Limits of limit sets in rank-one symmetric spaces

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Abstract

We consider the question of continuity of limit sets for sequences of geometrically finite subgroups of isometry groups of rank-one symmetric spaces, and prove analogues of classical (Kleinian) theorems in this context. In particular we show that, assuming strong convergence of the sequence of subgroups, the limit sets vary continuously with respect to Hausdorff distance, and if the sequence is weakly type-preserving, the sequence of Cannon-Thurston maps also converges uniformly to a limiting Cannon-Thurston map. Our approach uses the theory of extended geometrically finite representations, developed recently by the second author.

1 Introduction

Let X be a noncompact rank-one symmetric space, for example real, complex, or quaternionic hyperbolic space. Consider a finitely generated group Γ, and let $\rho : \Gamma \to \mathrm{Isom}(X)$ be a discrete faithful representation. Its image $\rho(\Gamma)$ has a limit set, denoted Λ_ρ:

Definition 1.1 (Limit set, see e.g. [DSU17, Chapter 7])**.** The limit set Λ_G of a subgroup $G \subset \mathrm{Isom}(X)$ is the set of accumulation points in ∂X of any G-orbit in X. It is a compact subset of ∂X, and any G-invariant closed subset of $\mathrm{Isom}(X)$ with cardinality at least 2 contains Λ_G.

The limit set has classically received a great deal of attention in this setting, see e.g. [CI99]. When deforming ρ to a nearby representation ρ', the limit set will vary. The continuity of the assignment $\rho' \mapsto \Lambda_{\rho'}$ (with respect to the Hausdorff topology on compact subsets of ∂X) is not guaranteed, even in the classical setting where X is 3-dimensional real hyperbolic space $\mathbb{H}^3_{\mathbb{R}}$. A crucial observation is that the notion of convergence of representations

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with respect to the compact-open topology on maps $\Gamma \to \mathrm{Isom}(X)$ (usually called *algebraic convergence*) is not by itself enough to guarantee convergence of the limit sets; the convergence $\rho_n \to \rho$ should also be *geometric* (see Definition 4.3). The combination of algebraic and geometric convergence is called *strong* convergence.

When $X = \mathbb{H}^3_{\mathbb{R}}$, several different results relate the convergence of limit sets $\Lambda_{\rho_n} \to \Lambda$ to strong convergence $\rho_n \to \rho$; see e.g. [JM90, McM99, AC00, Eva04], or refer to [Mar16] for a survey. We prove the following result for general rank-one X, in the situation where the limiting representation ρ is *geometrically finite*:

Theorem 1.2 (Convergence of limit sets)**.** *Let Γ be a finitely generated group, let X be a rank-one symmetric space, and let $\rho : \Gamma \to \mathrm{Isom}(X)$ be a faithful geometrically finite representation. Let (ρ_n) be a sequence of faithful representations of Γ such that ρ_n converges strongly to ρ.*

Then, for all sufficiently large n, ρ_n is geometrically finite, and the limit sets Λ_{ρ_n} converge to Λ_ρ with respect to Hausdorff distance on ∂X.

Remark 1.3.

(a) When $X = \mathbb{H}^3_{\mathbb{R}}$, Theorem 1.2 follows from work of Jørgensen-Marden [JM90], and when $X = \mathbb{H}^n_{\mathbb{R}}$, the theorem is covered by a result of McMullen [McM99, Thm. 4.1]. In both of these results, the hypothesis of strong convergence $\rho_n \to \rho$ can be relaxed to one of *relative* strong convergence. We are also able to relax the hypothesis in this way; see Proposition 4.4.

(b) Again in the case $X = \mathbb{H}^3_{\mathbb{R}}$, Jørgensen-Marden proved a partial converse to Theorem 1.2, and in this case McMullen also proved results implying continuity of the Hausdorff dimension of the sequence of limit sets under certain circumstances. We will not pursue either of these directions in this paper.

(c) The hypothesis in Theorem 1.2 that each ρ_n is faithful is likely unnecessary. Indeed, McMullen's result for $\mathbb{H}^n_{\mathbb{R}}$ does not make this assumption (although his proof does assume that each $\rho_n(\Gamma)$ is torsion-free). We expect to explore this case further in future work.

1.1 Cannon-Thurston maps

In [MS13], Mj-Series proved another version of continuity for limit sets of sequences of geometrically finite representations in $\mathbb{H}^3_{\mathbb{R}}$, in terms of uniform continuity of the associated *Cannon-Thurston maps*. In general, when $\Gamma_1 \to \Gamma_2$ is an isomorphism of Kleinian groups, a *Cannon-Thurston map* is a continuous equivariant map $\Lambda_{\Gamma_1} \to \Lambda_{\Gamma_2}$ between the limit sets of Γ_1 and Γ_2.

If Γ_1 and Γ_2 are arbitrary Kleinian groups, then such a map is not guaranteed to exist. However, there is always a Cannon-Thurston map $\Lambda_{\Gamma_1} \to \Lambda_{\Gamma_2}$ if Γ_1, Γ_2 are geometrically finite and the isomorphism $\Gamma_1 \to \Gamma_2$ is *weakly type-preserving*, meaning every parabolic element in Γ_1 is taken to a parabolic element of Γ_2. One says that a sequence (ρ_n) of faithful geometrically finite representations $\Gamma \to \mathrm{Isom}(\mathbb{H}^3_{\mathbb{R}})$ is weakly type-preserving if each isomorphism $\rho_n \circ \rho_1^{-1}$ is weakly type-preserving (see also Definition 5.2); in this case there is a sequence of Cannon-Thurston maps $\Lambda_{\rho_1} \to \Lambda_{\rho_n}$ between the limit sets of these representations.

We prove the following uniform continuity result for Cannon-Thurston maps associated to geometrically finite representations:

Theorem 1.4 (Convergence of Cannon-Thurston maps)**.** *Let X be a rank-one symmetric space, let Γ be a finitely generated group, and let $(\rho_n)_{n\in\mathbf{N}}$ be a weakly type-preserving sequence of faithful geometrically finite representations, converging relatively strongly to a geometrically finite representation ρ.*

Then the sequence of Cannon-Thurston maps $\mathrm{CT}_{1,n} : \Lambda_{\rho_1} \to \Lambda_{\rho_n}$ exists and converges uniformly to a Cannon-Thurston map $\mathrm{CT}_{1,\infty} : \Lambda_{\rho_1} \to \Lambda_{\rho}$.

In the special case $X = \mathbb{H}^3_{\mathbb{R}}$, Theorem 1.4 exactly recovers the aforementioned result of Mj-Series (see [MS13, Thm. A]).

Remark 1.5. One consequence of our proof of Theorem 1.2 will be that, if $(\rho_n)_{n\in\mathbf{N}}$ is a sequence of faithful representations converging strongly to a geometrically finite representation ρ, then a subsequence of (ρ_n) is weakly type-preserving (see Proposition 5.3). Thus the context of Theorem 1.4 is no more restricted than that of Theorem 1.2.

1.2 Proof strategy

Our proof of Theorem 1.2 and Theorem 1.4 does not use conformal dynamics, but rather relies on the notion of *extended geometrically finite* (EGF) representations and *peripherally stable* deformations developed by the second author in [Wei22]. Interestingly, these notions were originally developed to deal with generalizations of geometrical finiteness in higher-rank semisimple Lie groups, but we show in this paper that the techniques also apply fruitfully in rank one. In particular, the approach gives an alternative proof of the classical (Kleinian) versions of our main theorems.

The organisation of the paper is as follows. In the following Section 2 we present the definition of EGF representations in the context of a rank-one symmetric space X, and explain the relation to the usual definition of geometrical finiteness. Even in the rank-one case, the two notions are not exactly identical, but they are close enough to be essentially equivalent for our concerns.

In Section 3, we discuss deformations of EGF representations, and present the notion of a peripherally stable subspace. Here we recall the main theorem of [Wei22] (see Theorem 3.2), which states that small deformations of EGF representations in peripherally stable subspaces are still EGF, and that their limit sets deform (semi-)continuously in these subspaces. In this section we also briefly explain the connection between peripheral stability and a *relative automaton*, an important technical tool developed in [Wei22] which is useful in later sections of this paper.

In Section 4, we give some reminders about the Chabauty topology, and then prove the equivalence between peripheral stability, strong convergence, and relative strong convergence (Proposition 4.4). This is the most technical section of the paper; once we have established this equivalence, we are able to give quick proofs of Theorem 1.2 and Theorem 1.4 in Section 5, as corollaries of Theorem 3.2.

2 Geometrical finiteness and extended geometrical finiteness

Let Γ be a finitely generated group and let $\rho : \Gamma \to \mathrm{Isom}(X)$ be a geometrically finite representation. In this situation, Γ is a relatively hyperbolic group (see [Bow12]), relative to its collection $\mathcal{H}$ of maximal parabolic subgroups, i.e. the set of stabilizers of points in ∂X fixed by a parabolic isometry in $\rho(\Gamma)$. We say that $(\Gamma, \mathcal{H})$ is a *relatively hyperbolic pair*. Recall that such a pair is equipped with its *Bowditch boundary* $\partial(\Gamma, \mathcal{H})$, which in this situation is equivariantly homeomorphic to the limit set of $\rho(\Gamma)$ (see the beginning of Section 2.2).

The second author developed the notion of extended geometrical finiteness (or EGF) in [Wei22] for representations of a general relatively hyperbolic group $(\Gamma, \mathcal{H})$ into a semisimple Lie group G. One goal of this section is to prove that, when G is a rank-one Lie group, and all of the peripheral subgroups $H \in \mathcal{H}$ of Γ are virtually nilpotent, then EGF representations of Γ are precisely the same thing as geometrically finite representations (see Proposition 2.11 below). We will also explain the connection between the equivariant homeomorphism $\partial(\Gamma, \mathcal{H}) \simeq \Lambda_\rho$ and the objects appearing in the definition of an EGF representation.

In general it is possible to construct EGF representations which are not geometrically finite (see Example 2.7). However, in the context of this paper, the peripheral subgroups of our relatively hyperbolic group Γ will always be virtually nilpotent, and so EGF representations and geometrically finite representations will be equivalent. The main reason we work with EGF representations is that they come equipped with some extra structure which turns out to be very useful when considering the behavior of deformations of geometrically finite groups.

2.1 EGF representations in rank-one symmetric spaces

Below, we give the definition of an EGF representation $\rho : \Gamma \to \mathrm{Isom}(X)$ when Γ is a relatively hyperbolic group and X is a rank-one symmetric space. The full definition of an EGF representation into an arbitrary-rank semisimple Lie group G is more complicated. That level of generality is not relevant for this paper, so we refer the interested reader to [Wei22] for details.

Definition 2.1. Let $(\Gamma, \mathcal{H})$ be a relatively hyperbolic pair, with Bowditch boundary $\partial(\Gamma, \mathcal{H})$. A representation $\rho : \Gamma \to \mathrm{Isom}(X)$ is *extended geometrically finite* if there is a closed ρ-invariant subset $\Lambda \subset \partial X$ and a ρ-equivariant map $\phi : \Lambda \to \partial(\Gamma, \mathcal{H})$ (called a *boundary extension*) satisfying the following condition:

For any sequence $\gamma_n \in \Gamma$ satisfying $\gamma_n^{\pm 1} \to z_\pm \in \partial(\Gamma, \mathcal{H})$, any compact subset $K \subset \partial X \setminus \phi^{-1}(z_-)$, and any open subset $U \subset \partial X$ containing $\phi^{-1}(z_+) \subset U$, we have $\rho(\gamma_n)K \subset U$ for all sufficiently large n.

Remark 2.2. The Bowditch boundary of a relatively hyperbolic group is really an invariant of a relatively hyperbolic *pair*, i.e. a relatively hyperbolic group together with a choice of a collection of peripheral subgroups (which is not always uniquely determined). So the definition of an EGF representation depends on the choice of collection $\mathcal{H}$. This is important in this paper, since sometimes we will want to consider different peripheral structures on the same group.

Definition 2.1 is our primary definition of an EGF representation in this paper because it is fairly close to the original definition given in [Wei22]. However, in our current setting, there is an alternative formulation of the definition given in terms of convergence group actions. Recall that, if M is a Hausdorff space, an action $\Gamma \to \mathrm{Homeo}(M)$ is a *discrete convergence action* if, for every sequence of pairwise distinct elements $\gamma_n \in \Gamma$, one can extract a subsequence and find points $a, b \in M$ so that the restrictions $\gamma_n|_{M \setminus \{a\}}$ converge to the constant map b, uniformly on compacts in $M \setminus \{a\}$.

Proposition 2.3. *Suppose that $(\Gamma, \mathcal{H})$ is a relatively hyperbolic pair. A representation $\rho : \Gamma \to \mathrm{Isom}(X)$ is EGF if and only if there is a closed invariant set $\Lambda \subset \partial X$ and a ρ-equivariant map $\phi : \Lambda \to \partial(\Gamma, \mathcal{H})$ such that the induced action on the space $\partial X / \sim_\phi$ is a discrete convergence action, where $\sim_\phi$ is the equivalence relation identifying points in the same fiber of ϕ.*

This proposition is a straightforward consequence of the fact that any discrete subgroup of $\mathrm{Isom}(X)$ acts as a discrete convergence group on ∂X (see [Tuk94]).

2.2 EGF representations and geometrically finite representations

In sections 6 and 9 of [Bow12], Bowditch uses the Beardon-Maskit definition of geometrical finiteness (see [BM74], [Bow95]) to prove the following facts about any geometrically finite representation of Γ:

- If $\mathcal{H}$ is the collection of maximal subgroups of Γ sent by ρ to parabolic subgroups of $\mathrm{Isom}(X)$, then $(\Gamma, \mathcal{H})$ is a relatively hyperbolic pair.
- The limit set of $\rho(\Gamma)$ is canonically homeomorphic to the Bowditch boundary $\partial(\Gamma, \mathcal{H})$ of the pair.

Proposition 2.3 makes it clear that any geometrically finite representation is also an EGF representation for this relatively hyperbolic structure: the closed invariant set Λ can be taken to be the limit set of the group $\rho(\Gamma)$, and $\phi : \Lambda \to \partial(\Gamma, \mathcal{H})$ is given by the canonical homeomorphism. Since there is a preferred choice of peripheral structure in this case, we make the following definition:

Definition 2.4. For any representation $\rho : \Gamma \to \mathrm{Isom}(X)$ of a finitely generated group, the collection of the *ρ-parabolic subgroups* is the collection of maximal subgroups of Γ sent by ρ to parabolic subgroups of $\mathrm{Isom}(X)$.

With this terminology, if ρ is a geometrically finite representation, then it is also EGF with respect to the ρ-parabolic subgroups, with a homeomorphic boundary extension. In fact, the converse also holds:

Theorem 2.5 (See [Wei22, Theorem 1.10])**.** *Let $\rho : \Gamma \to \mathrm{Isom}(X)$ be a representation of a finitely generated group, and let $\mathcal{H}$ the collection of ρ-parabolic subgroups. Then ρ is geometrically finite if and only if $(\Gamma, \mathcal{H})$ is a relatively hyperbolic pair, and ρ is an EGF representation with an injective boundary extension $\phi : \Lambda \to \partial(\Gamma, \mathcal{H})$. In this case, the set Λ is the limit set of $\rho(\Gamma)$.*

On the other hand, it is in general possible (even in the rank-one setting) to construct EGF representations whose boundary extensions are *not* injective. This can occur for two different reasons, which we cover below.

The first reason that injectivity may fail is essentially an artifact of the definition: the closed invariant set Λ is *not* uniquely determined by the conditions given in Definition 2.1. If $\rho(\Gamma)$ is non-elementary, then Λ must always contain the limit set of $\rho(\Gamma)$, but it is sometimes also possible to find a larger set which still satisfies the definition.

Example 2.6. Consider a geometrically finite Fuchsian group $\Gamma \subset \mathrm{SL}(2, \mathbb{R}) \subset \mathrm{SL}(2, \mathbb{C})$ which is not convex cocompact and hence has parabolic elements; $\mathrm{SL}(2, \mathbb{Z})$ does the trick. Pick an invariant system of disjoint horoballs in $\mathbb{H}^2$

centered at the parabolic points. Now, identify $\mathbb{H}^2 \subset \partial\mathbb{H}^3 \simeq \mathbb{CP}^1$ with the upper half plane in $\mathbb{C}$. Let Λ be the union of $\mathbb{RP}^1$ and the horoballs, and let $\phi : \Lambda \to \mathbb{RP}^1$ be the map which sends horoballs to their center and non-cusp points to themselves. One may then check that ϕ is a boundary extension, but Λ is not the limit set of Γ (which is $\mathbb{RP}^1$).

Fortunately, we can safely brush this particular issue aside in the rank-one setting: it turns out that it always possible to choose the invariant set Λ in Definition 2.1 to be the limit set of $\rho(\Gamma)$ (see Proposition 2.8 below).

Even with this assumption, though, it is still possible that the boundary extension ϕ could fail to be injective. This is a feature, rather than a bug, in the definition of an EGF representation: it allows us to take into account the fact that the chosen collection $\mathcal{H}$ of peripheral subgroups in Γ may *not* be precisely the same as the collection of ρ-parabolic subgroups. This is especially relevant when we consider deformations of geometrically finite representations which do *not* preserve the natural choice of peripheral subgroups, and we want to define Cannon-Thurston maps between the limit sets of these representations (see Section 5.2).

In some cases, however, even the "most natural" choice of peripheral subgroups gives rise to a boundary extension for an EGF representation which is still not injective. This will occur precisely when the image of the EGF representation is a geometrically infinite group.

Example 2.7. Consider a finitely generated geometrically infinite discrete group $\Gamma \subset \mathrm{PO}(3,1)$. By including $\mathrm{PO}(3,1)$ into $\mathrm{PO}(4,1)$, we may view Γ as a geometrically infinite discrete group preserving an isometrically embedded $\mathbb{H}^3$ in $\mathbb{H}^4$; the limit set of Γ in $\partial\mathbb{H}^4$ is contained in the embedded 2-sphere at the boundary of this $\mathbb{H}^3$. Then let Γ' be a conjugate of Γ in $\mathrm{PO}(4,1)$ by some isometry taking this 2-sphere completely off of itself.

Using Klein-Maskit "ping-pong" combination theorems (see e.g. [Mas88]), one can show that, possibly after replacing Γ and Γ' with finite-index subgroups, the subgroup $\langle \Gamma, \Gamma' \rangle \subset \mathrm{PO}(4,1)$ is naturally isomorphic to the abstract free product $\Gamma * \Gamma'$. The free product is a relatively hyperbolic group, relative to the collection of conjugates of Γ, Γ', and one may use the same ping-pong techniques to prove that the limit set of $\langle \Gamma, \Gamma' \rangle$ surjects equivariantly onto the Bowditch boundary of the free product, so that the preimage of each parabolic point is the limit set of some conjugate of Γ or Γ'. This gives the boundary extension for an EGF representation with geometrically infinite image.

Since the example above is geometrically infinite, Theorem 2.5 says that there is no choice of injective boundary extension for this representation. However, even in this case, the boundary extension is well-behaved in the sense that all the non-injectivity occurs at the peripheral subgroups. This is true in general, due to the following consequence of [Wei22, Proposition 4.8]:

Proposition 2.8. *Let* $(\Gamma, \mathcal{H})$ *be a relatively hyperbolic pair, let* $\rho : \Gamma \to \mathrm{Isom}(X)$ *be a non-elementary discrete representation, and let* Λ_ρ *be the limit set of* $\rho(\Gamma)$*.*

If ρ *is an EGF representation, then there is a unique boundary extension* $\phi : \Lambda_\rho \to \partial(\Gamma, \mathcal{H})$*. Moreover, for this boundary extension:*

1. *If* $z \in \partial(\Gamma, \mathcal{H})$ *is a conical limit point, then the fiber* $\phi^{-1}(z)$ *is a singleton;*

2. *If* $z \in \partial(\Gamma, \mathcal{H})$ *is a parabolic point, then* $\phi^{-1}(z)$ *is the limit set of* $\rho(\mathrm{Stab}_\Gamma(z))$*.*

Proof. Since $\rho(\Gamma)$ is non-elementary, any closed invariant subset of ∂X contains Λ_ρ, so in particular $\Lambda_\rho \subset \Lambda$ for any subset Λ as in Definition 2.1.

Proposition 4.8 in [Wei22] asserts that one can choose Λ and the boundary extension $\phi : \Lambda \to \partial(\Gamma, \mathcal{H})$ so that, for every conical limit point $z \in \partial(\Gamma, \mathcal{H})$, $\phi^{-1}(z)$ is a singleton. This implies that each such $\phi^{-1}(z)$ is in Λ_ρ: we can always find a sequence $\gamma_n \in \Gamma$ so that $\gamma_n \to z$ in the compactification $\Gamma \cup \partial(\Gamma, \mathcal{H})$. Then, Definition 2.1 implies that for some nonempty open subset $U \subset \partial X$, the sequence $\rho(\gamma_n)\overline{U}$ converges to the singleton $\phi^{-1}(z)$ and so this singleton must lie in Λ_ρ.

Proposition 4.8 in [Wei22] also asserts that the boundary extension above can be chosen so that, for each parabolic point $p \in \partial(\Gamma, \mathcal{H})$, the fiber $\phi^{-1}(p)$ consists of accumulation points of sequences of a particular form. Precisely, the proposition states that there is an open subset $C_p \subset \partial X$ so that $\phi^{-1}(p)$ is exactly the closure of the set of accumulation points of sequences $\rho(\gamma_n)x$, for $x \in C_p$ and γ_n a sequence of pairwise distinct elements in $\mathrm{Stab}_\Gamma(p)$. Now, any such accumulation point must lie in the limit set of $\rho(\mathrm{Stab}_\Gamma(p))$.

Every point in $\partial(\Gamma, \mathcal{H})$ is either a conical limit point or a parabolic point. So, the two cases above show that the fiber above every point in $\partial(\Gamma, \mathcal{H})$ is contained in Λ_ρ, and that this fiber is completely determined by the representation ρ and satisfies the conditions in the statement of the proposition. □

Convention 2.9. Since the boundary extension determined by Proposition 2.8 is unique, for the rest of this paper, we will always refer to "the" boundary extension for an EGF representation when we mean the extension determined by the proposition.

Remark 2.10. In light of Proposition 2.8, one may well ask why the definition of an EGF representation allows for different boundary extensions for the same representation of the same group with the same peripheral structure. The answer is that the uniqueness property in Proposition 2.8 is more subtle in the higher-rank setting, which makes it harder to determine a "best" choice for the set Λ in the definition.

2.3 EGF representations with nilpotent peripheral subgroups

If a representation $\rho : \Gamma \to \mathrm{Isom}(X)$ is geometrically finite, then every parabolic subgroup of Γ maps to a discrete subgroup of $\mathrm{Isom}(X)$ fixing a unique point in ∂X, and is thus virtually nilpotent.

Proposition 2.11. *Let $(\Gamma, \mathcal{H})$ be a relatively hyperbolic pair, such that each $H \in \mathcal{H}$ is virtually nilpotent. For any representation $\rho : \Gamma \to \mathrm{Isom}(X)$, the following are equivalent:*

(i) The representation $\rho : \Gamma \to \mathrm{Isom}(X)$ is geometrically finite, and $\mathcal{H}$ contains the collection of ρ-parabolic subgroups of Γ.

(ii) The representation ρ is EGF with respect to $\mathcal{H}$.

Proof. First we prove (i) $\implies$ (ii). Theorem 2.5 implies that a geometrically finite representation is always extended geometrically finite with respect to its collection $\mathcal{H}'$ of parabolic subgroups. Then, as $\mathcal{H}' \subseteq \mathcal{H}$, we may apply a special case of a relativization theorem of Wang [Wan23, Thm. 1.8] to see that ρ is also EGF with respect to $\mathcal{H}$.

Next, we prove (ii) $\implies$ (i). Suppose that $\rho : \Gamma \to \mathrm{Isom}(X)$ is EGF with respect to $\mathcal{H}$, with boundary extension $\phi : \Lambda \to \partial(\Gamma, \mathcal{H})$. The image of each peripheral subgroup $H \in \mathcal{H}$ is a virtually nilpotent discrete subgroup in $\mathrm{Isom}(X)$, which means that its limit set $\Lambda(\rho(H))$ consists of either zero, one, or two points. If the limit set contains zero points then H is finite, which is disallowed by the definition of an EGF representation (our convention is that all peripheral subgroups of a relatively hyperbolic group are infinite). So, we know that each $\Lambda(\rho(H))$ contains either one or two points.

If the limit set of H contains two points, then H is virtually cyclic and $\rho(H)$ is quasi-isometrically embedded, i.e. convex cocompact. Put another way, the restriction of ρ to H is geometrically finite with respect to an *empty* collection of cusp subgroups of H, so we can apply the other direction of the relativization theorem cited above (see also [Wei22, Thm 1.15]) to see that ρ is also an EGF representation with respect to the peripheral structure

$$\mathcal{H}' := \{H \in \mathcal{H} : |\Lambda(\rho(H))| = 1\}.$$

Proposition 2.8 then implies that the boundary extension with respect to this peripheral structure is injective. Thus, by Theorem 2.5, ρ is geometrically finite, and the collection $\mathcal{H}' \subseteq \mathcal{H}$ is precisely the collection of parabolic subgroups of Γ. □

We have now finished our review of most important properties of EGF representations and their relation to geometrical finiteness. However, to prove some of the results in this paper, we need to work with an additional technical tool: a *relative automaton*.

2.4 The relative automaton for an (extended) geometrically finite representation

The automaton discussed in this section is constructed (in a more general setting) in [Wei22], building upon ideas and results going back to the computational approach to hyperbolic groups. Below we will state several results regarding the construction. All of these results also apply in the more general context, but we will just state versions appropriate for the present setting.

Fix a finitely generated group Γ, let $\rho : \Gamma \to \mathrm{Isom}(X)$ be a geometrically finite representation and consider $\mathcal{H}$ the collection of ρ-parabolic subgroups. Let Λ_ρ denote the limit set of $\rho(\Gamma)$. There are finitely many orbits of parabolic points in Λ_ρ, so we fix once and for all a finite subset $\Pi \subset \Lambda_\rho$, containing exactly one point from each of these parabolic orbits. We also fix a metrization of ∂X; this can be taken to be a visual metric, but the precise choice does not matter.

Definition 2.12. A *relative automaton* associated to ρ consists of the following data:

- a finite directed graph $\mathcal{G}$, whose vertex set Z is a subset of the limit set of Γ;
- a pair of mappings $z \mapsto W(z)$ and $z \mapsto L(z)$ defined on Z, where $W(z)$ is an open subset of ∂X and $L(z)$ (the *label set*) is a subset of Γ.

The main result of Section 6 of [Wei22] tells us that, for the geometrically finite representation ρ, it is always possible to construct a relative automaton for ρ which satisfies all of the following properties:

(A1) The closure of each subset $W(z)$ is a proper subset of ∂X.

(A2) There is a fixed $\varepsilon > 0$ so that, for each directed edge $z \to y$ in $\mathcal{G}$ and each $\alpha \in L(z)$, we have an inclusion

$$\rho(\alpha)\overline{N_\varepsilon(W(y))} \subset W(z).$$

(A3) If $z \in Z$ is a parabolic point (so that $z = \rho(g)p$ for some $p \in \Pi$ and $g \in \Gamma$), then $L(z)$ is a subset of the coset $g\,\mathrm{Stab}_\Gamma(p)$.

(A4) If $z \in Z$ is *not* a parabolic point, then the label set $L(z)$ is a singleton.

(A5) For every edge $z \to y$ in $\mathcal{G}$, if z is a parabolic point, equal to $\rho(g)p$ for $p \in \Pi$, then $W(z)$ contains z, and $\overline{W(y)}$ does not contain p.

(A6) There is a uniform constant $R > 0$ so that, for each element $\gamma \in \Gamma$, we can find a directed vertex path $z_1 \to \ldots \to z_{n+1}$ in $\mathcal{G}$ and elements $\alpha_i \in L(z_i)$ so that the product

$$\alpha_1 \cdots \alpha_n$$

lies within distance R of γ. Here, Γ is equipped with the word metric induced by some fixed choice of finite generating set.

Remark 2.13. The reader may also wish to refer to [MMW24, Section 3] for a somewhat simpler version of the construction in [Wei22] in a slightly different context.

The relative automaton above contains all the information needed to reconstruct the limit set of ρ. The use of automata to encode limit sets of Kleinian groups has a long history, tracing back to Sullivan's original "symbolic coding" argument for structural stability of convex cocompact groups [Sul85]. The same idea has also been used to compute visualizations of limit sets of Kleinian groups; see e.g. [MPR94]. The *relative* automaton we consider here is very convenient to work with when deforming ρ in the space of EGF representations. We will use it in the next section, where we deal with properties of this deformation space.

3 Deformations of EGF representations and peripheral stability

The previous section dealt with the connection between the concepts of extended geometrical finiteness and geometrical finiteness for a single representation ρ. Now we want to understand families of such representations, so we will review a key *relative stability* property of EGF representations. Roughly, this property says that if ρ' is a small deformation of an EGF representation $\rho : \Gamma \to \mathrm{Isom}(X)$, and the restriction of ρ' to peripheral subgroups satisfies a certain condition, then ρ' is also EGF. This technical condition is called *peripheral stability.*

As in the previous section, we will define a version of peripheral stability which makes sense for a geometrically finite representation $\rho : \Gamma \to \mathrm{Isom}(X)$ when X is a rank one symmetric space. This will be simpler than the full definition when ρ is an EGF representation into a general semisimple Lie group; the definitions are equivalent in the present context.

Note that, although we consider several notions of convergence for representations throughout this text, we will always understand the space $\mathrm{Hom}(\Gamma, \mathrm{Isom}(X))$ to be equipped with the algebraic (or compact-open) topology.

3.1 Peripheral stability

Let Γ be a finitely generated group and let $\rho : \Gamma \to \mathrm{Isom}(X)$ be a faithful and geometrically finite representation. Let $\mathcal{P} \subset \partial X$ be the collection of all cusp points for ρ, meaning that the collection $\mathcal{H}$ of ρ-parabolic subgroups is precisely the set $\{\mathrm{Stab}_\Gamma(p), p \in \mathcal{P}\}$.

Definition 3.1. A subset $O \subseteq \mathrm{Hom}(\Gamma, \mathrm{Isom}(X))$ is *peripherally stable* about ρ if the following holds:

Let $p \in \mathcal{P}$, let U be a neighborhood of p in ∂X, let $K \subset \partial X$ be a compact subset of $\partial X \setminus \{p\}$, and let F be a finite subset of $\mathrm{Stab}_\Gamma(p)$ such that

$$\rho(\mathrm{Stab}_\Gamma(p) \setminus F)K \subset U.$$

Then there is a relatively open subset O' of O (depending on U, F, and K) such that for all $\rho' \in O'$, we have

$$\rho'(\mathrm{Stab}_\Gamma(p) \setminus F)K \subset U.$$

In the current context, the relative stability property for EGF representations can be stated as follows. For the result below, fix an arbitrary metrization of ∂X.

Theorem 3.2 (See [Wei22, Theorem 1.4])**.** *Let $\rho : \Gamma \to \mathrm{Isom}(X)$ be a geometrically finite representation, let $\mathcal{H}$ be the associated collection of ρ-parabolic subgroups, and let $\phi : \Lambda \to \partial(\Gamma, \mathcal{H})$ be the associated boundary extension. Let $O \subseteq \mathrm{Hom}(\Gamma, \mathrm{Isom}(X))$ be a peripherally stable subspace about ρ.*

Then, for any $\varepsilon > 0$ and any compact subset $Z \subset \partial(\Gamma, \mathcal{H})$, there is a relatively open subset $O' \subset O$ satisfying the following: if $\rho' \in O'$, then ρ' is also an EGF representation with boundary extension $\phi' : \Lambda' \to \partial(\Gamma, \mathcal{H})$ satisfying

$$(\phi')^{-1}(Z) \text{ lies inside an } \epsilon\text{-neighborhood of } \phi^{-1}(Z).$$

Moreover, the set Λ' can be taken to be the limit set of $\rho'(\Gamma)$, so that ϕ' is the unique EGF boundary extension described by Proposition 2.8.

Remark 3.3. The "moreover" part of Theorem 3.2 is not stated explicitly as part of the cited result in [Wei22]. However, this statement follows directly from the proof in that paper. Indeed, in [Wei22], the construction of the unique limit set described in Proposition 2.8 is carried out by applying the relative stability theorem to the (trivially) peripherally stable subspace $\{\rho\} \subset \mathrm{Hom}(\Gamma, \mathrm{Isom}(X))$; see [Wei22, Remark 9.18].

3.2 Peripheral stability and relative automata

The next lemma translates the peripheral stability condition into a stability property for the relative automaton discussed previously in Section 2.4. This is another way to motivate the definition of peripheral stability, since the automaton is a key tool used both for the proof of Theorem 3.2 in [Wei22] and for some results later in this paper.

Lemma 3.4. *Let $O \subseteq \mathrm{Hom}(\Gamma, \mathrm{Isom}(X))$ be a peripherally stable subspace about ρ, and let $\mathcal{G}$ be a relative automaton satisfying the properties listed above.*

There is an open neighborhood $O' \subseteq O$ of ρ and a constant $\varepsilon' > 0$ such that, for every $\rho' \in O'$, every directed edge $z \to y$ in $\mathcal{G}$, and every $\alpha \in L(z)$, we have

$$\rho'(\alpha)\overline{N_{\varepsilon'}(W(y))} \subset W(z).$$

Proof. Property (A2) says that the desired inclusions are all satisfied when $\rho' = \rho$. So we just need to check that the desired condition is relatively open in O for each edge in $\mathcal{G}$, since there are finitely many such. First, if z is not a parabolic point in Λ_ρ, then $L(z)$ is a singleton by (A4) and so the condition is already open in $\mathrm{Hom}(\Gamma, \mathrm{Isom}(X))$.

On the other hand, if z is a parabolic point, then by property (A3), each $\alpha \in L(z)$ can be written $\alpha = g_z\alpha'$, with $\alpha' \in \mathrm{Stab}_\Gamma(p)$ for $p \in \Pi$ and $g_z \in \Gamma$ depending only on z. (Here $\mathrm{Stab}_\Gamma(p)$ is the stabilizer with respect to the ρ action.)

By property (A5), the set $W(z)$ contains z, which means that $\rho(g_z^{-1})W(z)$ contains p. By property (A5), we know that $\overline{W(y)}$ does not contain p; this means that there is also some $\varepsilon' > 0$ so that $\overline{N_{\varepsilon'}(W(y))}$ does not contain p. Since there are only finitely many edges $z \to y$, this ε' can be chosen independently of z; we can also assume that ε' is smaller than the constant ε from condition (A2).

As $\mathrm{Stab}_\Gamma(p)$ acts properly discontinuously on $\partial X \setminus \{p\}$, for all but finitely many $\alpha' \in \mathrm{Stab}_\Gamma(p)$ we have $\rho(\alpha')\overline{N_{\varepsilon'}(W(y))} \subset \rho(g_z^{-1})W(z)$. The peripheral stability assumption then implies that there is an open neighborhood O' of ρ in O, so that for all but finitely many $\alpha' \in \mathrm{Stab}_\Gamma(p)$, every $\rho' \in O'$ satisfies

$$\rho'(\alpha')\overline{N_{\varepsilon'}(W(y))} \subset \rho'(g_z^{-1})W(z).$$

This means that for all but finitely many exceptional $\alpha \in L(z)$, every $\rho' \in O'$ satisfies

$$\rho'(\alpha)\overline{N_{\varepsilon'}(W(y))} \subset W(z). \tag{1}$$

However we also know that $\rho(\alpha)\overline{N_{\varepsilon'}(W(y))} \subset W(z)$ for every $\alpha \in L(z)$, so by further shrinking O' we can also ensure that for every $\rho' \in O'$, (1) holds for the finitely many exceptional α as well. ☐

4 Strong convergence and peripheral stability

As noted in the introduction, in the classical (Kleinian) context, algebraic convergence of a sequence of representations $\rho_n : \Gamma \to \mathrm{Isom}(X)$ does not guarantee that the sequence of limit sets Λ_{ρ_n} converges to the limit set of the limiting representation. Typically one must also assume that the sequence (ρ_n) also converges *strongly*, meaning that the sequence of subgroups $\rho_n(\Gamma)$

converges in the Chabauty topology on the space of closed subgroups of $\mathrm{Isom}(X)$.

In this section, after briefly reviewing the notion of strong convergence for geometrically finite representations, we prove that it is consistent with the notion of peripheral stability for EGF representations.

4.1 Chabauty topology and strong convergence

We refer to [BdlHK09] (see also [dlH08] and [BP92, Section E.1]) for a general reference on the Chabauty topology.

Let $\mathcal{CL} := \mathcal{CL}(\mathrm{Isom}(X))$ be the set of closed subgroups of $\mathrm{Isom}(X)$. The Chabauty topology on $\mathcal{CL}$ is generated by the basis of open subsets, for $C \in \mathcal{CL}$ a closed subgroup, $K \subset \mathrm{Isom}(X)$ a compact subset and U an open neighborhood of the identity in $\mathrm{Isom}(X)$:

$$V_{K,U,C} := \{D \in \mathcal{CL} \mid D \cap K \subset CU \text{ and } C \cap K \subset DU\}.$$

Equipped with this topology, $\mathcal{CL}$ is a compact space.

An important fact is that the subset of discrete subgroups of $\mathrm{Isom}(X)$ is open in $\mathcal{CL}$ [BdlHK09, Prop 3.4]. Following the proof of this fact in [BdlHK09] actually yields a slightly stronger statement, given below:

Proposition 4.1. *Let d be any metric inducing the compact-open topology on $\mathrm{Isom}(X)$. For any $R > 0$, the set of closed subgroups $G < \mathrm{Isom}(X)$ satisfying $\min_{g \in G\setminus\{e\}} d(g, e) > R$ is open in $\mathcal{CL}$.*

We also have the following useful criterion for convergence in $\mathcal{CL}$:

Proposition 4.2 ([BP92, Proposition E.1.2])**.** *A sequence $(C_n)_{n\in\mathbf{N}}$ of elements of $\mathcal{CL}$ converges to $C \in \mathcal{CL}$ if and only if both conditions below hold:*

(C1) Any accumulation point of a sequence $(c_n)_{n\in\mathbf{N}}$, where each c_n belongs to C_n, belongs to C.

(C2) Each point of C is the limit of a sequence $(c_n)_{n\in\mathbf{N}}$, where each c_n belongs to C_n.

The notion of strong convergence combines algebraic convergence and convergence in the Chabauty topology. We note that convergence in the Chabauty topology is also classically called *geometric convergence*.

Definition 4.3. Let $(\rho_n)_{n\in\mathbf{N}}$ be a sequence in $\mathrm{Hom}(\Gamma, \mathrm{Isom}(X))$.

- We say that the sequence (ρ_n) converges *strongly* to ρ if it converges algebraically to ρ and if the subgroups $\rho_n(\Gamma)$ converge to $\rho(\Gamma)$ in the Chabauty topology on the space of subgroups of $\mathrm{Isom}(X)$.
- If $\rho : \Gamma \to \mathrm{Isom}(X)$ is geometrically finite, and $\mathcal{H}$ is the set of ρ-parabolic subgroups, we say the sequence (ρ_n) converges *relatively strongly* to ρ if it converges algebraically to ρ and if, for any $H \in \mathcal{H}$, the sequence $\rho_n(H)$ converges to $\rho(H)$ in the Chabauty topology.

4.2 Equivalence between strong convergence and peripheral stability

The proposition below links the notions of strong convergence for geometrically finite representations and peripheral stability for EGF representations:

Proposition 4.4. *Let $\rho \in \mathrm{Hom}(\Gamma, \mathrm{Isom}(X))$ be a geometrically finite representation, and let $\mathcal{H}$ be the family of ρ-parabolic subgroups. Suppose that the sequence $(\rho_n)_{n\in\mathbf{N}}$ of representations in $\mathrm{Hom}(\Gamma, \mathrm{Isom}(X))$ converges algebraically to ρ, and that the restriction of ρ_n to each cusp group $H \in \mathcal{H}$ is faithful. Then the following are equivalent:*

(1) The sequence (ρ_n) converges strongly to ρ;

(2) The sequence (ρ_n) converges relatively strongly to ρ;

(3) The family $\{\rho_n, n \in \mathbf{N}\}$ is a peripherally stable deformation of ρ.

The first implication (1) $\implies$ (2) is not difficult and we prove it below. The second implication (2) $\implies$ (3) relies on Gromov hyperbolicity of the space X. We will deal with it in the following Section 4.2.1, using the language of CAT(-1) geometry. The last implication (3) $\implies$ (1) is more involved, since it relies on the relative automaton discussed in Section 2.4. We will tackle it in Section 4.2.2.

Proof of (1) $\implies$ (2). We assume that (ρ_n) converges strongly to ρ. Denote by G_n, resp. G, the images $\rho_n(\Gamma)$, resp. $\rho(\Gamma)$. Fix $H \in \mathcal{H}$, and let $H_n := \rho_n(H)$ and $H_\infty := \rho(H)$. Recall that H is virtually nilpotent and infinite. Since ρ_n is faithful on H, this implies that each H_n preserves a subset of ∂X containing either one or two points. Let $\{p_n^1, p_n^2\}$ be this subset (allowing for the possibility that $p_n^1 = p_n^2$).

As ρ_n converges algebraically to ρ, for each $h \in H$, we have that $\rho_n(h) \in H_n$ converges to $\rho(h)$ in H_∞. So condition (C2) for the Chabauty convergence of H_n to H_∞ is fulfilled. It also follows that both p_n^1 and p_n^2 converge to p. If $p_n^1 = p_n^2 = p_n$, this holds because $\rho_n(h)$ fixes p_n, and for any infinite-order $h \in H$ we know that $\rho(h)$ uniquely fixes p. Otherwise, H is virtually infinite cyclic. So for every h in a finite-index subgroup of H, we have $\rho_n(h)p_n^i = p_n^i$, and again when h has infinite order the fixed points of $\rho_n(h)$ must converge to the unique fixed point of $\rho(h)$.

We want to prove (C1) for the convergence of H_n to H_∞. So suppose that an element $g \in \mathrm{Isom}(X)$ is an accumulation point of some sequence h_n in H_n. First, the Chabauty convergence of G_n to G ensures that $g \in G$. Moreover, as $h_n \cdot \{p_n^1, p_n^2\} = \{p_n^1, p_n^2\}$ for all n, we can pass to the limit to see that $g \cdot p = p$. But $H_\infty = \mathrm{Stab}_G(p)$, so that $g \in H_\infty$. $\square$

4.2.1 Relative strong convergence implies peripheral stability

To prove the implication (2) $\implies$ (3) in Proposition 4.4, we need to use the fact that the space X is $\mathrm{CAT}(-1)$, up to a rescaling of the metric. This assumption is actually necessary, since the analogous implication does *not* hold if we instead only assume that ρ is an EGF representation into some higher-rank semisimple Lie group. Indeed, [Wei22, Example 9.3] exhibits a continuous family $\{\rho_t : 0 \leq t \leq \varepsilon\}$ of representations of the free group $\mathbf{F}_2 \simeq \mathbb{Z} * \mathbb{Z}$ into $\mathrm{SL}(4, \mathbb{R})$ converging to an EGF representation ρ_0, so that the family ρ_t is relatively strongly convergent but *not* peripherally stable. In this example, the limiting representation ρ_0 even has an injective boundary extension, meaning it is a *relative Anosov* representation (another related generalization of geometrical finiteness in higher rank).

In the cited example, the peripheral subgroups of $\mathbf{F}_2$ are the conjugates of the cyclic free factors, and the restriction of ρ_t to each of these factors converges strongly to a unipotent representation of $\mathbb{Z}$. The problem in the example occurs because the limit set of $\rho_t(\mathbb{Z})$ does not vary continuously at $t = 0$.

This problem no longer occurs in rank one, which follows from our next lemma. Recall that any virtually nilpotent subgroup G of $\mathrm{Isom}(X)$ is either elliptic, parabolic or hyperbolic. For such a subgroup, we denote by $\mathrm{Fix}(G)$ the set of fixed points in $\overline{X}$ of G and by $\mathcal{C}(G)$ the convex hull in $\overline{X}$ of $\mathrm{Fix}(G) \cup \Lambda_G$.

Lemma 4.5. *Let G be a virtually nilpotent group and let (ν_n) be a sequence of representations $\nu_n : G \to \mathrm{Isom}(X)$, converging algebraically to a representation $\nu : G \to \mathrm{Isom}(X)$ whose image is a nontrivial parabolic subgroup.*

Then the convex hulls $\mathcal{C}(\nu_n(G))$ converge to Λ_ν in the Hausdorff topology on $\overline{X}$.

Proof. First, since ν is nontrivial, ν_n is nontrivial for sufficiently large n and has virtually nilpotent image. Thus $\nu_n(G)$ is a nontrivial elliptic, parabolic, or hyperbolic subgroup, implying that $\mathcal{C}(\nu_n(G))$ is nonempty.

By assumption ν is parabolic, so Λ_ν is a singleton $\{p\}$, and there is an element $g \in G$ so that the unique fixed point of $\nu(g)$ in $\overline{X}$ is p. By algebraic convergence we know that $\nu_n(g) \to \nu(g)$, so any limit of fixed points of $\nu_n(g)$ in $\overline{X}$ is fixed by $\nu(g)$, hence equal to p. Since the fixed points of $\nu_n(g)$ contain the fixed points of $\nu_n(G)$, this proves that $\mathrm{Fix}(\nu_n(G)) \to \{p\}$ and therefore $\mathcal{C}(\nu_n(G)) \to \{p\}$. □

Below we state the key lemma we need for our proof of the implication at hand. This lemma says that isometries of X whose fixed points all lie close to p and remain bounded far from p belong to a compact subset of $\mathrm{Isom}(X)$.

Lemma 4.6. *Let $p \in \partial X$, and let K and K' be two compact subsets of ∂X not containing p. Then there exists an open neighborhood U of p in $\overline{X}$, disjoint from K and K', such that the set of isometries $g \in \mathrm{Isom}(X)$ satisfying $\mathrm{Fix}(g) \subset U$ and $g(K) \cap K' \neq \emptyset$ is relatively compact in $\mathrm{Isom}(X)$.*

The idea behind this lemma is that isometries which fix points close to p should behave roughly like elements preserving a horosphere S through p, and the stabilizer of S in G acts properly on S.

Before proving the lemma, we set up some notation. Whenever $\mathcal{C}$ is a convex subset of $\overline{X}$, and $k \in \overline{X} \setminus \mathcal{C}$, we let $[k; \mathcal{C}]$ denote the geodesic segment between k and its projection $\pi_{\mathcal{C}}(k)$ on $\mathcal{C}$.

Now fix a point $p \in \partial X$ as in the lemma, and let S be a horosphere centered at p. For any convex subset $\mathcal{C} \subset \overline{X}$ and any $k \in \overline{X} \setminus \mathcal{C}$, let $s(k; \mathcal{C})$ denote the point on the intersection $[k; \mathcal{C}] \cap S$ closest to k (assuming this intersection exists).

The first step in the proof of Lemma 4.6 is the following, which says that these "projections" $s(k; \mathcal{C})$ stay in a compact subset of X when k stays far away from p and $\mathcal{C}$ is close to p.

Lemma 4.7. *Let K be a compact subset of ∂X disjoint from p. Then there exists a neighborhood U of p in $\overline{X}$ such that:*

- *for any convex subset $\mathcal{C}$ of U and any $k \in K$, the geodesic segment $[k; \mathcal{C}]$ intersects S at least once, and*
- *the set $\{s(k; \mathcal{C}) \in X, k \in K, \mathcal{C} \subset U\}$ has compact closure in X.*

Proof. For any $k \in K$, the geodesic $[k; \{p\}]$ intersects S transversely exactly once, at the point denoted by $s(k; \{p\})$. Moreover, the induced map $\partial X \setminus \{p\} \to S \setminus \{p\}$ is a homeomorphism, so the image of K is compact in $S \setminus \{p\} \subset X$.

Now, near any pair $(k; \mathcal{C})$ such that the geodesic $[k; \mathcal{C}]$ intersects S transversely, the assignment $(k; \mathcal{C}) \mapsto s(k; \mathcal{C})$ is locally a well-defined continuous map, because the geodesic $[k; \mathcal{C}]$ varies continuously with k and $\mathcal{C}$. This proves the claim. □

Proof of Lemma 4.6. Fix a neighborhood U of p in ∂X that verifies the conclusion of the previous Lemma 4.7 for both K and K'. Without loss of generality we may assume that U is a convex subset of $\overline{X}$. Let $T \subset S \setminus \{p\}$ be a compact set containing all $s(k; \mathcal{C})$ for $k \in K \cup K'$ and convex subsets $\mathcal{C} \subset U$, and let D_T be the diameter of T. Consider an isometry g such that the convex hull $\mathcal{C}$ of its fixed points lies in U, and such that there exists $k \in K$ with $k' = g(k) \in K'$. Then g sends the geodesic $[k; \mathcal{C}]$ to $[k', \mathcal{C}]$. We want to prove that $g(s(k; \mathcal{C}))$ is uniformly close to $s(k'; \mathcal{C})$ on the geodesic $[k', \mathcal{C}]$.

First, consider the case where $\mathcal{C}$ is a singleton in $U \subset \partial X$. Fix an origin $o \in X$. The stabilizer of o is compact and acts transitively on ∂X. So, up to a uniformly compact conjugation, and if necessary a shrinking of U, we may assume without loss of generality that $\mathcal{C} = \{p\}$. Then g stabilizes S, and we actually have $g(s(k;\mathcal{C})) = s(k';\mathcal{C})$.

The other possibility is that $\mathcal{C}$ is not a singleton, so that $\mathcal{C} \cap X$ is nonempty. In this case, we can define $t(k;\mathcal{C})$ to be the distance from $s(k;\mathcal{C})$ to the projection $\pi_{\mathcal{C}}(k)$ of k on $\mathcal{C}$. Up to shrinking U another time to a smaller subset, we may assume that, for all $k \in K$, $\mathcal{C}$ included in U, we have $t(k;\mathcal{C}) > 2D_T$.

For simplicity, for the rest of the proof, we let $s(k)$, $t(k)$ and $\pi(k)$ denote the points $s(k;\mathcal{C})$, $t(k;\mathcal{C})$ and $\pi_{\mathcal{C}}(k)$, for any $k \in K$. The points $\pi(k)$ and $\pi(k')$ are the projections on $\mathcal{C}$ of $s(k)$ and $s(k')$ respectively, and the distance between those last two points is at most D_T.

As the projection map to a convex set does not increase distances in a CAT(-1)-space, we obtain:

$$d(\pi(k), \pi(k')) \leq d(s(k), s(k')) \leq D_T.$$

The distance $t(k')$ between $s(k')$ and its projection $\pi(k')$ then satisfies (by the triangle inequality):

$$\begin{aligned} |t(k') - t(k)| &\leq d(\pi(k'), \pi(k)) + d(s(k), s(k')) \\ &\leq 2D_T. \end{aligned}$$

So $g(s(k))$ and $s(k')$ are two points on the geodesic $[k', \mathcal{C}]$, whose distance from $\mathcal{C}$ is respectively $t(k)$ and $t(k')$. Thus their relative distance is less than $2D_T$.

In either case above, we conclude that $g(s(k))$ lies at distance at most $2D_T$ from $s(k') \in T$. The $2D_T$-neighborhood T' of T is a compact set in X. The isometry g sends a point of T to T'. As the action of $\mathrm{Isom}(X)$ on X is proper, the set of isometries h verifying $h(T) \cap T' \neq \emptyset$ is compact. This finishes the proof of the lemma. □

Equipped with these results, we can proceed with the proof that strong convergence implies peripheral stability, under the assumptions of Proposition 4.4. Recall that we consider a sequence ρ_n of representations that converges algebraically to a geometrically finite representation ρ. We assume moreover that the representations ρ_n are all faithful when restricted to the parabolic peripheral subgroups. We fix such a subgroup H and, by assumption (2), we have that $H_n := \rho_n(H)$ converges in the Chabauty topology to $H_\infty := \rho(H)$, which is a nontrivial parabolic subgroup in $\mathrm{Isom}(X)$.

Proof of (2) $\implies$ (3). Suppose that the sequence (ρ_n) is not peripherally stable and that the subgroup H is a witness for this: there exists a neighborhood U of p in ∂X, a compact $K \subset \partial X$ disjoint from U and a divergent sequence $\gamma_n \in H$ such that $\rho_n(\gamma_n)(K)$ has an accumulation point outside U. So there is a compact K' disjoint from U such that $\rho_n(g_n)(K) \cap K' \neq \emptyset$.

Since $\rho(H)$ is a nontrivial parabolic subgroup and $\rho_n \to \rho$ algebraically, we may apply Lemma 4.5. This means that Λ_{H_n} converges to $\{p\}$ in the Hausdorff topology on ∂X. In particular, for sufficiently large n, Λ_{H_n} is contained in U.

Now we can apply Lemma 4.6, which tells us that the elements $\rho_n(\gamma_n)$ remain in a compact subset of $\mathrm{Isom}(X)$. Up to extraction, we can assume they converge, and by the relative strong convergence assumption (2), the limit is $\rho(\gamma)$ for some $\gamma \in H$. Thus $\rho_n(\gamma^{-1}\gamma_n)$ converges to the identity, and since $\rho(H)$ is discrete, Proposition 4.1 implies that $\rho_n(\gamma^{-1}\gamma_n) = e$, hence $\rho_n(\gamma) = \rho_n(\gamma_n)$, for sufficiently large n. Since ρ_n is faithful when restricted to H, we have $\gamma_n = \gamma$, which contradicts the assumption that the sequence γ_n is divergent. This proves that the family (ρ_n) is peripherally stable around ρ. □

The last step of the proof of Proposition 4.4 is carried out in the following section.

4.2.2 Peripheral stability implies strong convergence

Proof of (3) $\implies$ (1). The proof of this implication is similar to the proof of Proposition 4.6 in [Wei22]. Assume that (3) holds: the family (ρ_n) is peripherally stable around ρ. Algebraic convergence $\rho_n \to \rho$ ensures that condition (C2) for Chabauty convergence holds. To show that (C1) also holds, it suffices to prove the following:

Claim 1. *For any sequence (γ_n) of pairwise distinct elements in Γ, the sequence $(\rho_n(\gamma_n))$ leaves every compact subset of* $\mathrm{Isom}(X)$.

So, fix such a sequence (γ_n) in Γ. It will be enough to show that every subsequence of (γ_n) has a further subsequence which leaves every compact subset of $\mathrm{Isom}(X)$, so we can extract subsequences throughout the rest of the argument.

Fix a finite generating set S for Γ, and let $\mathrm{Cay}(\Gamma, S, \Pi)$ denote the *relative Cayley graph* for Γ, i.e. the Cayley graph of Γ with respect to the generating set

$$S \cup \bigcup_{p \in \Pi} \mathrm{Stab}_\Gamma(p).$$

The path metric on this graph induces a metric on Γ; note that this metric is *not* quasi-isometric to the word metric induced by a finite generating set.

We now consider two cases, depending on the behavior of our sequence (γ_n) with respect to the metric coming from the relative Cayley graph:

Case 1: the sequence (γ_n) is unbounded in $\mathrm{Cay}(\Gamma, S, \Pi)$**.** For this case, we use the relative automaton $\mathcal{G}$ described in Section 2.4. Using property (A6) of the automaton, for each n, write

$$\gamma_n = \alpha_1^{(n)} \cdots \alpha_{m(n)}^{(n)} \beta^{(n)},$$

where $\beta^{(n)} \in \Gamma$ has length at most R with respect to the path metric on $\mathrm{Cay}(\Gamma, S)$, and each element $\alpha_i^{(n)}$ lies in a set $L(z_i)$ for a vertex path $z_1^{(n)} \to z_2^{(n)} \to \ldots \to z_{m(n)+1}^{(n)}$ in the automaton $\mathcal{G}$. Since $\beta^{(n)}$ has uniformly bounded length, it suffices to prove that the sequence

$$\rho_n(\gamma_n(\beta^{(n)})^{-1}) = \rho_n(\alpha_1^{(n)} \cdots \alpha_{m(n)}^{(n)})$$

leaves every compact subset of $\mathrm{Isom}(X)$.

Properties (A3) and (A4) of the automaton tell us that each element $\alpha_i^{(n)}$ has uniformly bounded length with respect to the metric on $\mathrm{Cay}(\Gamma, S, \Pi)$. So, our assumption for this case tells us that the length $m(n)$ of the vertex path is unbounded. After extracting a subsequence, we can assume that this length tends to infinity.

Now, consider the sequence of subsets

$$\rho_n(\alpha_1^{(n)} \cdots \alpha_{m(n)}^{(n)}) W(z_{m(n)+1}^{(n)}). \tag{2}$$

Lemma 3.4 implies that there is a uniform positive constant $\varepsilon_0 > 0$ so that, for every sufficiently large n and every every $1 \le i \le m(n)$, we have

$$\rho_n(\alpha_i^{(n)}) N_{\varepsilon_0}(W(z_{i+1}^{(n)})) \subset W(z_i^{(n)}). \tag{3}$$

After extracting a subsequence, we can assume that the vertex $z_1^{(n)}$ is independent of n; we write this vertex as z_1. By applying [Wei22, Proposition 7.11], we can see that the uniform strong nesting in (3) implies that, with respect to a fixed choice of metric on the open subset $W(z_1)$, the diameter of the set defined in (2) tends to zero as $m(n)$ tends to infinity. (This step is where we apply property (A1) of the automaton, since otherwise the cited proposition in [Wei22] does not apply.)

We can extract a further subsequence so that the vertex $z_{m(n)+1}^{(n)}$ does not depend on n, and write this vertex as z'. Then, as $W(z')$ is nonempty and open (so in particular has positive diameter), it follows that $\rho_n(\alpha_1^{(n)} \cdots \alpha_{m(n)}^{(n)})$ does not accumulate to any point in $\mathrm{Homeo}(\partial X)$, and therefore leaves every compact subset of $\mathrm{Isom}(X)$ as desired.

Case 2: the sequence (γ_n) is bounded in $\mathrm{Cay}(\Gamma, S, \Pi)$. In this case, we choose a shortest representative for each γ_n with respect to the relative generating set $S \cup \bigcup_{p \in \Pi} \mathrm{Stab}_\Gamma(p)$. Such a word has the form of an alternating product

$$\gamma_n = g_0^{(n)} h_1^{(n)} g_1^{(n)} \cdots h_{k(n)}^{(n)} g_{k(n)}^{(n)},$$

where, for each fixed $0 \le i \le k$, $(g_i^{(n)})$ is a bounded sequence in Γ (with respect to the word metric induced by S), and each $h_i^{(n)}$ lies in $\mathrm{Stab}_\Gamma(p_i^{(n)})$ for some $p_i^{(n)}$ in Π. Since the length of this word is uniformly bounded, we can extract a subsequence and assume that $k(n)$ is a fixed number k, independent of n. By repeatedly combining adjacent terms in this word and extracting further subsequences, we can also assume that for each fixed $1 \le i \le k$, the sequence $(h_i^{(n)})$ is unbounded (with respect to the word metric coming from S), the parabolic point $p_i^{(n)}$ is a point p_i independent of n, and $\rho(g_i)p_i \neq p_{i-1}$.

Let K be a compact subset of $\partial X \setminus \{\rho(g_k^{-1})p_k\}$ with nonempty interior. We claim that the sequence $\rho_n(\gamma_n)K$ converges to a singleton; since K has nonempty interior this will ensure that $\rho_n(\gamma_n)$ leaves every compact subset of $\mathrm{Homeo}(\partial X)$, hence of $\mathrm{Isom}(X)$.

To prove the claim, we induct on k. In the case $k = 1$, since K is a compact subset of $\partial X \setminus \{\rho(g_1^{-1})p_1\}$, and g_1 is fixed, we know that for all sufficiently large n, the set $\rho_n(g_1)K$ is a compact subset of $\partial X \setminus \{p_1\}$. Then, since $h_1^{(n)}$ is unbounded in $\mathrm{Stab}_\Gamma(p_1)$, the peripheral stability assumption implies that for any neighborhood U of p_1 in ∂X, for all sufficiently large n we have $\rho_n(h_1^{(n)} g_1)K \subset U$. So, $\rho_n(h_1^{(n)} g_1)K$ converges to the singleton $\{p_1\}$, and so the sequence $\rho_n(g_0 h_1^{(n)} g_1)K$ converges to the singleton $\{g_0 p_1\}$.

When $k > 1$, the exact same reasoning implies that the sequence of sets $\rho_n(h_k^{(n)} g_k)K$ converges to the singleton $\{p_k\}$. In particular, since we know that $\rho(g_{k-1})p_k \neq p_{k-1}$, we know that for sufficiently large n we also have $\rho_n(g_{k-1})p_k \neq p_{k-1}$, and thus (also for large n) the set $\rho_n(h_k^{(n)} g_k)K$ lies in a fixed compact subset of $\partial X \setminus \{\rho(g_{k-1}^{-1})p_{k-1}\}$. So applying induction we see that the sequence of sets

$$\rho_n(\gamma_n)K = \rho_n(g_0 h_1^{(n)} \cdots h_{k-1}^{(n)} g_{k-1}) \rho_n(h_k^{(n)} g_k)K$$

again converges to a singleton, and we are done.

$\square$

5 Convergence of limit sets and Cannon-Thurston maps

We are now able to prove Theorem 1.2 and Theorem 1.4 from the introduction. We have already done the difficult part, which was to connect the framework of EGF representations and peripheral stability to the notions of geometrical finiteness and strong convergence; with this relationship established, both of our main theorems are straightforward corollaries of the relative stability theorem for EGF representations (Theorem 3.2).

5.1 Limit sets

First we will prove Theorem 1.2, whose statement is subsumed by the following:

Theorem 5.1 (Convergence of limit sets)**.** *Let X be a noncompact rank-one symmetric space, let Γ be a finitely generated group, and let $(\rho_n)_{n\in\mathbf{N}}$ be a sequence of faithful representations converging algebraically to a geometrically finite representation ρ.*

Then, ρ_n converges to ρ strongly if and only if $\rho_n \to \rho$ relatively strongly. In this case:

(1) For all sufficiently large n, the representation ρ_n is geometrically finite. Further, for any subgroup $H \subset \Gamma$, $\rho_n(H)$ is parabolic only if $\rho(H)$ is parabolic.

(2) The limit sets Λ_n of $\rho_n(\Gamma)$ converge in the Hausdorff topology to the limit set Λ of $\rho(\Gamma)$.

Proof. The equivalence of strong convergence and strong relative convergence for $\rho_n \to \rho$ is given by Proposition 4.4. So, now suppose that the convergence $\rho_n \to \rho$ is relatively strong.

We first show that (1) holds. As we have already observed several times, the limiting representation ρ is EGF with respect to the collection $\mathcal{H}$ of its ρ-parabolic subgroups, which are virtually nilpotent. Since (ρ_n) converges relatively strongly and ρ_n is faithful, the family (ρ_n) is peripherally stable, with respect to $\mathcal{H}$ around ρ, by Proposition 4.4. Then by Theorem 3.2, ρ_n is EGF for sufficiently large n, again with respect to the collection $\mathcal{H}$. Thus, by Proposition 2.11, the representations ρ_n are geometrically finite for large n, and the collection $\mathcal{H}_n$ of ρ_n-parabolic subgroups is contained in $\mathcal{H}$. This proves (1).

To prove (2), let $\epsilon > 0$ be fixed. One may write the limit set Λ of $\rho(\Gamma)$ as a union of finitely many compact subsets with diameter at most ϵ, with respect to a chosen visual metric on ∂X. Applying the boundary extension $\phi : \Lambda \to \partial(\Gamma, \mathcal{H})$ to each of these sets, we obtain a finite collection $\mathcal{Z}$ of compact subsets of $\partial(\Gamma, \mathcal{H})$, so that $\bigcup_{Z \in \mathcal{Z}} Z = \partial(\Gamma, \mathcal{H})$, and so that each $\phi^{-1}(Z)$ has diameter at most ϵ.

Now let $\phi_n : \Lambda_n \to \partial(\Gamma, \mathcal{H})$ be the boundary extension for the EGF representation ρ_n, with respect to the peripheral structure $\mathcal{H}$; recall from Proposition 2.8 that the domain of this boundary extension is the limit set Λ_n of $\rho_n(\Gamma)$. Applying the inclusion in Theorem 3.2, we see that for all sufficiently large n and all $Z \in \mathcal{Z}$, the set $\phi_n^{-1}(Z)$ is contained in an ϵ-neighborhood of $\phi^{-1}(Z)$, and thus $\Lambda_n = \bigcup_{Z \in \mathcal{Z}} \phi_n^{-1}(Z)$ is within Hausdorff distance 2ϵ of $\Lambda = \bigcup_{Z \in \mathcal{Z}} \phi^{-1}(Z)$. Since $\epsilon > 0$ was arbitrary this completes the proof. □

5.2 Cannon-Thurston maps

We now turn to Theorem 1.4, which concerns the convergence of Cannon-Thurston maps for a strongly convergent sequence (ρ_n) of geometrically finite representations. In order to define Cannon-Thurston maps, we need the notion of a weakly type-preserving sequence of representations:

Definition 5.2. A sequence of representations $(\rho_n)_{n \in \mathbf{N}}$ from Γ to $\mathrm{Isom}(X)$ is *weakly type-preserving* if the collection $\mathcal{H}_n$ of ρ_n-parabolic subgroups of Γ always contains the collection $\mathcal{H}_1$ of ρ_1-parabolic subgroups of Γ.

Suppose that a sequence of representations (ρ_n) converges relatively strongly to a geometrically finite representation ρ, whose collection of ρ-parabolic subgroups is $\mathcal{H}$. By Theorem 5.1, after forgetting a finite number of indices, we may assume that each ρ_n is geometrically finite, and that the collection of ρ_n-parabolic subgroups $\mathcal{H}_n$ is a subset of $\mathcal{H}$. Since each $\mathcal{H}_n$ and $\mathcal{H}$ consists of a finite number of conjugacy classes, after extracting a further subsequence we can assume that $\mathcal{H}_1 \subseteq \mathcal{H}_n$ for all n. Thus we have shown:

Proposition 5.3. *Suppose that ρ is a faithful geometrically finite representation and (ρ_n) is a sequence of faithful representations converging strongly to ρ. Then a subsequence of (ρ_n) is weakly type-preserving, and consists of geometrically finite representations.*

Now let (ρ_n) be any weakly type-preserving sequence of faithful geometrically finite representations, and let $\mathcal{H}_n$ be the collection of ρ_n-parabolic subgroups. The inclusion $\mathcal{H}_1 \subseteq \mathcal{H}_n$ implies (via Proposition 2.11) that the representation ρ_1 is actually EGF with respect to $\mathcal{H}_n$ for every n. Thus we have a Γ-equivariant boundary extension $\phi_{1,n} : \Lambda_1 \to \partial(\Gamma, \mathcal{H}_n)$. We also know from Theorem 2.5 that the boundary extension $\phi_{n,n} : \Lambda_n \to \partial(\Gamma, \mathcal{H}_n)$ for ρ_n is a Γ-equivariant homeomorphism. So we may make the following definition:

Definition 5.4. For a weakly type-preserving sequence (ρ_n) of faithful geometrically finite representations, we define the Cannon-Thurston maps $\mathrm{CT}_{1,n} : \Lambda_1 \to \Lambda_n$ by the composition $\phi_{n,n}^{-1} \circ \phi_{1,n}$.

If the sequence (ρ_n) converges strongly to a geometrically finite representation ρ, then by Theorem 5.1, $\mathcal{H}_n$ is eventually a subset of the collection $\mathcal{H}$ of parabolic subgroups for ρ. In particular, as $\mathcal{H}_1 \subseteq \mathcal{H}_n$ for all n, we have $\mathcal{H}_1 \subseteq \mathcal{H}$. Thus (again by Proposition 2.11) there is an EGF boundary extension $\phi_{1,\infty} : \Lambda_1 \to \partial(\Gamma, \mathcal{H})$, and we may also define the Cannon-Thurston map $\mathrm{CT}_{1,\infty} : \Lambda_1 \to \Lambda$ via the composition $\phi^{-1} \circ \phi_{1,\infty}$, where $\phi : \Lambda \to \partial(\Gamma, \mathcal{H})$ is the EGF boundary extension for ρ.

With this notation established, we can now prove Theorem 1.4 from the introduction; we give a restatement of this result below. This result can be thought of as a more precise version of the convergence of limit sets expressed in Theorem 5.1.

Theorem 5.5 (Convergence of Cannon-Thurston maps)**.** *Let X be a non-compact rank-one symmetric space, let Γ be a finitely generated group, and let $(\rho_n)_{n\in\mathbf{N}}$ be a weakly type-preserving sequence of faithful geometrically finite representations, converging relatively strongly to a geometrically finite representation ρ.*

Then the sequence of Cannon-Thurston maps $\mathrm{CT}_{1,n} : \Lambda_1 \to \Lambda_n$ converges uniformly to the Cannon-Thurston map $\mathrm{CT}_{1,\infty} : \Lambda_1 \to \Lambda_n$.

Proof. First, if ρ_1 is elementary, meaning Λ_1 contains one or two points, then the statement is easy. We assume from now on that ρ_1 is non-elementary.

Using the notation established above, we first observe that, since $\mathcal{H}_n \subseteq \mathcal{H}$ for all sufficiently large n, we can once again apply Proposition 2.11 to see that that ρ_n is EGF with respect to $\mathcal{H}$. Thus there is an EGF boundary extension $\phi_{n,\infty} : \Lambda_n \to \partial(\Gamma, \mathcal{H})$ and a Cannon-Thurston map $\mathrm{CT}_{n,\infty} : \Lambda_n \to \Lambda_\infty$ given by $\phi^{-1} \circ \phi_{n,\infty}$.

We claim that the composition $\mathrm{CT}_{n,\infty} \circ \mathrm{CT}_{1,n}$ is precisely the Cannon-Thurston map $\mathrm{CT}_{1,\infty}$. As $\rho_1(\Gamma)$ is non-elementary, Λ_1 contains at least three points. Then there is some $\gamma \in \Gamma$ which has an attracting fixed point for the ρ_1-action of Γ on Λ_1. Any (ρ_1, ρ)-equivariant continuous map $\Lambda_1 \to \Lambda$ must take this attracting fixed point to the attracting fixed point of $\rho(\gamma)$ (if $\rho(\gamma)$ is loxodromic) or to the unique fixed point of $\rho(\gamma)$ (if $\rho(\gamma)$ is parabolic). Moreover, since $\rho_1(\Gamma)$ is non-elementary, every $\rho_1(\Gamma)$-orbit in Λ_1 is dense. So any equivariant continuous map $\Lambda_1 \to \Lambda$ is uniquely determined on a dense set and therefore must agree with the Cannon-Thurston map. Since the composition $\mathrm{CT}_{n,\infty} \circ \mathrm{CT}_{1,n}$ is such an equivariant continuous map, we obtain

$$\mathrm{CT}_{n,\infty} \circ \mathrm{CT}_{1,n} = \mathrm{CT}_{1,\infty},$$

as claimed. We sum up the situation in the commutative diagram of Figure 1.

Now, let $\epsilon > 0$ be fixed. As in the proof of Theorem 1.2, we may cover $\partial(\Gamma, \mathcal{H})$ with a finite collection $\mathcal{Z}$ of compact sets so that, for each $Z \in \mathcal{Z}$, the preimage $\phi^{-1}(Z)$ has diameter at most ϵ. By Theorem 3.2, for all sufficiently

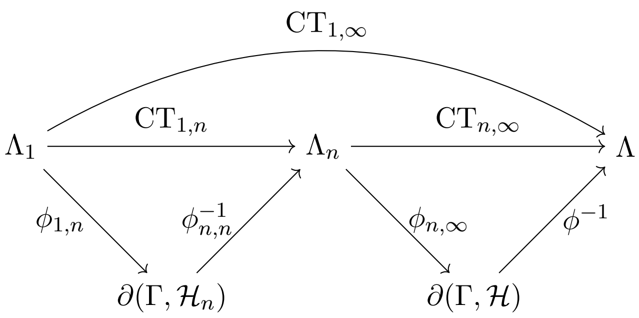

Figure 1: Synthetic view of boundary extensions and Cannon-Thurston maps.

large n (depending only on ϵ and $\mathcal{Z}$), for every $Z \in \mathcal{Z}$ the preimage $\phi_{n,\infty}^{-1}(Z)$ lies in an ϵ-neighborhood of $\phi^{-1}(Z)$. Then, for any $x \in \Lambda_1$, we may choose $Z \in \mathcal{Z}$ so that $\phi_{1,\infty}(x) \in Z$, and therefore $\mathrm{CT}_{1,\infty}(x) = \phi^{-1} \circ \phi_{1,\infty}(x) \in \phi^{-1}(Z)$.

On the other hand, since $\phi_{1,\infty} = \phi \circ \mathrm{CT}_{1,\infty}(x) \in Z$, we have

$$\phi \circ \mathrm{CT}_{n,\infty} \circ \mathrm{CT}_{1,n}(x) \in Z,$$

or equivalently $\phi_{n,\infty} \circ \mathrm{CT}_{1,n}(x) \in Z$. Thus $\mathrm{CT}_{1,n}(x) \in \phi_{n,\infty}^{-1}(Z)$ and the distance between $\mathrm{CT}_{1,n}(x)$ and $\mathrm{CT}_{1,\infty}(x)$ is at most 2ϵ, independent of x. $\square$

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