Limit sets in higher-dimensional hyperbolic spaces

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Hyperbolic geometry is one of the most basic (and yet most interesting) examples of a *curved* or *non-Euclidean* geometry; for centuries, mathematicians dismissed it as an impossibility, but its study has become central to the modern understanding of two- and three-dimensional manifolds. This is because it is possible to work with many interesting low-dimensional spaces by relating them to certain nice *subgroups* of the isometry group of n-dimensional hyperbolic space $\mathbb{H}^n$.

Any group Γ of isometries of $\mathbb{H}^n$ has an associated *limit set*: a (typically) fractal-like object embedded into the "boundary at infinity" of hyperbolic space, which encodes the nature of the "limiting behavior" of Γ. Limit sets of isometry groups can often be very wild, but much is still unknown about the sorts of spaces which can appear this way—especially in the boundary of $\mathbb{H}^n$ when $n > 3$.

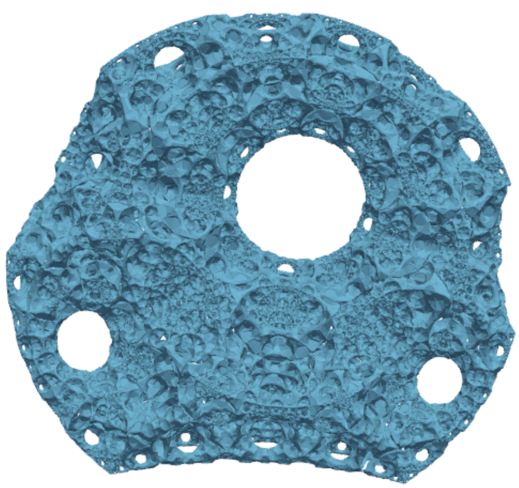

Figure 1: A limit set appearing in the boundary of $\mathbb{H}^4$, homeomorphic to a *Pontryagin sphere*.

In this project, our aim will be to produce visualizations of some interesting limit sets in the boundary of 4-dimensional hyperbolic space (this boundary can be identified with a 3-dimensional sphere); these limit sets are associated to subgroups very recently discovered by Douba–Lee–Marquis–Ruffoni. No previous exposure to hyperbolic spaces should be required—we will spend much of the semester learning the basics of hyperbolic geometry, using both theoretical and computational approaches.

Prerequisites

- Should be very comfortable with linear algebra (we may have to spend some time learning linear algebra topics beyond what is covered in Math 217)
- Group theory (Math 412 or 493 or equivalent)
- Some programming experience (Python would be ideal, but C/C++, Matlab, Mathematica, etc. all fine)
- Useful, but not required: knowledge of basic algebraic topology (fundamental groups and covering spaces), some exposure to the theory of manifolds.