Expansion/contraction dynamics for non-strictly convex projective manifolds

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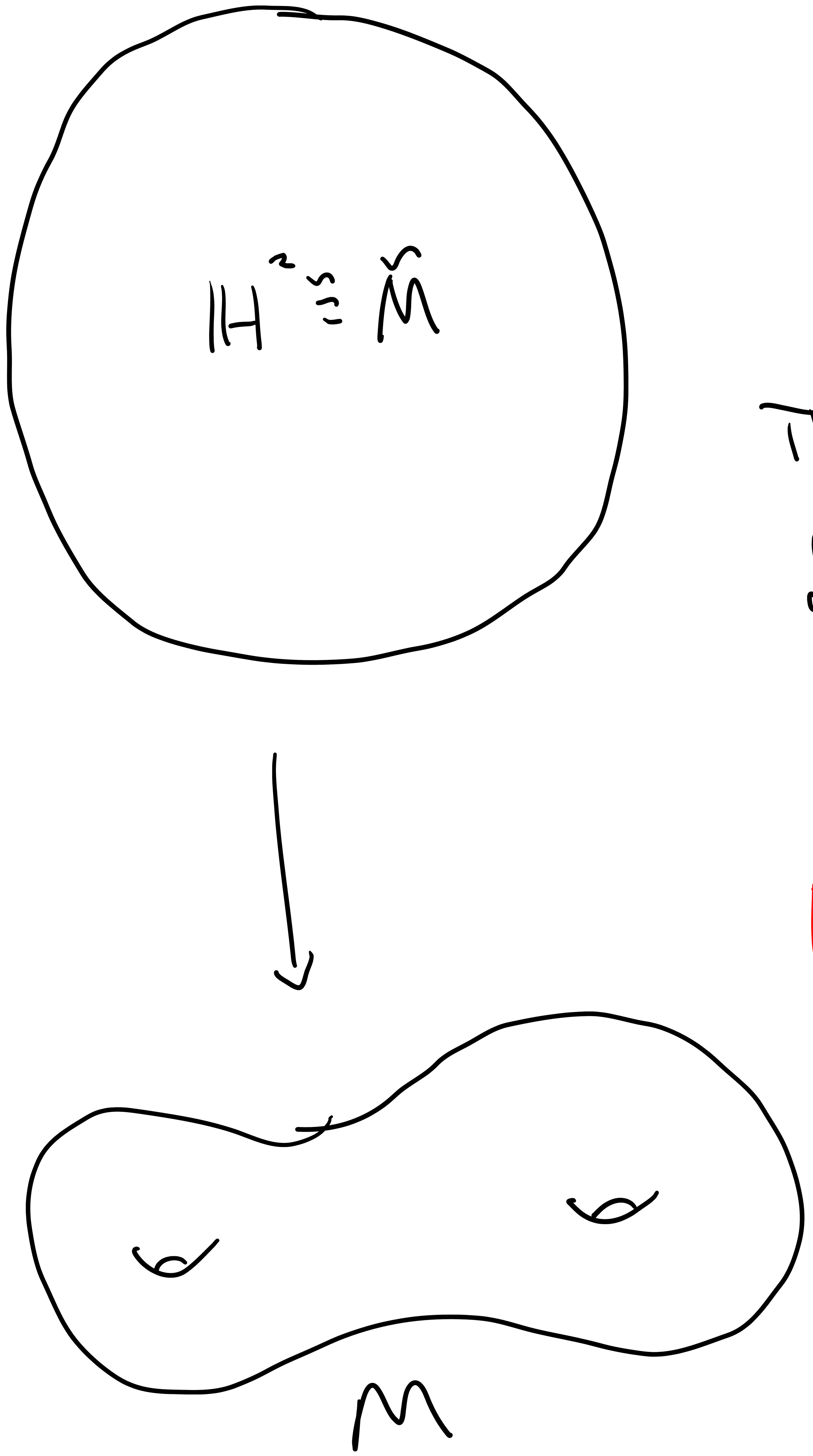
Convex hyperbolic manifolds:

M compact hyperbolic d-manifold (possibly with boundary).

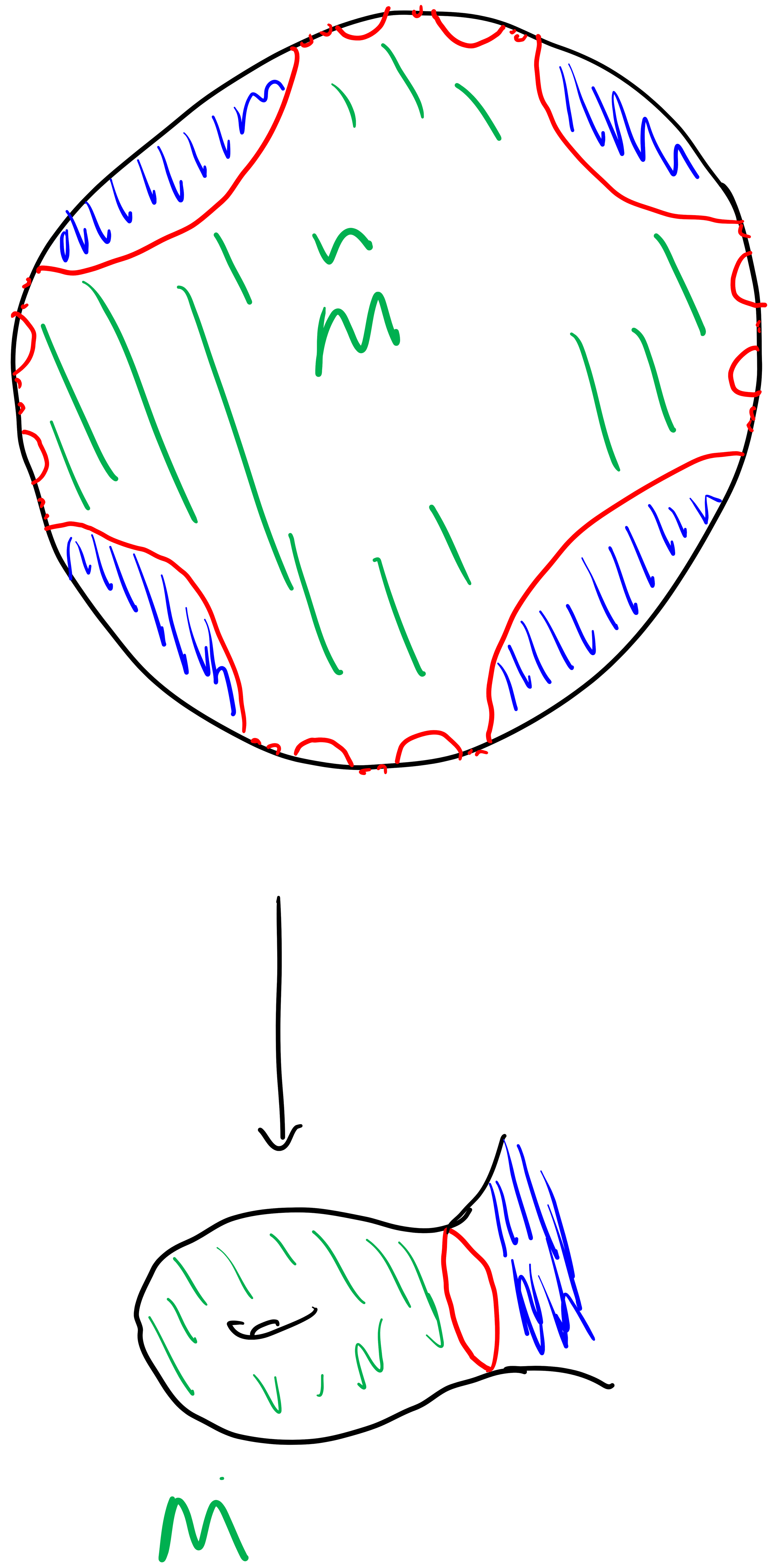
Def: M is convex if $\tilde{M} \subset \mathbb{H}^d$ is a convex subset of $\mathbb{H}^d$.



$$\pi_1 M \cong \Gamma \subset PO(d,1)$$

discrete, acts cocompactly on $\tilde{M}$ in $\mathbb{H}^d$

Γ is convex cocompact.



Thm (Sullivan):

$\Gamma \subset PO(d,1)$ discrete group.

(virtually)

$\Gamma \overset{v}{\cong} \pi_1 M$, M compact convex hyperbolic manifold

Γ is Gromov-hyperbolic and $\exists$ equivariant embedding $\phi : \partial\Gamma \to \partial\mathbb{H}^d$

$\phi(\partial\Gamma) = \partial_i \hat{M}$

Γ acts with expansion dynamics on a closed invariant subset $\Lambda \subset \partial\mathbb{H}^d$.

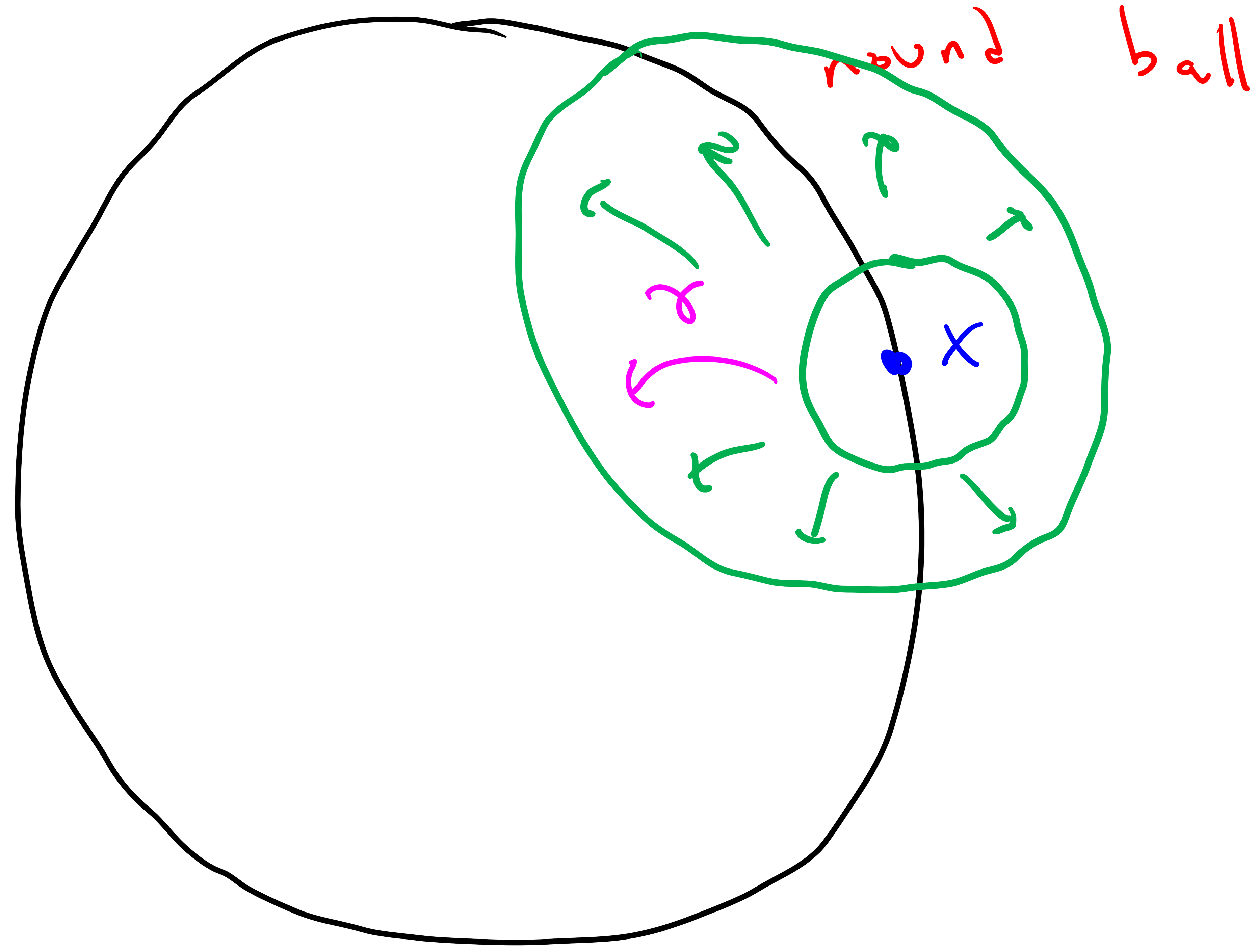
limit set $= \partial_i \hat{M}$

Expansion dynamics:

$$\mathbb{H}^d \subset \mathbb{RP}^d$$

$$\circlearrowright \qquad \circlearrowright$$

$$PO(d,1) \subset PGL(d+1,\mathbb{R})$$



Fix a metric on $\mathbb{RP}^d$.

For each $x \in \Lambda \subset \partial \mathbb{H}^d$, we can find $\gamma \in \Gamma$ and $U \subset \mathbb{RP}^d$ and a constant $C > 1$ so that

$$d(\gamma a, \gamma b) > C \cdot d(a,b)$$

for all $a, b \in U$.

Γ convex cocompact in $PO(d,1)$

geometric structures

$\Gamma \cong \pi_1 M$, M compact convex ~~hyperbolic~~ manifold

projective manifold

Γ is Gromov-hyperbolic and $\exists$ equivariant embedding $\phi : \partial\Gamma \to \partial\mathbb{H}^d$

Geometric group theory

dynamics

Γ acts with expansion dynamics on a closed invariant subset $\Lambda \subset \partial\mathbb{H}^d$.

What if $\Gamma \subset PGL(d+1, \mathbb{R})$?

Closed manifold M has a convex projective structure if

$M \cong \Omega / \Gamma$, where:

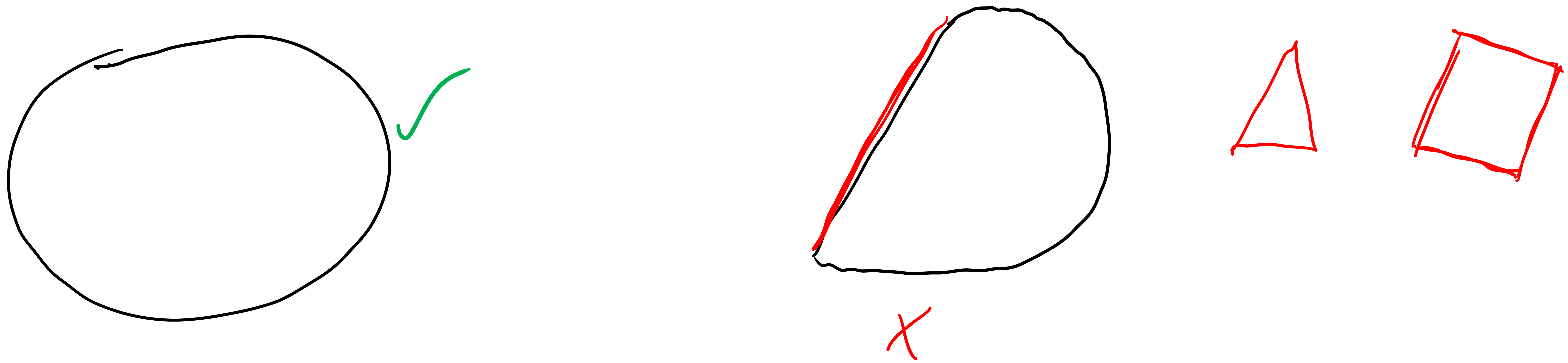
- $\Omega \subset \mathbb{RP}^d$ is a properly convex open set

  Ω is bounded and convex in an affine chart of $\mathbb{RP}^d$.

- $\Gamma \subset PGL(d+1, \mathbb{R})$ is discrete and preserves Ω.

Ex: closed hyperbolic manifolds

Def: Ω is strictly convex if $\partial\Omega$ contains no projective segments.



Thm: (Danciger - Guéritaud - Kassel):

Γ discrete subgroup of $PGL(d+1, \mathbb{R})$, Γ preserves a properly and strictly convex domain Ω.

Γ is convex cocompact $\iff$

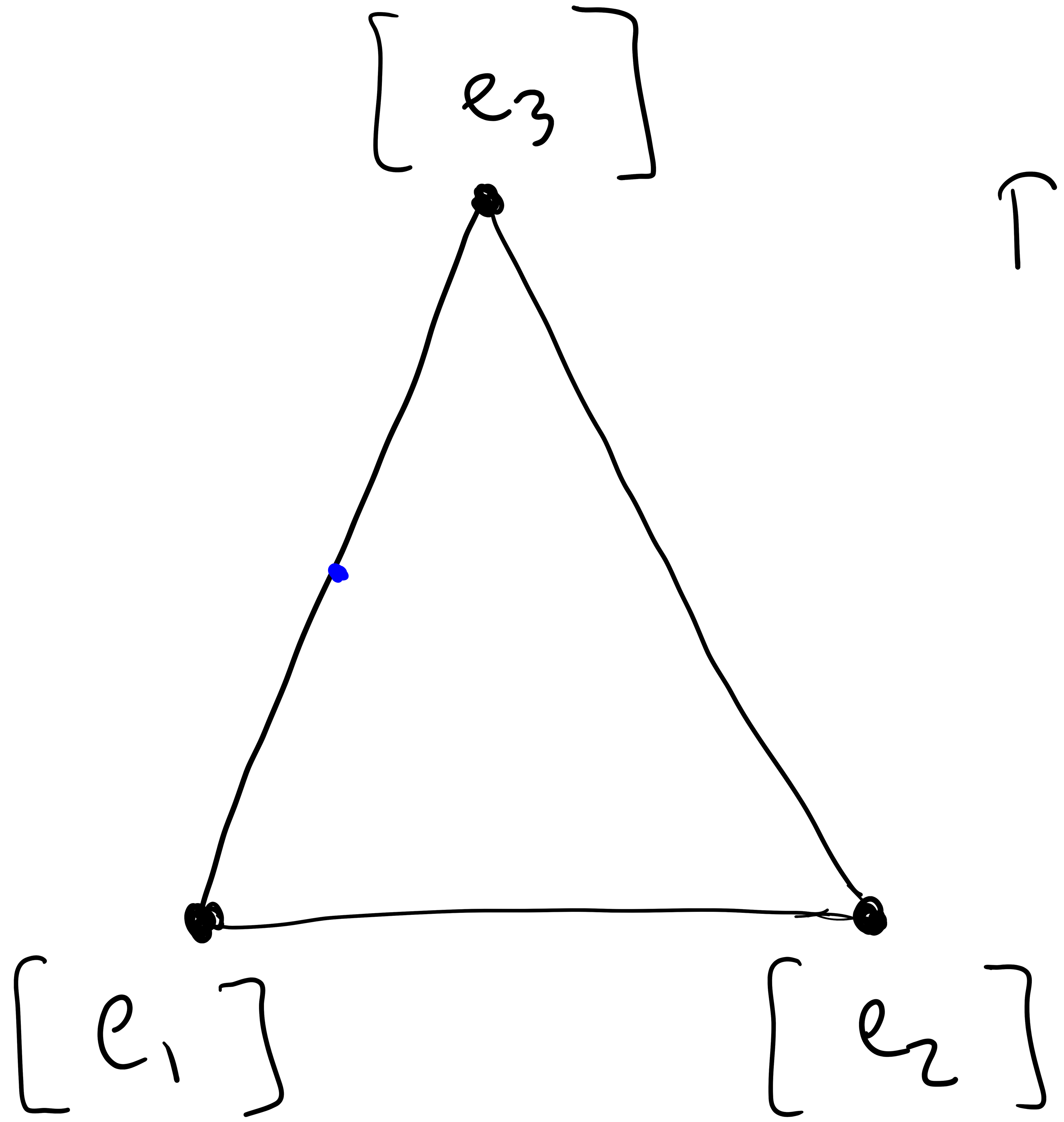
Γ is Gromov-hyperbolic and $\exists$ equivariant boundary embedding $\phi : \partial\Gamma \longrightarrow \mathbb{R}P^d$ and Γ acts with expansion dynamics on $\phi(\partial\Gamma)$

GGT + dynamics

Γ acts w/ compact quotient on closed convex subset $C \subset \Omega$.

$\Gamma \hookrightarrow PGL(d+1, \mathbb{R})$ is a P_1-Anosov representation.

What if Ω is <u>not</u> strictly convex?



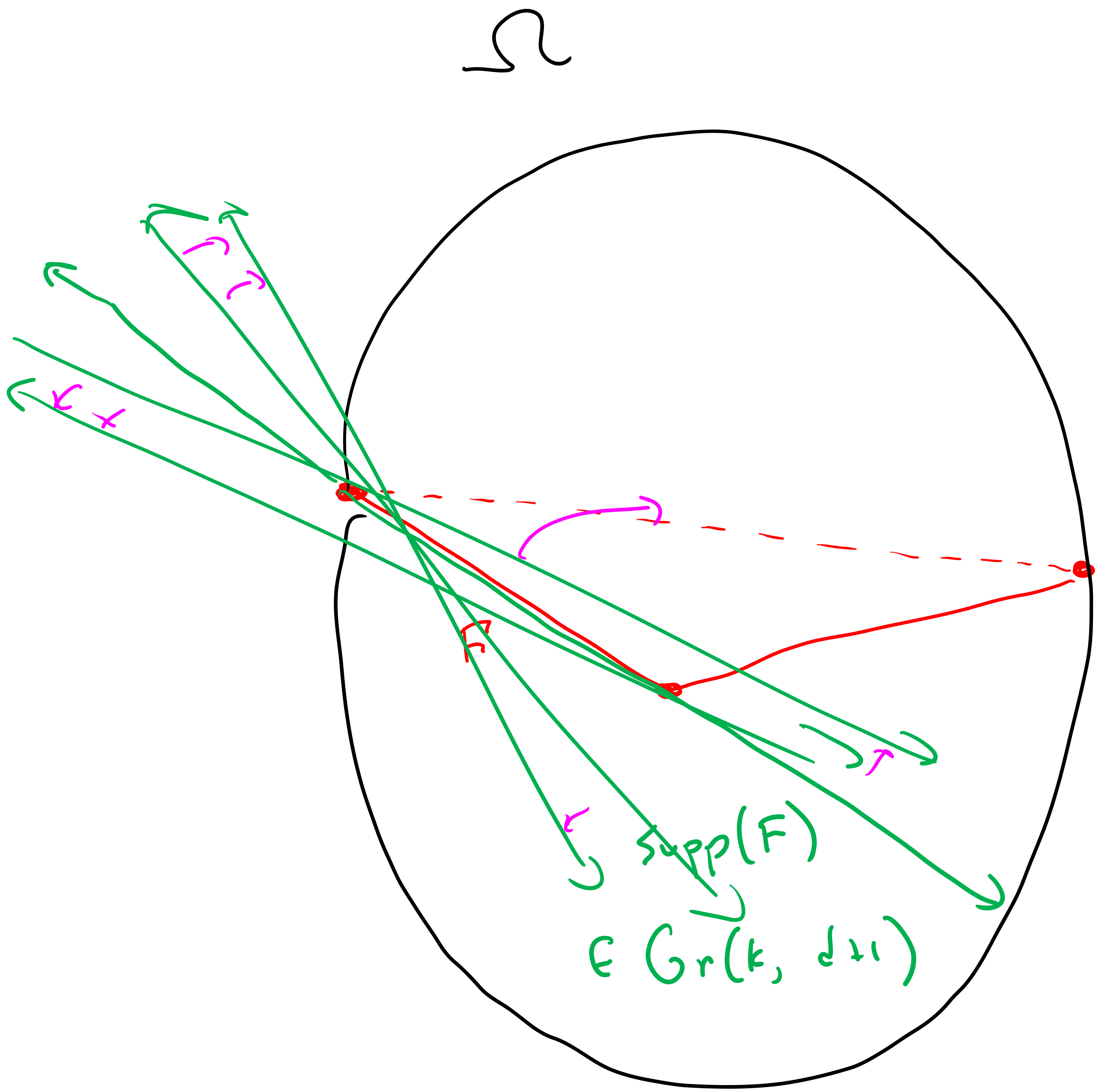
$\Gamma \subset PGL(3, \mathbb{R})$

$$\Gamma : \left\{ \begin{pmatrix} 2^a & & \\ & 2^b & \\ & & 1 \end{pmatrix} : a, b \in \mathbb{Z} \right\}$$

quotient is a torus.

- $\Gamma \cong \mathbb{Z}^2$ which is <u>not</u> Gromov-hyperbolic
- Γ does <u>not</u> act w/ expansion dynamics on $\partial\Delta$.

Expansion on Grassmannians:



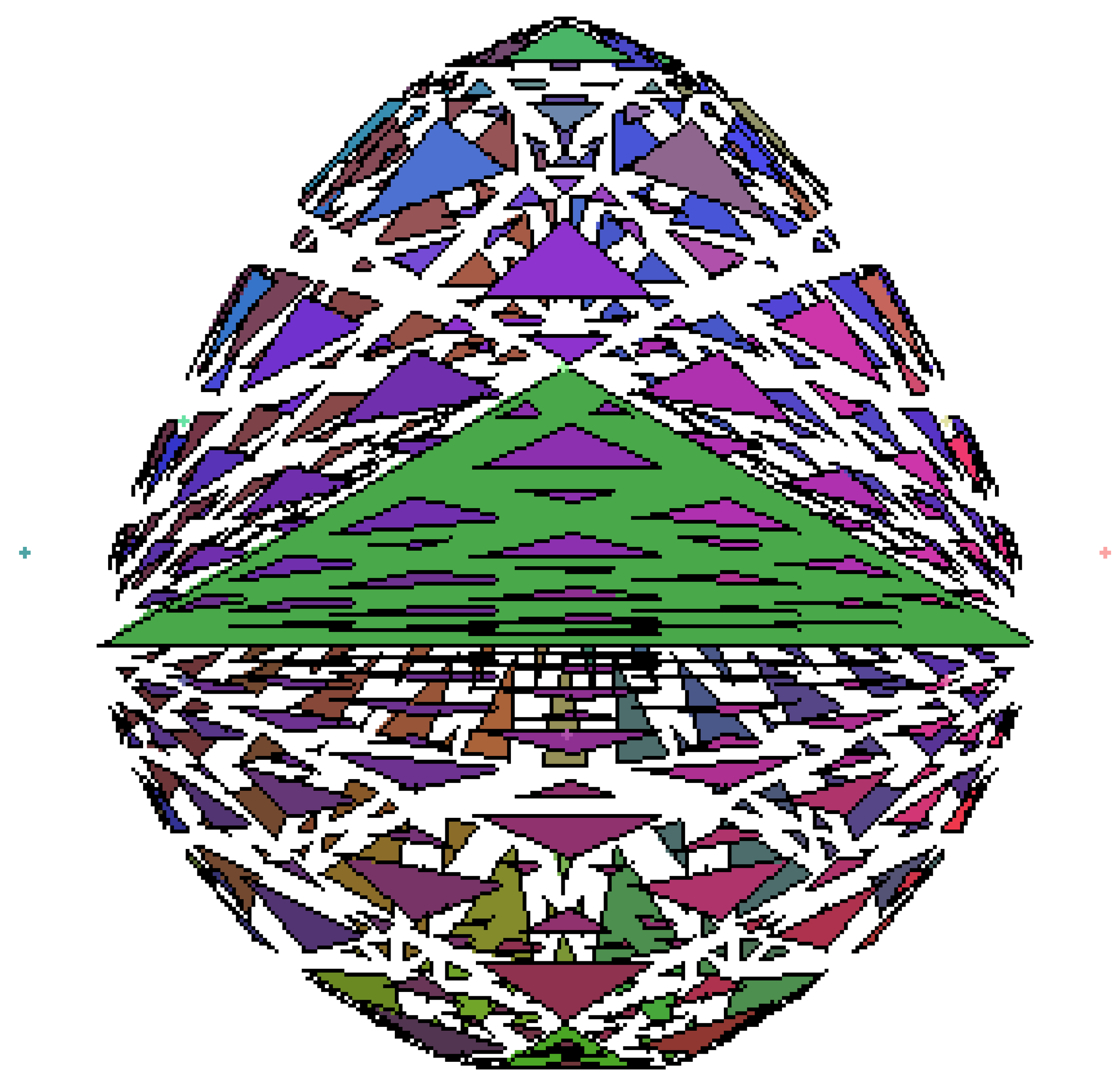
Thm (W.):

Ω properly convex domain, $\Gamma \subset PGL(d+1, \mathbb{R})$ discrete, preserves Ω.

Γ acts cocompactly on Ω $\iff$ Γ acts with uniform expansion dynamics on the faces of Ω: for each face $F \subset \partial\Omega$, there exists $\gamma \in \Gamma$ expanding in a nbhd. of $\mathrm{supp}(F)$ in $Gr(k, d+1)$, where $k = \dim(F)$.

(also a version for manifolds w/ bdry)

A 3-dim example (Benoist):



Recovering a boundary embedding:

Γ hyperbolic relative to free abelian subgroups. Has a natural Bowditch boundary.

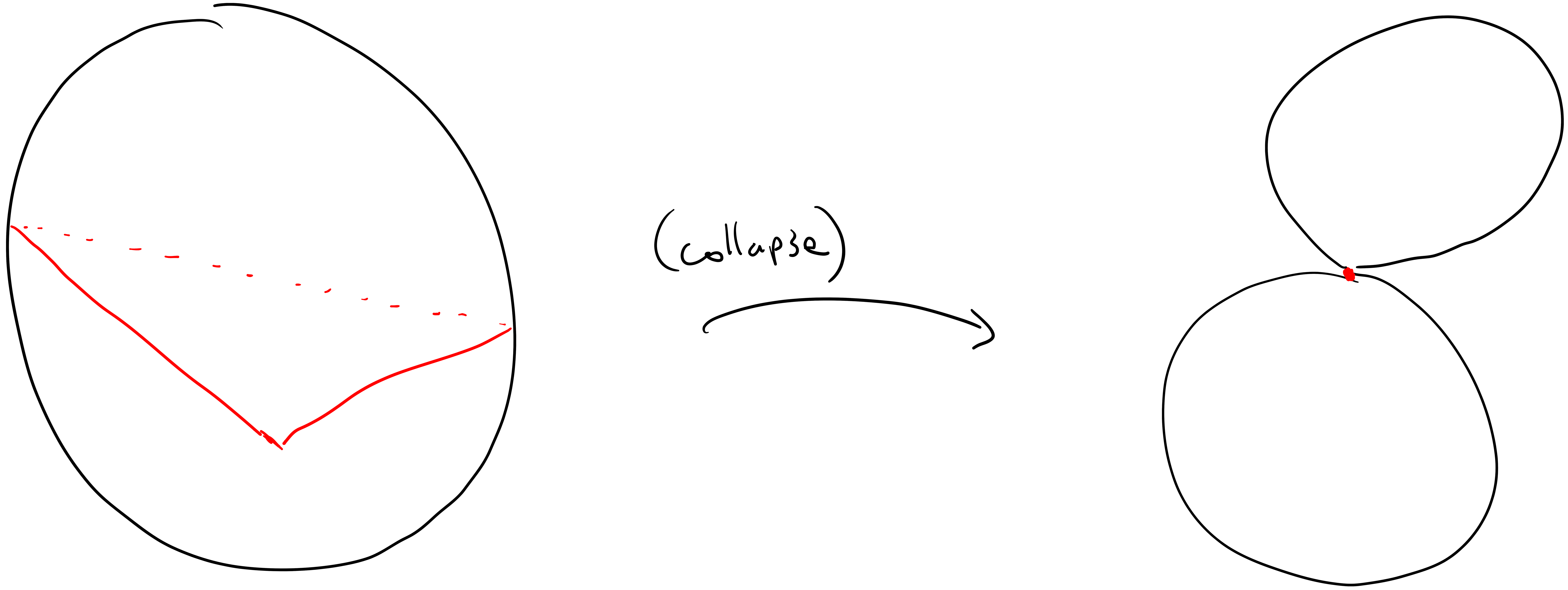
Ex: $\Gamma \subset PO(3,1)$ is holonomy of a finite-volume hyperbolic 3-manifold.

Cusp groups = peripheral subgroups

Bowditch boundary = $\partial \mathbb{H}^3 \cong S^2$

Thm (W.): $\Gamma \subset PGL(d+1, \mathbb{R})$ discrete, preserves Ω, hyperbolic relative to free abelian subgroups.

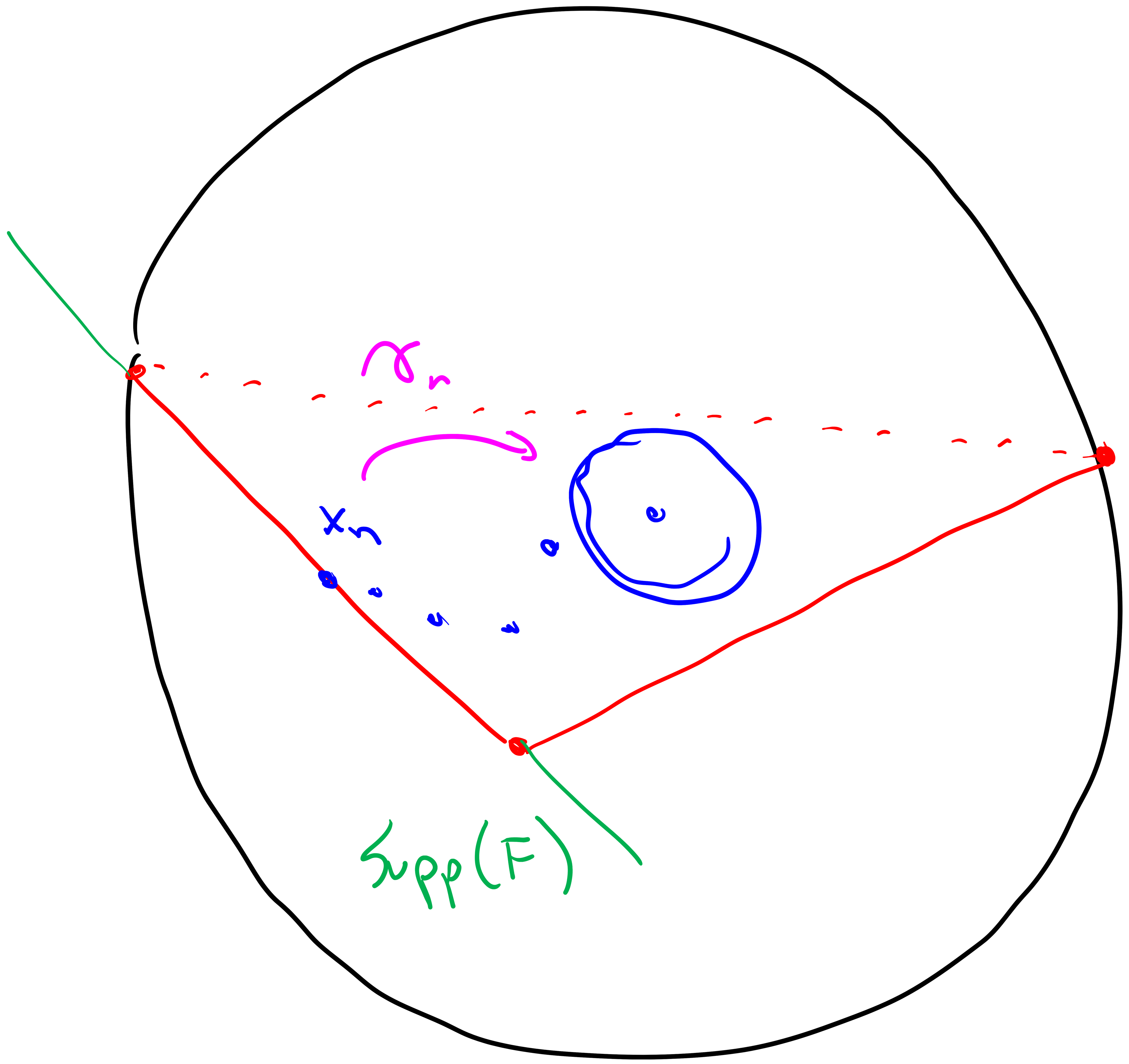
Ω/Γ compact $\iff$ $\exists$ equivariant homeomorphism from $\partial_B \Gamma$ to $\partial\Omega/\sim$



Each $\mathbb{Z}^k$ subgroup fixes a simplex in Ω

Idea: use result of Yaman: relatively hyperbolic group actions on Bowditch boundary are characterized by topological dynamics.

use expansion/contraction dynamics on quotient to see that it must be Bowditch boundary.



Find γ_n so $\gamma_n x_n \in K$ for compact fundamental domain K.

$\gamma_n = k_n g_n$, $k_n \in$ compact subset of PGL

g_n

Thank you for listening!