M427L: Exam 2 review

Chapter 4

1. The acceleration, initial velocity, and initial position of a particle traveling through space are given by
$$\vec{a}(t) = \langle 2, -6, -4\rangle, \quad \vec{v}(0) = \langle -5, 1, 3\rangle, \quad \vec{r}(0) = (6, -2, 1).$$
The particle's path intersects the the yz plane at exactly two points. Find those two points.

   Solution We integrate the x, y, z components separately to first find velocity as a function of time:
$$\begin{aligned}\vec{v}(t) &= v(0) + \int_0^t \vec{a}(s)ds \\ &= \langle -5 + 2t, 1 - 6t, 3 - 4t\rangle.\end{aligned}$$
   Then we integrate again to find position:
$$\begin{aligned}\vec{r}(t) &= \vec{r}(0) + \int_0^t \vec{v}(s)\ ds \\ &= \langle 6 + \int_0^t (-5 + 2s)ds, -2 + \int_0^t (1 - 6s)ds, 1 + \int_0^t (3 - 4s)ds\rangle \\ &= \langle 6 - 5t + t^2, -2 + t - 3t^2, 1 + 3t - 2t^2\rangle.\end{aligned}$$
   This path intersects the yz plane when $x = 0$, so we solve $t^2 - 5t + 6 = 0$ by factoring $(t - 2)(t - 3) = 0$ yielding $t = 2, t = 3$. Plugging in, we see that the two points we want are
$$(0, -12, -1), \qquad (0, -26, -8).$$

2. If $c(t)$ is the *helix* $c(t) = (\cos t, \sin t, 4t)$, find a function $\ell(s)$ representing the length of the curve c from $t = 0$ to $t = s$.

   Solution The length of a curve c from $t = 0$ to $t = s$ is given by
$$\int_0^s ||c'(t)||dt.$$
   We find
$$c'(t) = \langle -\sin t, \cos t, 4\rangle,$$
   so $||c'(t)|| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 16} = \sqrt{17}$. So we have
$$\ell(s) = \int_0^s \sqrt{17}\ dt = s\sqrt{17}.$$

3. Sketch a vector field whose curl is not the zero function *and* whose divergence is not the zero function. Write down an equation for a vector field (possibly not the same one) which satisfies the same properties.

Solution There are a lot of ways to do this one. Here's one nice observation: if F_1, F_2 are two different vector fields, then $\nabla \cdot (F_1 + F_2) = \nabla \cdot F_1 + \nabla \cdot F_2$, and similarly for curl. So we can pick vector fields F_1, F_2 such that $\nabla \cdot F_1 \neq 0$, $\nabla \times F_2 \neq 0$, and $\nabla \cdot F_2 = 0$ and $\nabla \times F_1 = 0$, and then take $F = F_1 + F_2$.

One easy choice is $F_1 = x\mathbf{i} + y\mathbf{j}$ and $F_2 = y\mathbf{i} - x\mathbf{j}$. Then take $F = F_1 + F_2 = (x+y)\mathbf{i} + (y-x)\mathbf{j}$.

4. Write down a formula for $\nabla \cdot (f\vec{F})$, where $f : \mathbb{R}^3 \to \mathbb{R}$ is a function and $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is a vector field. (You can write this down in terms of f and its partial derivatives, and $\vec{F} = (F_1, F_2, F_3)$ and the partial derivatives of these quantities).

Solution Write $\vec{F} = (F_1, F_2, F_3)$, so $f\vec{F} = (fF_1, fF_2, fF_3)$. Then we have

$$\begin{aligned}\nabla \cdot (f\vec{F}) &= \frac{\partial fF_1}{\partial x} + \frac{\partial fF_2}{\partial y} + \frac{\partial fF_3}{\partial z} \\ &= \frac{\partial f}{\partial x}F_1 + f\frac{\partial F_1}{\partial x} + \frac{\partial f}{\partial y}F_2 + f\frac{\partial F_2}{\partial y} + \frac{\partial f}{\partial z}F_3 + f\frac{\partial F_3}{\partial z}.\end{aligned}$$

This is a good enough answer, but we can do better. Reorganizing terms, this is the same as

$$\frac{\partial f}{\partial x}F_1 + \frac{\partial f}{\partial y}F_2 + \frac{\partial f}{\partial z}F_3 + f\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) = (\nabla f) \cdot F + f\nabla \cdot F.$$

Chapter 5

1. Evaluate the integral

$$\iint_R (xy)^2 \cos x^3 \; dA,$$

where R is the rectangle $[0, \pi] \times [0, 1]$.

Solution

$$\begin{aligned}\int_0^\pi \int_0^1 (xy)^2 \cos x^3 \; dy\,dx &= \int_0^\pi x^2 \cos x^3 \frac{y^3}{3}\bigg|_{y=0}^{y=1} dx \\ &= \frac{1}{3}\int_0^\pi x^2 \cos x^3 \; dx.\end{aligned}$$

Using the substitution $u = x^3$, $du = 3x^2 dx$, we rewrite this as

$$\frac{1}{3}\int_{x=0}^{x=\pi} \frac{1}{3} \cos u \; du = \frac{1}{9}\sin(u)\bigg|_{x=0}^{x=\pi} = \frac{1}{9}\sin(\pi^3).$$

2. Let D be the region of $\mathbb{R}^2$ given by the half-disk centered at $(0, 2)$ with radius 1, to the right of the y-axis. Evaluate the integral

$$\iint_D (y-2) \cdot x \; dA.$$

Solution We first need to set up our bounds. In this region, the y-values vary from 1 to 3. The circle giving part of the boundary of the disk has equation $x^2 + (y-2)^2 = 1$. Solving for x, we get $x = \sqrt{1-(y-2)^2}$. So we get:

$$\begin{aligned}
&\int_1^3 \int_0^{\sqrt{1-(y-2)^2}} (y-2)x \, dx \, dy \\
&= \int_1^3 (y-2)\frac{x^2}{2}\Big|_{x=0}^{x=\sqrt{1-(y-2)^2}} dy \\
&= \frac{1}{2}\int_1^3 (y-2)(1-(y-2)^2) \, dy.
\end{aligned}$$

To make our life a little easier we use the substitution $u = (y-2)$ and get

$$\begin{aligned}
\frac{1}{2}\int_{-1}^1 u(1-u^2) \, dy &= \frac{1}{2}\int_{-1}^1 u - u^3 \, du \\
&= \frac{1}{2}\left(\frac{u^2}{2} - \frac{u^4}{4}\right)\Big|_{-1}^1 = 0.
\end{aligned}$$

3. Let R be the region in $\mathbb{R}^3$ bounded by the coordinate planes (the xy, yz, and xz planes) and the plane $2x + 2y + z = 5$. Evaluate the integral

$$\iiint_R x^2 z - 2yz^2 \, dV.$$

Solution First we need to set up our bounds of integration. We'll integrate $dxdydz$, so z bounds come first; our region of integration lies in the strip $0 \le z \le 5$. For a fixed z value, our region of integration lies in the strip $0 \le y \le \frac{5-z}{2}$, and for fixed y and z, we integrate over the interval $0 \le x \le \frac{5-z-2y}{2}$.

So we want to integrate

$$\begin{aligned}
\int_0^5 \int_0^{\frac{5-z}{2}} \int_0^{\frac{5-z-2y}{2}} x^2 z - 2yz^2 \, dxdydz &= \int_0^5 \int_0^{\frac{5-z}{2}} \left(\frac{x^3}{3}z - 2xyz^2\right)\Big|_{x=0}^{x=\frac{5-z-2y}{2}} dydz \\
&= \frac{1}{3}\int_0^5 \int_0^{\frac{5-z}{2}} \left(\frac{5-z-2y}{2}\right)^3 - 2yz^2\frac{5-z-2y}{2} \, dydz.
\end{aligned}$$

We'll split this up into two integrals. For the first integral, we make the substitution $u = 5 - z - 2y$, $du = 2dy$:

$$\begin{aligned}
\frac{1}{3}\int_0^5 \int_{y=0}^{y=\frac{5-z}{2}} \frac{1}{16}u^3 \, dudz &= \frac{1}{192}\int_0^5 u^4\Big|_{u=5-z}^{u=0} dz \\
&= \frac{-1}{192}\int_0^5 (5-z)^4 \, dz.
\end{aligned}$$

Then we make the substitution $u = 5 - z$, $dz = -du$ to write this as

$$\frac{-1}{192}\int_5^0 -u^4\; du = \frac{-1}{192 \cdot 5}u^5\bigg|_0^5 = \frac{-5^4}{192}.$$

For the second integral, we multiply through to get

$$\begin{aligned}
\frac{-1}{3}\int_0^5\int_0^{\frac{5-z}{2}} z^2 y(5-z) - 2y^2\; dy\, dz &= \frac{-1}{3}\int_0^5 \left(z^2(5-z)\frac{y^2}{2} - \frac{2}{3}y^3\right)\bigg|_{y=0}^{y=\frac{5-z}{2}} dz \\
&= -\frac{1}{3}\int_0^5 z^2(5-z)\frac{(5-z)^2}{8} - \frac{2}{3}\frac{(5-z)^3}{8}\; dz \\
&= -\frac{1}{24}\int_0^5 z^2(125 - 75z + 15z^2 - z^3) - \frac{2}{3}(5-z)^3\; dz \\
&= -\frac{1}{24}\left(\frac{125}{3}z^3 - \frac{75}{4}z^4 + 3z^5 + \frac{5}{2}z^6 - \frac{1}{7}z^7 + \frac{1}{4}(5-z)^4\right)\bigg|_0^5.
\end{aligned}$$

This can be simplified to get a fraction, which I will leave for you to do.

4. Evaluate the integral

$$\int_0^4\int_{y/2}^2 e^{x^2}\; dx\, dy$$

by changing the order of integration.

Solution After inspection we see that the region of integration is a triangle with vertices $(0,0)$, $(2,4)$, and $(2,0)$. So our x-bounds are $0, 2$, and for a fixed x-value, y ranges from 0 to $2x$:

$$\int_0^2\int_0^{2x} e^{x^2}\; dy\, dx = \int_0^2 e^{x^2} 2x\; dx.$$

We make the substitution $u = x^2$ to write this as

$$\begin{aligned}
\int_{x=0}^{x=2} e^u\; du &= e^u\bigg|_{x=0}^{x=2} \\
&= e^u\bigg|_0^4 = e^4 - 1.
\end{aligned}$$

Chapter 6

1. Let $T(u,v) = (u^2 - v^2, 2uv)$, and let D' be the region of $\mathbb{R}^2$ given by $\{(u,v) : u^2 + v^2 \leq 1, u \geq 0, v \geq 0\}$. Describe the region $D = T(D')$, and evaluate

$$\iint_D dx\, dy.$$

Solution The region D' is bounded by three curves: the line segment L_1 from $(0,0)$ to $(1,0)$, the line segment L_2 from $(0,0)$ to $(0,1)$, and the quarter-circle A given by the intersection of the unit circle with the first quadrant. Our region D should also be bounded by three curves, which we can find by applying the map T to these three curves.

On L_1, v is identically zero, and u ranges from 0 to 1. So, the y-coordinate of $T(L_1)$ is $0 \cdot u = 0$, and the x-coordinate $u^2 - v^2$ ranges from 0 to 1. So $T(L_1)$ is the line segment from $(0,0)$ to $(1,0)$.

Similarly, $T(L_2)$ is the line segment from $(0,0)$ to $(-1,0)$.

$T(A)$ must be some curve from $(-1,0)$ to $(1,0)$. In fact, $T(A)$ lies on the unit circle! To see this, note that if $u^2 + v^2 = 1$, then

$$\begin{aligned} x^2 + y^2 &= (u^2 - v^2)^2 + (2uv)^2 = u^4 - 2u^2v^2 + v^4 + 4u^2v^2 \\ &= u^4 + 2u^2v^2 + v^4 = (u^2 + v^2)^2 \\ &= 1. \end{aligned}$$

So $T(A)$ lies on the curve $x^2 + y^2 = 1$, and since $u, v \geq 0$, $y = 2uv \geq 0$.

So, the region D is the half of the unit circle lying in the half-plane $y \geq 0$.

The integral

$$\iint_D dxdy$$

is just the area of D, so it evaluates to $\pi/2$.

Alternatively, we can evaluate this integral using the change-of-coordinates theorem: we know that

$$\iint_D dx\, dy = \iint_{D'} \left| \frac{\partial T}{\partial(u,v)} \right| du\, dv.$$

Here $\dfrac{\partial T}{\partial(u,v)}$ is the Jacobian (matrix of partials)

$$\begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} 2u & -2v \\ 2v & 2u \end{pmatrix},$$

so $\left| \dfrac{\partial T}{\partial(u,v)} \right| = 4(u^2 + v^2)$.

To evaluate

$$\iint_{D'} 4(u^2 + v^2)\, du\, dv,$$

we can switch to polar coordinates, and rewrite this as

$$\iint_{D'} 4r^2\, r\, dr\, d\theta.$$

Since we are integrating over one quarter of the unit disk, we let θ vary from 0 to $\pi/2$, let r range from 0 to 1, and get

$$\int_0^{\pi/2}\int_0^1 4r^3\,dr\,d\theta = \int_0^{\pi/2} r^4\Big|_0^1 d\theta$$
$$= \int_0^{\pi/2} d\theta = \pi/2.$$

2. Let A be the annulus $\{(x,y) : 1 \le (x^2+y^2) \le 4\}$. Find the integral

$$\iint_A xy + y^2\;dx\,dy.$$

Solution We switch to polar coordinates ($x = r\cos\theta$, $y = r\sin\theta$) and use the change-of-coordinates rule $dx\,dy = r\,dr\,d\theta$, to write the integral as

$$\int_0^{2\pi}\int_1^2 (r^2\cos\theta\sin\theta + r^2\sin^2\theta)r\;drd\theta$$
$$= \int_0^{2\pi} \frac{r^3}{3}\Big|_1^2 \cdot (\cos\theta\sin\theta + \sin^2\theta)d\theta$$
$$= \frac{7}{3}\int_0^{2\pi} (\cos\theta\sin\theta + \sin^2\theta)d\theta.$$

Using the identities $\sin(2\theta) = 2\cos\theta\sin\theta$ and $\cos(2\theta) = 1 - 2\sin^2\theta$, we rewrite this as

$$\frac{7}{3}\int_0^{2\pi} \frac{\sin(2\theta)}{2} + \frac{1-\cos 2\theta}{2}d\theta.$$

The $\sin 2\theta$ and $\cos 2\theta$ terms disappear when we integrate (since we are integrating from 0 to 2π), so we are left with

$$\frac{7}{3}\int_0^{2\pi}\frac{1}{2}d\theta = \frac{7\pi}{3}.$$

3. Find the volume of the solid in $\mathbb{R}^3$ bounded below by the paraboloid $z = x^2 + y^2$ and above by the cone $z = \sqrt{x^2+y^2}$.

Solution We switch to cylindrical coordinates, where the paraboloid is given by $z = r^2$ and the cone is given by $z = r$. These two surfaces intersect at the circle given by the equations $r = z = 1$, so the solid lies above the unit disk D in $\mathbb{R}^2$. So, the volume of the solid is given by

$$\iint_D r - r^2\;dA = \iint_D (r - r^2)\;r\,dr\,d\theta.$$

To parameterize the unit disk we let r vary from 0 to 1 and θ vary from 0 to 2π, giving

$$\int_0^{2\pi}\int_0^1 r^2 - r^3\;dr\,d\theta = \int_0^{2\pi}\left(\frac{r^3}{3} - \frac{r^4}{4}\right)\Big|_0^1 d\theta$$
$$= \frac{1}{12}\int_0^{2\pi} d\theta = \frac{\pi}{6}.$$

4. Let E be the ellipsoid

$$\frac{x^2}{2} + \frac{y^2}{3} + z^2 \leq 1.$$

Evaluate the integral

$$\iiint_E \frac{xy + z}{3} \, dV.$$

Solution We apply two changes of coordinates: first we transform this into an integral over the unit ball, and then use spherical coordinates to evaluate that integral.

We pick coordinates u, v, w so that $x^2/2 = u^2, y^2/3 = v^2$, and $z^2 = w^2$. Then, in (u, v, w) coordinates, the ellipsoid E is just the set $\{(u, v, w) : u^2 + v^2 + w^2 \leq 1\}$, or the unit ball B. Explicitly, we have

$$x = \sqrt{2} \cdot u, \quad y = \sqrt{3} \cdot v, \quad z = w.$$

Then, the change-of-coordinates theorem says that

$$\iiint_E \frac{xy + z}{3} \, dV = \iiint_B \frac{\sqrt{6}uv + z}{3} \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| \, du \, dv \, dw.$$

The Jacobian matrix is given by

$$\begin{pmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{pmatrix} = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

so the Jacobian determinant is $\sqrt{6}$, meaning we want to evaluate the integral

$$\sqrt{6} \iiint_B \frac{\sqrt{6}uv + w}{3} \, du \, dv \, dw.$$

To do this integral we switch to spherical coordinates. This is most convenient if we take $u = \rho \sin\phi \cos\theta$, $w = \rho \sin\phi \sin\theta$, $v = \rho \cos\phi$. In these coordinates, we have $du \, dv \, dw = \rho^2 \sin\phi d\rho \, d\phi \, d\theta$, meaning we are integrating

$$\frac{\sqrt{6}}{3} \iiint_B \left(\sqrt{6}\rho^2 \sin\phi \cos\phi \cos\theta + \rho^2 \sin\phi \sin\theta \right) \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi.$$

We can save ourselves some trouble by integrating with respect to θ first, since we want to take:

$$\frac{\sqrt{6}}{3} \int_0^1 \int_0^\pi \int_0^{2\pi} \left(\sqrt{6}\rho^2 \sin\phi \cos\phi \cos\theta + \rho^2 \sin\phi \sin\theta \right) \rho^2 \sin\phi \, d\theta \, d\phi \, d\rho$$

Since the integral of $\sin\theta$ and $\cos\theta$ from zero to 2π is zero, the whole integral vanishes and we just get 0.

Actually, we could have seen this without doing any work at all! The ellipsoid E is symmetric about the reflections $z \mapsto -z$, $x \mapsto -x$, and $y \mapsto -y$. We let $E^{z \geq 0}$ be the half-ellipsoid we get by intersecting E with the half-space $z \geq 0$. Similarly, we define $E^{z\leq 0}$ and $E^{x\geq 0}$ and $E^{x\leq 0}$. Then our original integral splits into four pieces:

$$\iiint_E \frac{xy+z}{3}\,dV = \iiint_{E^{x\geq 0}} \frac{xy}{3}\,dV + \iiint_{E^{x\leq 0}} \frac{xy}{3}\,dV + \iiint_{E^{z\geq 0}} \frac{z}{3}\,dV + \iiint_{E^{z\leq 0}} \frac{z}{3}\,dV.$$

Using the change-of-variables theorem, you can arrange the first two integrals to cancel with each other (and the last two as well).

Chapter 7

1. Evaluate the path integral

$$\int_c f(x,y,z)\,ds$$

where $c : [0,\pi] \to \mathbb{R}^3$ is the curve $t \mapsto (\sin t, \cos t, t)$ and $f(x,y,z) = x+y+z$.

Solution The path integral is given by

$$\int_0^\pi f(c(t))||c'(t)||\,dt,$$

so we compute

$$\begin{aligned} ||c'(t)|| &= ||\langle \cos t, -\sin t, 1\rangle|| \\ &= \sqrt{\cos^2 t + \sin^2 t + 1} \\ &= \sqrt{2}. \end{aligned}$$

So we want:

$$\begin{aligned} &\sqrt{2}\int_0^\pi (\sin t + \cos t + t)\,dt \\ &= \sqrt{2}\left(-\cos t + \sin t + t^2/2\right)\Big|_0^\pi \\ &= \sqrt{2}\left(2 + \pi^2/2\right). \end{aligned}$$

2. Let C be the boundary of the unit square $[0,1]\times[0,1]$, oriented counterclockwise, and let F be the vector field $y^2\mathbf{i} - xy\mathbf{j}$. Evaluate the line integral

$$\int_C F \cdot d\mathbf{r}.$$

Solution We need to parameterize the curve C, which is cut into four pieces C_1, C_2, C_3, C_4 corresponding to the edges of the square. We'll start at the origin, which means that if we parameterize each of our curves on the unit interval $[0,1]$, we have:

$$\begin{aligned} C_1(t) &= (t,0) \\ C_2(t) &= (1,t) \\ C_3(t) &= (1-t,1) \\ C_4(t) &= (0,1-t). \end{aligned}$$

Then $C_1'(t) = (1, 0)$, $C_2'(t) = (0, 1)$, $C_3'(t) = (-1, 0)$, and $C_4'(t) = (0, -1)$. We have four integrals to evaluate.

$$\int_0^1 F(C_1(t)) \cdot C_1'(t)\, dt = \int_0^1 (0, 0) \cdot (1, 0)\, dt = 0$$
$$\int_0^1 F(C_2(t)) \cdot C_2'(t)\, dt = \int_0^1 (t^2, t) \cdot (0, 1)\, dt = \int_0^1 t\, dt = \frac{1}{2}$$
$$\int_0^1 F(C_3(t)) \cdot C_3'(t)\, dt = \int_0^1 (1, 1-t) \cdot (-1, 0)\, dt = \int_0^1 -1\, dt = -1$$
$$\int_0^1 F(C_4(t)) \cdot C_4'(t)\, dt = \int_0^1 ((1-t)^2, 0) \cdot (0, -1)\, dt = 0.$$

The line integral is given by the sum of these four line integrals, meaning that $\int_C F \cdot d\mathbf{r} = -1/2$.