Dehn filling in semisimple Lie groups

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in preparation,
work in progress with Jeff Danciger

Motivation

Goal: find new examples of interesting discrete subgroups/geometric structures by deforming known examples

Inspiration:

Theorem (Thurston)

Let M be a finite-volume hyperbolic 3-manifold with one cusp. All but finitely many Dehn fillings of M admit hyperbolic structures.

Gives a rich source of examples of lattices in $\mathrm{PSL}(2, \mathbb{C})$.

What about discrete subgroups in other Lie groups?

Dehn filling in 3-manifolds

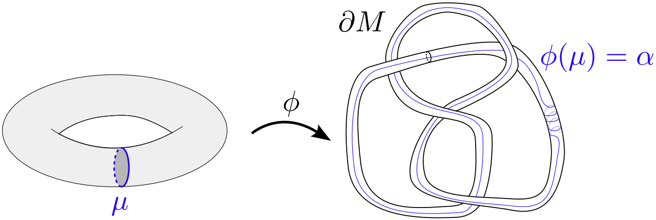
Let M be a 3-manifold, homeomorphic to interior of compact $\overline{M}$ with $\partial\overline{M} \simeq T^2$. Assume $\pi_1\partial\overline{M}$ injects into $\pi_1\overline{M} \simeq \pi_1 M$.

Definition

A *Dehn filling* of M is a 3-manifold

$$M_\phi = \overline{M} \sqcup_\phi S,$$

where S is a solid torus and $\phi : \partial S \to \partial\overline{M}$ is a homeomorphism.



At the group-theoretic level: $\pi_1 M_\phi \simeq \pi_1 M / \langle\langle [\alpha] \rangle\rangle$

Hyperbolic Dehn filling: proof idea

Let $\Gamma = \pi_1 M$ for M hyperbolic with one cusp. We have:

- Holonomy $\rho : \Gamma \to \mathrm{PSL}(2, \mathbb{C})$
- Character variety $X(\Gamma, \mathrm{PSL}(2, \mathbb{C}))$

Step 1. Prove: $X(\Gamma, \mathrm{PSL}(2, \mathbb{C}))$ is a 1-dimensional complex manifold in a neighborhood U of $[\rho]$.

Step 2. Show: countably many representations in U descend to holonomies of hyperbolic structures on Dehn fillings of M.

Step 1: **flexibility** of holonomy
Step 2: **(relative) stability** of holonomy

Main theorem in this talk generalizes **Step 2.**
We also carry out (an analog of) **Step 1** in specific contexts.

Dehn filling in rank one

- M hyperbolic 3-manifold with one cusp, $\Gamma = \pi_1 M$
- G is a rank-one Lie group
- $\rho : \Gamma \to G$ faithful and *geometrically finite*.

Theorem (Corollary of W. (in preparation), see also McMullen 1999)

Let ρ_n be a sequence in $\mathrm{Hom}(\Gamma, G)$. *If:*

I. $\rho_n \to \rho$ *in* $\mathrm{Hom}(\Gamma, G)$, *in compact-open topology*

II. $\rho_n(\pi_1 \partial M) \to \rho(\pi_1 \partial M)$ *in Chabauty topology*

then for all sufficiently large n:

(a) ρ_n *has geometrically finite (in particular, discrete) image*

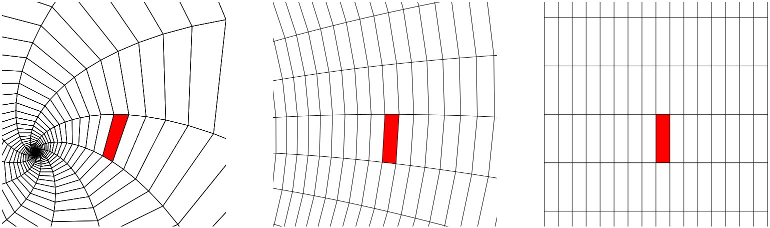
(b) $\ker(\rho_n) = \langle\langle \ker(\rho_n|_{\pi_1 \partial M}) \rangle\rangle$

(c) *If $\rho_n(\pi_1 \partial M)$ is convex cocompact, then so is $\rho_n(\pi_1 M)$.*

Gives a way to generalize Step 2 from the previous slide.

Recovering 3-dimensional hyperbolic Dehn filling

Fix holonomy representation $\rho : \pi_1 M \to \mathrm{PSL}(2, \mathbb{C})$.
Sequences $\rho_n \to \rho$ give *affine structures* on ∂M:



- Check: along Dehn filling sequences, holonomy of affine structures converge in Chabauty topology.
- This proof does *not* use a triangulation or a fundamental domain for the original 3-manifold!

Higher rank version

- Γ is *relatively hyperbolic*, relative to subgroups $H < \Gamma$
- G a semisimple Lie group, $P < G$ (symmetric) parabolic
- $\rho : \Gamma \to G$ is extended geometrically finite.

Theorem (W., in preparation)

If a sequence ρ_n satisfies:

I. $\rho_n \to \rho$ *in compact-open topology on* $\mathrm{Hom}(\Gamma, G)$

II. *For each peripheral subgroup $H < \Gamma$, restriction $\rho_n|_H$ satisfies "uniform convergence of orbit maps in G/P"*

then for all sufficiently large n:

(a) $\rho_n : \Gamma/\ker(\rho_n) \to G$ *is extended geometrically finite (in particular, has discrete image).*

(b) $\ker(\rho_n) = \langle\langle \bigcup_H \ker(\rho_n|_H) \rangle\rangle$

(c) *If $\rho_n(H)$ is P-Anosov for all H, then so is $\rho_n(\Gamma)$.*

Exotic Dehn fillings in $\mathbb{H}^3_{\mathbb{C}}$ and $\mathbb{RP}^3$

$M = \pi_1 M$ for M cusped hyperbolic 3-manifold.

$\rho : \pi_1 M \to \mathrm{PO}(3,1) \to \mathrm{PU}(3,1)$ gives geometrically finite action of $\pi_1 M$ on $\mathbb{H}^3_{\mathbb{C}}$, preserving isometrically embedded $\mathbb{H}^3_{\mathbb{R}} \subset \mathbb{H}^3_{\mathbb{C}}$.

Theorem (Danciger-W., in progress)

For infinitely many Dehn fillings $M_{p,q}$ of M, there is a deformation of ρ in $\mathrm{Hom}(\pi_1 M, \mathrm{PU}(3,1))$ which induces a convex cocompact representation of $\pi_1 M_{p,q}$. This representation does not factor through a copy of $\mathrm{PO}(3,1) \hookrightarrow \mathrm{PU}(3,1)$.

Corollary (Danciger-W., in progress)

For infinitely many closed hyperbolic 3-manifolds M, the space of convex cocompact representations $\pi_1 M \to \mathrm{PU}(3,1)$ is not connected.

Similar results for convex projective 3-manifolds (Ballas-Danciger-Lee-Marquis); higher-rank version of theorem describes these deformations.

Very exotic Dehn fillings in $\mathbb{H}^3_{\mathbb{C}}$

Let M be figure-8 knot complement. We have $\pi_1 \partial M \simeq \mathbb{Z} \oplus \mathbb{Z}$.

Consider $\rho : \pi_1 M \to \mathrm{PO}(3,1) \to \mathrm{PU}(3,1)$.

- Paupert-Thistlethwaite: explicit 1-dimensional family of deformations of $[\rho]$ in $X(\pi_1 M, \mathrm{PU}(3,1))$ coming from deformations of $\mathrm{Bi}(3) > \pi_1 M$.

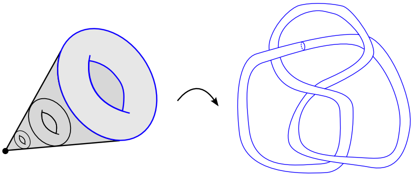
Theorem (Danciger-W., in progress)

There exists a sequence of representations ρ_n converging to ρ in $\mathrm{Hom}(\pi_1 M, \mathrm{PU}(3,1))$ so that, for sufficiently large n:

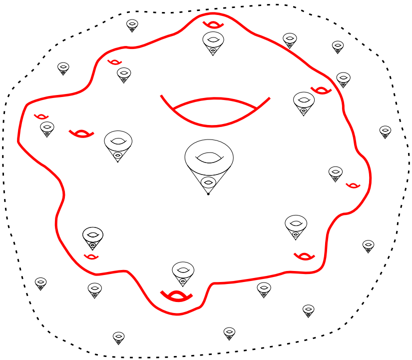
- $\ker(\rho_n) = \langle\langle n\mathbb{Z} \oplus n\mathbb{Z} \rangle\rangle$
- $\rho_n(\pi_1 M)$ *is a convex cocompact subgroup of* $\mathrm{PU}(3,1)$.

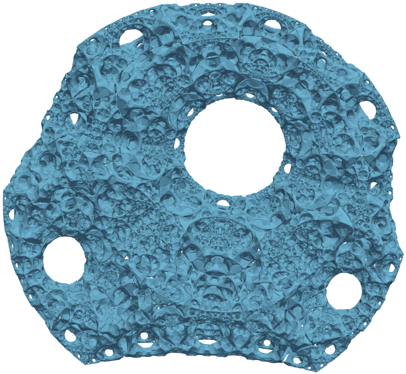
Note: $\Gamma / \ker(\rho_n)$ is *not* isomorphic to the fundamental group of a 3-manifold!

$\Gamma / \ker(\rho_n)$ is fundamental group of a locally CAT(-1) space X:



Universal cover $\tilde{X}$:





Limit sets converge to $\partial\mathbb{H}^3_{\mathbb{R}} \simeq S^2$ embedded in $\partial\mathbb{H}^3_{\mathbb{C}} \simeq S^5$.

For visualization, project to a 3d slice of $\partial\mathbb{H}^3_{\mathbb{C}}$ containing $\partial\mathbb{H}^3_{\mathbb{R}}$: