c(\pm\infty)$. Since each group element g_n preserves H_-, g_n acts by an affine map in our chosen affine chart $A = \mathbb{P}(\mathbb{R}^d) - H_-$. Via the identification $\Psi : W_- \to A$, the action of g_n on A (i.e. the map $\Psi^{-1} \circ g_n \circ \Psi$) is identified with the linear map $\phi(g_n) : W_- \to W_-$ given by

$$\phi(g_n)v = \frac{g_n v}{\lambda_+(g_n)}. \tag{22}$$

Now we analyze the linear map $\phi(g_n)$. With respect to the decomposition $W_- = W_0 \oplus [v_-]$, we can write $\phi(g_n)$ as

$$\phi(g_n)(x,y) = \left(\frac{g_n x}{\lambda_+(g_n)}, \frac{\lambda_-(g_n)}{\lambda_+(g_n)}y\right) \tag{23}$$

where $x \in W_0$ and $y \in [v_-]$.

Now, for each n, consider the intersection $g_n\Omega \cap A = k_n\Omega \cap A$. In coordinates given by Ψ, $\Omega \cap A$ is the graph of the function $f : W_0 \to \mathbb{R}$. Then, in the Ψ-coordinates, $g_n\Omega \cap A$ is the graph of the convex function $f_n : W_0 \to \mathbb{R}$ given by

$$f_n(v) = \frac{\lambda_-(g_n)}{\lambda_+(g_n)} f\left(\frac{g_n^{-1}v}{\lambda_+(g_n^{-1})}\right).$$

This holds because $\phi(g_n)(x, f(x)) = \left(\frac{g_n x}{\lambda_+(g_n)}, \frac{\lambda_-(g_n)}{\lambda_+(g_n)}f(x)\right)$ and $\lambda_+(g_n) = \frac{1}{\lambda_+(g_n^{-1})}$. Further, as the action of g_n preserves H_+, the graph of f_n is a convex hypersurface through the origin with a supporting hyperplane $\{(w,0) : w \in W_0\}$ at the origin.

Claim 6.10.1. *There exists a constant $0 < C < \infty$ such that: for any $n \in \mathbb{N}$ and any $x \in S_n$,*

$$1/C \leq \frac{\lambda_-(g_n)}{\lambda_+(g_n)} f(x) < C.$$

Proof of Claim. To prove this claim, fix $x \in S_n$ for some $n \in \mathbb{N}$. Letting h by the function from Definition 6.7, the point $c(h(x))$ has coordinates $(0, f(x))$ in the coordinates given by Ψ. Thus, applying the coordinate formula (23) for $\phi(g_n)$, we see that $g_n c(h(x))$ has coordinates

$$\left(0, \frac{\lambda_-(g_n)}{\lambda_+(g_n)} f(x)\right).$$

Since $\{\gamma_n\}$ tracks c, there is compact set $K \subset \Omega$ such that $\gamma_n^{-1}c(n) \in K$ for any $n \in \mathbb{N}$. Letting $K' \subset \Omega$ be the closed 1-neighborhood of K in the Hilbert metric d_Ω, we see that $\gamma_n^{-1}c(h(x)) \in K'$ since $c(h(x)) \in [n-1, n]$.

Then, as $g_n = k_n\gamma_n^{-1}$ for k_n in a compact subset $Q \subset \mathrm{GL}(d, \mathbb{R})$, we must have

$$g_n c(h(x)) \in \bigcup_{q\in Q} qK'.$$

Each k_n takes Ω to some domain in a compact family of domains that are all supported by the hyperplanes H_-, H_+. So, we may assume that Q is chosen such that the union $\bigcup_{q\in Q} qK'$ lies in the set $\mathbb{P}(\mathbb{R}^d) - (H_- \cup H_-)$. This means that $g_n c(h(x))$ lies in a fixed compact subset of $A - H_+ = \mathbb{P}(\mathbb{R}^d) - (H_+ \cup H_-)$ which does not depend on n. As H_+ is identified with the horizontal coordinate plane in our chosen coordinates on the affine chart A, this means that the vertical coordinate of $g_n c(h(x))$ is bounded above and below, establishing the claim. □

Now we explain how the above Claim 6.10.1 immediately implies the inequality (14). Note that $-\mu_{1,d}(\gamma_n) = \mu_{1,d}(\gamma_n^{-1})$. Then, applying inequalities (18) and (19) above, we see that there is a constant B (independent of n) so that

$$-\mu_{1,d}(\gamma_n) - B \leq \log f(x) \leq -\mu_{1,d}(\gamma_n) + B,$$

which is the inequality (14) we wanted to show.

The argument for the proof of the second inequality (15) is similar. We first claim the following:

Claim 6.10.2. *There exists a constant $0 < C' < \infty$ such that: for any $n \in \mathbb{N}$ and any $x \in S_n$,*

$$1/C' < ||\phi(g_n)x|| < C'. \tag{24}$$

Proof of Claim. Since each S_n is a compact subset of W_0 not containing the origin, we can prove this claim by showing that, for any sequence $x_n \in S_n$, no subsequence of $\phi(g_n)x_n$ tends towards zero or infinity.

Consider any such sequence $x_n \in S_n$. Let $\Omega_n := g_n\Omega$. We know that the point with coordinates $(x_n, f(x_n))$ lies on the hypersurface $\partial\Omega \cap A$, so the points

$$(\phi(g_n)x_n, f_n(x_n)) = \phi(g_n)(x_n, f(x_n))$$

lies on the convex hypersurface $\partial\Omega_n \cap A$. Here, we are using the notation of (23) so that $\phi(g_n)x_n = \frac{g_n x_n}{\lambda_+(g_n)}$ (as $x_n \in W_0$).

As each domain Ω_n lies in a fixed compact subset of the space of properly convex domains, we may extract a subsequence so that the domains Ω_n converge to a properly convex domain Ω_∞, which is supported by the hyperplanes $H_\pm$ at $c(\pm\infty)$. Thus, the hypersurfaces $\partial\Omega_n \cap A$ converge to the convex hypersurface $\partial\Omega_\infty \cap A$. The convex functions $f_n : W_0 \to \mathbb{R}$ then converge pointwise to a convex function $f_\infty : W_0 \to \mathbb{R}$, whose graph (in Ψ-coordinates on A) is the hypersurface $\partial\Omega_\infty \cap A$.

After extracting a further subsequence, we can assume that the points

$$(\phi(g_n)x_n, f_n(x_n))$$

converge to a point in $\partial\Omega_\infty$ (a priori, this limit may not lie in the affine chart A). However, the previous claim gives us $\frac{1}{C} \leq f_n(x_n) \leq C$, i.e. there exist uniform upper and lower bounds on the vertical coordinates $f_n(x_n)$ of these points. In particular, this implies that the limit of the sequence $\{(\phi(g_n)x_n, f_n(x_n))\}$ lies on the graph of f_∞, and that the limit of the sequence $\{\phi(g_n)x_n\}$ lies in the subset $f_\infty^{-1}([1/C, C])$. Since f_∞ is convex, $\{\phi(g_n)x_n\}$ lies in a uniformly bounded subset of $W_0 \setminus \{0\}$. This proves the claim. $\square$

We can now use the claim to show inequality (15). For any $x \in W_0$ and any $g \in \mathrm{GL}(W_0)$, by definition we have $||x|| \cdot \mathbf{m}(g) \leq ||gx|| \leq ||x|| \cdot ||g||$. Since $\phi(g_n)$ is a linear map preserving W_0, this yields

$$||x|| \cdot \mathbf{m}(\phi(g_n)|_{W_0}) \leq ||\phi(g_n)x|| \leq ||x|| \cdot ||\phi(g_n)|_{W_0}||, \tag{25}$$

for every n. Then by applying our formula (22) for $\phi(g_n)$ we obtain

$$||x|| \frac{\mathbf{m}(g_n|_{W_0})}{\lambda_+(g_n)} \leq ||\phi(g_n)x|| \leq ||x|| \frac{||g_n|_{W_0}||}{\lambda_+(g_n)}. \tag{26}$$

Now, the inequalities (20) and (21) tell us that $\log ||g_n|_{W_0}||$ and $\log \mathbf{m}(g_n|_{W_0})$ are within uniformly bounded additive error of $-\mu_{d-1}(\gamma_n)$ and $-\mu_2(\gamma_n)$ respectively, and (18) tells us that $\lambda_+(g_n)$ is within uniformly bounded additive error of $-\mu_1(\gamma_n)$. Putting this together with (24) and (26), we see that there is another uniform constant $B > 0$ so that

$$-\mu_{1,d-1}(\gamma_n) - B \leq \log ||x|| \leq -\mu_{1,2}(\gamma_n) + B.$$

This establishes inequality (15).

Now we prove the last two inequalities of Lemma 6.10. Observe that each S_n contains $h^{-1}(n)$, which is a level set of the convex function f. Since f is uniquely minimized at the origin (see Remark 6.6), this means that each S_n contains the boundary of a convex open ball in W_0, containing the origin. Then restrict the continuous function $\phi(g_n)$ to each S_n, consider (25), and recall the definition of $\mathbf{m}(\cdot)$ and $\|\cdot\|$. It is clear that for each n, we can find a pair of points $x_2 = x_2(n)$ and $x_{d-1} = x_{d-1}(n)$ in S_n so that when $x = x_2$ (resp. $x = x_{d-1}$), the left-hand (resp. right-hand) inequality in (25) is actually an equality. In particular, this implies that the corresponding inequalities in (26) are equalities when $x = x_2$ or x_{d-1}. Then, we again use the fact that $\log ||g_n|_{W_0}||$ and $\log \mathbf{m}(g_n|_{W_0})$ are within bounded error of $-\mu_{d-1}(\gamma_n)$ and $-\mu_2(\gamma_n)$ to establish (16) and (17). $\square$

We will now use Lemma 6.10 to finish the proof of Theorem 1.12.

6.3. **Proof of Theorem 1.12.** Let $\beta = \beta_0$ and $\alpha = \alpha_0$ where α_0, β_0 are as in the statement of Theorem 1.12. We will first prove that $\beta < \infty \implies \beta(x, \Omega) \leq \beta$ and then show that $\beta(x, \Omega) < \infty \implies \beta \leq \beta(x, \Omega)$. This proves that $\beta = \beta(x, \Omega)$ when either side is finite or infinite.

Assume first that $\beta < \infty$. For each $u \in U$, choose some n so that $u \in S_n$. We let $\beta_n := \frac{\mu_{1,d}(\gamma_n)}{\mu_{1,2}(\gamma_n)}$. We apply Lemma 6.10: putting the left-hand side of (14) together with the right-hand side of (15), we have

$$\log f(u) \geq -\mu_{1,d}(\gamma_n) - B = -\mu_{1,2}(\gamma_n)\beta_n - B \geq \beta_n(\log ||u|| - B) - B.$$

Hence there is a uniform constant $D > 0$ such that $f(u) \geq D^{-\beta_n} ||u||^{\beta_n}$. Now, fix some $\beta < \beta' < \infty$. Since $\limsup_{n\to\infty} \beta_n = \beta < \beta'$, we have $||u||^{\beta_n} \geq ||u||^{\beta'}$ for u sufficiently close to zero. Thus for some $C > 0$ we have $f(u) \geq C||u||^{\beta'}$ in a small neighborhood of the origin. Hence $\partial\Omega$ is β'-convex at x. Since $\beta' > \beta$ was arbitrary, $\beta(x, \Omega) \leq \beta$ by definition of $\beta(x, \Omega)$.

Conversely, suppose that $\beta(x, \Omega) < \infty$, and fix $\beta' > \beta(x, \Omega)$. Now, for each $n \in \mathbb{N}$, choose $u_n \in S_n$ so that the inequality (17) holds. We know that $\partial\Omega$ is

β'-convex at x, so there is some $C > 0$ so that $f(u_n) \geq C||u_n||^{\beta'}$. Then we combine the right-hand inequality in (14) with (17) to obtain

$$-\mu_{1,d}(\gamma_n) + B \geq -\beta'(\mu_{1,2}(\gamma_n) + B) + \log C.$$

Replacing $\mu_{1,d}(\gamma_n)$ by $\beta_n \cdot \mu_{1,2}(\gamma_n)$ and rearranging, we obtain

$$\beta' \geq \beta_n \left(\frac{\mu_{1,2}(\gamma_n)}{\mu_{1,2}(\gamma_n) + B} \right) + \frac{B + \log C}{\mu_{1,2}(\gamma_n) + B}.$$

Since x is an extreme point in Ω, Proposition 4.3 implies that $\mu_{1,2}(\gamma_n) \to \infty$. So the above implies that $\beta' \geq \limsup_{n\to\infty} \beta_n = \beta$. Since $\beta' > \beta(x, \Omega)$ was arbitrary, the definition of $\beta(x, \Omega)$ implies that $\beta(x, \Omega) \geq \beta$. This concludes the proof that $\beta = \beta(x, \Omega)$.

The proof of $\alpha = \alpha(x, \Omega)$ is completely symmetric, using the opposite inequalities in (14), (15), and (16), and applying the fact that x is a C^1 point to see that $\mu_{d-1,d}(\gamma_n)$ tends to infinity. □

7. Boundary regularity does not imply Morse

In this section, we construct a specific example realizing Theorem 1.9: we will find a projective geodesic ray c in a divisible domain Ω so that c is tracked by a sequence $\{\gamma_n\}$ that is uniformly regular, but not *strongly* uniformly regular. Thus c is not Morse in either the group-theoretic sense or the sense of Kapovich-Leeb-Porti. But, by Theorem 1.12, its endpoint $c(\infty)$ in $\partial\Omega$ is still C^α-regular and β-convex.

7.1. **Convex divisible domains in dimension 3.** The starting point for our construction is a convex divisible domain Ω in $\mathbb{P}(\mathbb{R}^4)$ which is irreducible (meaning it is not projectively equivalent to the cone over a 2-dimensional domain in $\mathbb{P}(\mathbb{R}^3)$), but not strictly convex. Domains of this type were studied and classified by Benoist [Ben06a]. Benoist proved that when Γ is a torsion-free discrete group dividing such a domain, the quotient manifold $M = \Omega/\Gamma$ can be cut along a nonempty collection of incompressible tori so that each connected component is homeomorphic to a (non-compact) finite-volume hyperbolic 3-manifold. This means that $\Gamma \simeq \pi_1 M$ is a relatively hyperbolic group, relative to the collection $\mathcal{P}$ of fundamental groups of cutting tori. Moreover, it turns out that the cutting tori in Ω/Γ lift to properly embedded 2-simplices in Ω whose stabilizers act by a group of simultaneously diagonalizable matrices in $\mathrm{PGL}(4, \mathbb{R})$. Thus, each connected component of the geometric decomposition of Ω/Γ has the structure of a convex projective manifold.

Benoist also provided explicit constructions for examples of these domains, using the theory of projective actions of Coxeter groups. Additional examples were later constructed by Ballas-Danciger-Lee [BDL18] and Blayac-Viaggi [BV23].

7.2. **Construction.** For the rest of the section, we let Ω be one of the convex divisible domains in $\mathbb{P}(\mathbb{R}^4)$ as above, and let $\Gamma \subseteq \mathrm{Aut}(\Omega)$ divide Ω. Note that Ω is a *rank one* domain [Isl], so the dividing group Γ contains infinitely many *rank one automorphisms* (see Section 1.4.2); these are precisely the automorphisms which do not preserve any projective geodesic lying in a properly embedded triangle in Ω. Fix such a rank one automorphism $\gamma \in \Gamma$, and let α be the closed projective geodesic in Ω/Γ representing γ. In addition, fix a cutting torus T in the geometric decomposition of M.

Let us first give an informal sketch of the idea behind Theorem 1.9. The cyclic subgroup $\langle\gamma\rangle \subset \Gamma$ gives a Morse geodesic in the group Γ tracking a lift of α, along which both $\mu_{1,2}$ and $\mu_{1,4}$ tend (uniformly) to infinity. On the other hand, we can find a geodesic β in T (not necessarily closed), corresponding to a sequence of group elements $a_n \in \pi_1 T \subset \Gamma$ along which $\mu_{1,2}$ stays bounded but $\mu_{1,4}$ goes to infinity. We will produce a projective geodesic ray c in M that successively follows α and β for increasingly longer times. As c spends arbitrarily long times close to the torus T, it fails to be Morse. However, c picks up enough singular value gaps by looping around α to ensure that the ratio $\mu_{1,2}/\mu_{1,4}$ stays bounded away from zero in the limit.

We now turn to the details. First, we note the following.

Lemma 7.1. *The group element $\gamma \in \Gamma$ representing the geodesic α satisfies the following (equivalent) conditions:*

(i) The mapping $\mathbb{Z} \to \Gamma$ given by $j \mapsto \gamma^j$ is a Morse quasi-geodesic.
(ii) γ is biproximal, i.e. it has unique eigenvalues with maximum and minimum modulus.
(iii) There is a positive constant B_0 such that $\mu_{1,2}(\gamma^j) \geq B_0 \cdot |j|$ and $\mu_{3,4}(\gamma^j) \geq B_0 \cdot |j|$ for any $j \in \mathbb{Z}$.

Proof. Proposition 1.21 implies that both (i) and (ii) are equivalent to the fact that γ is a rank one automorphism. The equivalence of (ii) and (iii) follows from the relationship between the Jordan projection $\ell : \mathrm{GL}(d, \mathbb{R}) \to \mathbb{R}^d_{\geq 0}$ and Cartan projection $\mu : \mathrm{GL}(d, \mathbb{R}) \to \mathbb{R}^d_{\geq 0}$: if $\ell_1(g) \geq \ell_2(g) \geq \ldots \geq \ell_d(g)$ are the logarithms of the moduli of the eigenvalues of $g \in \mathrm{GL}(d, \mathbb{R})$, then $\ell_i(g) = \lim_{n\to\infty} \mu_i(g^n)/n$ (see e.g. [GGKW17, Section 2.4]). □

Next, let $A_0 \simeq \mathbb{Z}^2$ be the subgroup of Γ identified with $\pi_1 T \subset \pi_1 M \simeq \Gamma$.

Lemma 7.2. *There is a finite-index subgroup $A \subseteq A_0$ so that the subgroup $\Gamma' \subset \Gamma$ generated by $\{\gamma\} \cup A$ is naturally isomorphic to the abstract free product $\langle\gamma\rangle * A$.*

Moreover, this subgroup is strongly quasi-convex *in the sense of* [Tra19]*: there exists a function $M : \mathbb{R}^2_{\geq 0} \to \mathbb{R}_{\geq 0}$ so that any (K_1, K_2)-quasi-geodesic in Γ with endpoints in Γ' lies in the $M(K_1, K_2)$-neighborhood of Γ'.*

Proof. The first part of the lemma follows from a combination theorem for relatively quasi-convex subgroups of relatively hyperbolic groups ([MP09, Theorem 1.1]). To apply the combination theorem, we need to check that the group $\langle\gamma\rangle$ is relatively quasi-convex in Γ, which follows from Lemma 7.1 (i). This combination theorem also implies that every parabolic subgroup in Γ' is a finite-index subgroup of some conjugate of A_0. Consequently, the second part of the lemma follows from the characterization of strongly quasi-convex subgroups in relatively hyperbolic groups given by [Tra19, Theorem 1.9]. □

The next step is to construct the geodesic β in the torus T we alluded to previously. We know that the finite-index subgroup $A \subseteq \pi_1 T$ is generated by a pair of commuting diagonalizable matrices. So, we can choose a basis for $\mathbb{R}^4$ and find linearly independent vectors $(x_i) \in \mathbb{R}^4$, $(y_i) \in \mathbb{R}^4$ so that (with respect to this basis) A can be written as the group

$$\left\{\mathrm{diag}(e^{ux_1+vy_1}, e^{ux_2+vy_2}, e^{ux_3+vy_3}, e^{ux_4+vy_4}) \in \mathrm{PGL}(4, \mathbb{R}) : u, v \in \mathbb{Z}\right\}. \tag{27}$$

Now fix a finite generating set S_A for A, and let $|\cdot|_{S_A}$ denote the word metric on A induced by a choice of finite generating set for A.

Lemma 7.3. *There exists a constant $C > 0$ and a sequence $\{a_n\}$ in A so that $|a_n|_{S_A} = n$ but $\mu_{1,2}(a_n) < C$.*

Proof. We can view A as a lattice in a subgroup $\hat{A} \subset \mathrm{PGL}(4, \mathbb{R})$ isomorphic to $\mathbb{R}^2$; the group $\hat{A}$ is defined exactly as in (27), except that the parameters u, v are allowed to vary in $\mathbb{R}$ instead of $\mathbb{Z}$. We can additionally lift $\hat{A}$ to an (isomorphic) subgroup of $\mathrm{SL}(4, \mathbb{R})$, so that every element of $\hat{A}$ has positive eigenvalues. After choosing an appropriate inner product on $\mathbb{R}^4$, we may assume that the eigenspaces of A (hence of $\hat{A}$) are mutually orthogonal. Then, since the eigenvalues of any $a \in \hat{A}$ are positive, the eigenvalues of a are precisely the singular values of a.

Let $e_1, \ldots, e_4$ be the eigenvectors of $\hat{A}$, and let $\lambda_i(a)$ denote the eigenvalue of a on e_i for $i = 1, 2, 3, 4$. For each i, the mapping w_i given by

$$w_i(a) = \log \lambda_i(a)$$

is an element of the dual $\hat{A}^* \simeq (\mathbb{R}^2)^*$. Since A is discrete with rank 2, these four dual vectors must span $\hat{A}^*$, so their convex hull is a polygon P whose interior contains the origin. Pick an edge of this polygon, with endpoints w_i, w_j. We can pick a vector $v \in \hat{A}^{**} = \hat{A}$ which is positive on the chosen edge, but vanishes on the line through the origin in $\hat{A}^*$ parallel to the edge. It follows that v achieves its maximum on P on both w_i and w_j, hence $\mu_1(v) = \mu_2(v) = w_i(v) = w_j(v)$ and thus $\mu_{1,2}(v) = 0$. The same is true for any positive real multiple of v. Then, since A is a lattice in $\hat{A}$, we can find length-n points $a_n \in A$ which are uniformly close to the line $\{rv : r \in \mathbb{R}_{>0}\}$, giving us the desired sequence. □

The sequence $\{a_n\}$ corresponds to our geodesic β in the torus T. Next, we define a sequence in Γ that we will use to determine the projective geodesic in Theorem 1.9:

Definition 7.4. Define a sequence of words $\{w_k\}_{k \in \mathbb{N}}$ as follows:

$$w_k := \begin{cases} \mathrm{id}, & \text{if } k = 0 \\ a_1\gamma \ldots a_m\gamma^m, & \text{if } k = 2m \\ a_1\gamma \ldots a_m\gamma^m a_{m+1}, & \text{if } k = 2m+1 \end{cases} \tag{28}$$

7.3. **Proof of Theorem 1.9.** Fix a finite generating set S_A for A, and extend $S_A \cup \{\gamma\}$ to a finite generating set S for Γ. Fix a basepoint $x_0 \in \Omega$ and let $F : \Gamma \to \Omega$ be the orbit map defined by $F(g) = gx_0$, so that F is a quasi-isometry with respect to the word metric d_S on Γ induced by S. We first prove that if $\{w_k\}$ is the sequence in Definition 7.4, then we can extend $\{w_k\}$ to a sequence that tracks a projective geodesic ray; this ray will be the ray appearing in Theorem 1.9.

Lemma 7.5. *There exists $R > 0$ and a projective geodesic ray $[x_0, \xi)$ such that for any $k \geq 0$, $\mathrm{d}_\Omega(w_k x_0, [x_0, \xi)) \leq R$.*

Proof. Fix any $1 \leq k < l$. We first claim that there exists $R > 0$, independent of k, l, such that $\mathrm{d}_\Omega(w_k x_0, [x_0, w_l x_0]) < R$. Before proving this claim, let us explain how this claim immediately implies the lemma. Choose a subsequence of $\{w_l x_0\}$ such that it converges to a point $\xi \in \partial\Omega$. As $[x_0, w_l x_0] \to [x_0, \xi)$ uniformly on compact subsets of Ω, $\mathrm{d}_\Omega(w_k x_0, [x_0, \xi)) \leq \limsup_{l \to \infty} \mathrm{d}_\Omega(w_k x_0, [x_0, w_l x_0])$. Supposing that the claim holds, it is immediate that $\mathrm{d}_\Omega(w_k x_0, [x_0, \xi)) \leq R$ for all k.

Now we prove the claim. Fix a quasi-inverse F^{-1} for the quasi-isometry F, and consider the quasi-geodesic $F^{-1}([x_0, w_l]) \subset \Gamma$; we will show that for some uniform R we have

$$d_S(w_k, F^{-1}([x_0, w_l x_0])) < R.$$

We may assume that the quasi-geodesic $F^{-1}([x_0, w_l x_0])$ joins id to w_l. So, by strong quasi-convexity of Γ', this quasi-geodesic is within uniformly bounded Hausdorff distance of some quasi-geodesic q in the Cayley graph $\mathrm{Cay}(\Gamma', S_A \cup \{\gamma\})$. We may assume that q is continuous (see e.g. [BH99, Lemma III.H.1.11]). However, observe that if $k < l$ then w_k separates $\mathrm{Cay}(\Gamma', S_A \cup \{\gamma\})$ into two components, one containing id and the other containing w_l. In particular, q passes through w_k, which completes the proof of the claim. □

Fix a sequence $\{\gamma_n\}$ tracking the ray $[x_0, \xi)$ from the previous lemma; we may assume that the sequence $\{w_k\}$ is a subsequence of $\{\gamma_n\}$. From now on, we make this assumption about $\{\gamma_n\}$. We will show that $\{\gamma_n\}$ is both uniformly 1-regular and uniformly 3-regular. The first step is the following:

Lemma 7.6. *There exist constants $\hat{C}, \hat{D} > 0$ such that, for $i \in \{1, 3\}$, we have*

$$\liminf_{n\to\infty} \frac{\mu_{i,i+1}(w_k)}{k^2} > \hat{C} \quad \textit{and} \quad \limsup_{k\to\infty} \frac{\mu_{1,4}(w_k)}{k^2} < \hat{D}.$$

Proof. For concreteness, take $i = 1$; the proof when $i = 3$ is essentially the same. We first claim that:

Claim 7.6.1. *There is a positive constant C_0 such that: for any $k \in \{2m, 2m+1\}$,*

$$\mu_{1,2}(w_k) - \mu_{1,2}(w_{k-2}) \geq \mu_{1,2}(\gamma^m) - 2C_0. \tag{29}$$

Proof of Claim. Since $\{w_k\}$ is a subsequence of a tracking sequence, we may prove the claim by applying Lemma 4.10. First, suppose that $k = 2m$. Then, by Lemma 4.10, there exists a constant C_0—independent of k—such that:

$$\begin{aligned}\mu_{1,2}(w_k) &\geq \mu_{1,2}(w_{k-1}) + \mu_{1,2}(\gamma^m) - C_0 \\ &\geq \mu_{1,2}(w_{k-2}) + \mu_{1,2}(a_m) + \mu_{1,2}(\gamma^m) - 2C_0.\end{aligned}$$

Since $\mu_{1,2}(a_m) \geq 0$, $\mu_{1,2}(w_k) - \mu_{1,2}(w_{k-2}) \geq \mu_{1,2}(\gamma^m) - 2C_0$. This proves the claim for $k = 2m$. The case $k = 2m + 1$ is similar. □

Using (29) above, we have

$$\mu_{1,2}(w_k) \geq \sum_{j=1}^{m} (\mu_{1,2}(\gamma^j) - 2C_0),$$

for any $k \in \{2m, 2m+1\}$. By Lemma 7.1, there is a positive constant B_0 such that $\mu_{1,2}(\gamma^j) \geq B_0 \cdot j$ for any $j \geq 1$. Then for any $k \in \{2m, 2m+1\}$,

$$\mu_{1,2}(w_k) \geq \sum_{j=1}^{m} (B_0 \cdot j - 2C_0) = \frac{B_0}{2} m(m+1) - 2C_0 m.$$

Since $2m \leq k \leq 2m+1$, $\frac{m^2}{k^2} \to \frac{1}{4}$ while $\frac{m}{k^2} \to 0$ as $k \to \infty$. Thus, there exists a constant $\hat{C} > 0$ such that

$$\liminf_{k\to\infty} \frac{\mu_{1,2}(w_k)}{k^2} > \hat{C}.$$

This finishes the proof of the first part.

To prove the estimate for $\mu_{1,4}$, observe that the triangle inequality implies that, if $k = 2m$, then

$$\mathrm{d}_\Omega(x_0, w_k x_0) \leq \sum_{j=1}^{m} (\mathrm{d}_\Omega(x_0, a_j x_0) + \mathrm{d}_\Omega(x_0, \gamma^j x_0)).$$

Since both groups $\langle\gamma\rangle$ and A are quasi-isometrically embedded in Γ, and the orbit map for Γ is a quasi-isometry, both terms appearing in the sum above are uniformly linear in j. So there is a uniform constant $D > 0$ so that the sum is at most Dk^2. Then the desired bound follows from Proposition 4.1. □

Using the above lemma, we can show:

Lemma 7.7. *Any sequence $\{\gamma_n\}$ which tracks the geodesic ray $[x_0, \xi)$ is both uniformly 1-regular and uniformly 3-regular.*

Proof. We know that each point $\{w_k x_0\}$ lies within uniformly bounded distance of $[x_0, \xi)$, and that $\mathrm{d}_\Omega(w_k x_0, w_{k+1} x_0) = O(k)$. So, as $\{w_k\}$ is an unbounded subsequence of $\{\gamma_n\}$, it follows that for each n there is some $k = k(n) \in \mathbb{N}$ so that

$$\mathrm{d}_\Omega(\gamma_n x_0, w_k x_0) = O(k).$$

Then by Proposition 4.1 we also have $\mu_{1,4}(\gamma_n^{-1} w_k) = O(k)$. So, by Lemma 2.15,

$$|\mu_{1,2}(\gamma_n) - \mu_{1,2}(w_k)| = O(k), \quad \text{and} \quad |\mu_{1,4}(\gamma_n) - \mu_{1,4}(w_k)| = O(k).$$

So, it follows from the previous lemma that, $\liminf_{n\to\infty} \frac{\mu_{1,2}(\gamma_n)}{\mu_{1,4}(\gamma_n)} \geq \frac{\hat{C}}{\hat{D}} > 0$, i.e. $\{\gamma_n\}$ is uniformly 1-regular. The proof for 3-regularity is similar. □

Corollary 7.8. *Let $[x_0, \xi)$ be the geodesic ray in Lemma 7.5 that $w_k x_0$ embeds along. Then ξ is C^α-regular and β-convex for some $\alpha > 1$ and $\beta < \infty$.*

Proof. Since Ω has exposed boundary (see [Ben06a]), Theorem 1.12 applies and the previous lemma implies the result. □

Lemma 7.9. *The sequence $\{\gamma_n\}$ is not strongly uniformly 1-regular.*

Proof. Observe that $\{w_k\}$ is a subsequence of $\{\gamma_n\}$ and since strong uniform regularity passes to subsequences, it suffices to prove the claim for $\{w_k\}$. Suppose $k = 2m+1$. Then $w_{k-1}^{-1} w_k = a_{m+1}$. Recall that $\mu_{1,2}(a_{m+1})$ is uniformly bounded while $\mu_{1,4}(a_{m+1}) \to \infty$ linearly in m. Then $\frac{\mu_{1,2}(w_{k-1}^{-1} w_k)}{\mu_{1,d}(w_{k-1}^{-1} w_k)} \to 0$. So $\{w_k\}$ is not strongly uniformly 1-regular. □

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