M427L: Exam 1 review

Note: these are arranged basically in order of material covered. **However**, many questions from later sections *rely on knowledge from earlier sections*, so questions towards the end of this review sheet are (mostly) *harder* than the ones more towards the beginning.

Chapter 1

1. Find the area of a triangle with vertices at the points $(0, -1, 2)$, $(-1, -1, -2)$, and $(4, 1, 0)$.

2. Suppose that P is the point $(-1, 0, 4)$, $\vec{v}$ is the vector $\langle 2, 1, 1\rangle$ and L is the line passing through P given by $\{P + t\vec{v} : t \in \mathbb{R}\}$.

   Find an equation for the set of points $Q = (x, y, z)$ such that the angle between $\vec{PQ}$ and L is $\pi/4$.

3. Write down an equation for the plane $z = x$ in spherical and cylindrical coordinates.

4. Sketch the polar curve $r = \theta + 1$ for $0 \le \theta \le \pi/2$.

Chapter 2

1. Draw level curves for the function $f(x, y) = (100 - x^2 - y^2)^{1/2}$ for $c = 0, 4, 8, 10$. Then sketch the graph $z = f(x, y)$.

2. Compute the limit $\lim_{x,y\to(0,0)} \frac{x^2y-y^2}{y^2-x^4}$, or show that it does not exist.

3. Where is the function $\frac{1}{\sqrt{x^2-y^2}}$ continuous? (**Note:** the mathematically precise way to ask this question is to say "what is the maximal subset of $\mathbb{R}^2$ on which the formula $\frac{1}{\sqrt{x^2-y^2}}$ defines a continuous function?" If you don't like this version of the question, you can ignore it.)

4. Find the gradient of the function $f(x, y) = x\cos(y) - \sin(xy)$.

5. In *cylindrical coordinates*, the equation $z = e^r + r^2$ gives a surface in $\mathbb{R}^3$. Find an equation in *Cartesian coordinates* for the tangent plane to this surface, at the point P with *cylindrical coordinates* given by $(r = 1,\ \theta = \pi/6,\ z = e + 1)$.

6. Let C be the curve given by the intersection of the plane with equation $z = 4x - 5y$ with the sphere given by the equation $x^2 + y^2 + z^2 = 1$. Describe the curve C, and give an equation for the *tangent line* to C at a point on C where $x = 1/2$.

7. Use the chain rule to find the derivative $\frac{\partial f}{\partial s}$ when $f(x, y) = \cos(xy) - xy^2$, if $x = 3\cos(t) + \sin(s) - t$ and $y = 2s - t$.

8. Let $\vec{v}$ be the vector $\langle -1, 2, 0\rangle$, let $\vec{u} = \langle 0, 1, 1\rangle$, and let $\vec{w}$ be the vector $\langle 4, 4, -6\rangle$.
   If S is the *parameterized* surface given by
   $$S = t\vec{v} + s\vec{u} + s^2\vec{w}$$
   for $s, t \in \mathbb{R}$, find the *normal vector* to S at the point where $t = 1, s = 1$.

9. Let $f(x, y)$ be a function so that $\nabla f = \langle -1, 4\rangle$ at the point $(0, 1)$, and $f(0, 1) = -3$. Let $g(x, y) = e^{xy}$. Compute the directional derivative of $g(x, y) \cdot f(x, y)$ at $(0, 1)$, in the direction of the vector $\langle -1, -1\rangle$.

Chapter 3

1. Compute all of the second order partial derivatives for the function $f(x, y) = x - y^2\cos(x) + y^x$. Then write down a *second-order* approximation for $f(x, y)$ at the point $x = y = 1$ (i.e. a function of the form $g(x, y) = ax^2 + bxy + cy^2 + ex + dy + k$ which approximates f).

2. Find all the critical points of the function $f(x, y) = xy + 1/x + 1/y$ and determine if these critical points are local maxima, local minima, or saddle points.

3. Find all maxima and minima on the graph $z = (x^2 + 3y^2)e^{1-x^2-y^2}$.

4. Find the point on the plane $2x - y + 2z = 20$ nearest to the origin.

5. Find the extrema of $f(x, y) = 4x + 2y$, subject to the constraint $2x^2 + 3y^2 = 21$.

6. Find the maximum and minimum values of the function $g(x, y, z) = xyz$ on the unit ball $x^2 + y^2 + z^2 \leq 1$.

7. Let S be the sphere of radius 1 centered at $(1, 2, 3)$. Find the distance from S to the plane $x + y + z = 0$. (You will need to use Lagrange multipliers, and know the formula for the distance from a point to a plane).