M427L: final review

Note: this review sheet only has problems from the *last* part of the semester (so it does *not* have any material covered on the previous review sheets). You are still responsible for *all* of the material from this class on the final, so you may want to go back and look at earlier review sheets!

Chapter 7

1. Determine if the vector field

$$\vec{F}(x,y,z) = 2ye^{2xy}\vec{i} + (2xe^{2xy} + 6yz)\vec{j} + 3y^2\vec{k}$$

is the gradient of some function $f : \mathbb{R}^3 \to \mathbb{R}$. If it is, find an f so that $\vec{F} = \nabla f$.

Solution We can evaluate the curl of this vector field:

$$\nabla \times \vec{F} = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2ye^{2xy} & 2xe^{2xy} + 6yz & 3y^2 \end{pmatrix}$$

This evaluates to $\vec{i}(6y - 6y) + \vec{j}(0) + \vec{k}(2e^{2xy} + 4xye^{2xy} - 2e^{2xy} - 4xye^{2xy}) = 0$. Since the vector field is defined on all of $\mathbb{R}^3$, it must be a gradient field.

To find a function f so $\vec{F} = \nabla f$, we first integrate in a straight line from $(0,0,0)$ to $(x,0,0)$, then from $(x,0,0)$ to $(x,y,0)$, then from $(x,y,0)$ to (x,y,z).

The first integral is

$$\int_0^x 2ye^{2ty}dt,$$

but as $y = 0$ this evaluates to zero. The second integral is

$$\int_0^y (2xe^{2xt} + 6yz)dt.$$

As $z = 0$ we can ignore the second term. We treat x as a constant, so the antiderivative is e^{2xt} and the integral evaluates to $e^{2xy} - 1$. The third integral is

$$\int_0^z 3y^2 dt,$$

which evaluates to $3y^2z$. So altogether we get $f(x,y,z) = e^{2xy} + 3y^2z - 1$. (We can discard the constant term if we want.)

2. Let C be the curve given by the graph of the function $y = 2x^2$, where x ranges from 0 to 1, and let $\vec{F}$ be the vector field

$$\vec{F}(x,y) = (x^2 - y^2)\vec{i} + (x^2 + y^2)\vec{j}.$$

Evaluate the line integral

$$\int_C \vec{F} \cdot dr.$$

Solution The curve C is the graph of a function, so we can parameterize it as

$$r(x) = (x, 2x^2),$$

as x varies from 0 to 1. Then $r'(x) = (1, 4x)$. The line integral is given by

$$\begin{aligned}\int_0^1 \vec{F}(x) \cdot r'(x) dx &= \int_0^1 (x^2 - 4x^4) + (x^2 + 4x^4) 4x dx \\ &= \int_0^1 x^2 - 4x^4 + 4x^3 + 16x^5 dx \\ &= x^3/3 - 4x^5/5 + x^4 + 16x^6/6 \Big|_0^1 \\ &= 1/3 - 4/5 + 1 + 8/3 = 16/5.\end{aligned}$$

3. Find the area of the surface S given by the intersection of the unit sphere with the cone $z \geq \sqrt{3(x^2 + y^2)}$.

   Solution We can parameterize this region using spherical coordinates. The angle between the boundary of the cone $z \geq \sqrt{3(x^2+y^2)}$ and the z-axis is given by $\tan^{-1}(1/\sqrt{3}) = \pi/6$. In spherical coordinates (ϕ, θ), the area element of a unit sphere is given by $\sin\phi \; d\phi \; d\theta$, so we want to integrate

$$\int_0^{2\pi} \int_0^{\pi/6} \sin\phi \; d\phi \; d\theta = \int_0^{2\pi} - \cos\phi|_0^{\pi/6} d\theta = 2\pi \cdot (1 - \sqrt{3}/2).$$

4. Let S be the closed surface given by a cone of height 4, whose axis lies along the z axis and whose base lies in the xy plane and has radius 3. Write down (but do not evaluate) iterated integrals you could use to find the surface integral

$$\iint_S (3x + 2y) \; dA.$$

   Solution To parameterize this cone, we can use cylindrical coordinates. Since the cone has radius 3 and height 4, the "top" part is the graph of the function $z = 4 - 4r/3$, as r varies from 0 to 3 and θ varies from 0 to 2π. Then taking $x = r\cos\theta$ and $y = r\sin\theta$ gives a parameterization $\Phi(r, \theta)$.

   We have $\dfrac{\partial \Phi}{\partial r} = (\cos\theta, \sin\theta, -4/3)$ and $\dfrac{\partial \Phi}{\partial \theta} = (-r\sin\theta, r\cos\theta, 0)$, so the area element for the top part of the cone is given by the norm of

$$\det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & -4/3 \\ -r\sin\theta & r\cos\theta & 0 \end{pmatrix} = 4/3r\cos\theta\vec{i} + 4/3r\sin\theta\vec{j} + r\vec{k}.$$

   We evaluate the norm of this vector to get

$$\sqrt{(4/3)^2 r^2 + r^2} = r\sqrt{16/9 + 1} = 5r/3.$$

So the integral of this function over the top of the cone is given by

$$\int_{r=0}^{r=3}\int_0^{2\pi}[3(r\cos\theta)+2(r\sin\theta)]5r/3d\theta dr.$$

We also need to integrate the function over the base of the cone. In this case we can just use polar coordinates in the xy-plane; the area element is $r\ dr\ d\theta$, so we get

$$\int_{r=0}^{r=3}[3(r\cos\theta)+2(r\sin\theta)]r\ dr\ d\theta.$$

5. Let S be the portion of graph of the function $z=3x^2-2xy$ above the rectangle $[0,1]\times[0,2]$, oriented with upward-pointing normal. Find the flux of the vector field

$$\vec{F}(x,y,z)=(x-y)\vec{i}+xz\vec{j}-2yz\vec{k}$$

through the surface S.

Solution Since S is the graph of a function, we can parameterize it by taking

$$\Phi(x,y)=(x,y,3x^2-2xy)$$

as x,y ranges over the rectangle $[0,1]\times[0,2]$. We evaluate the cross product $\frac{\partial\Phi}{\partial x}\times\frac{\partial\Phi}{\partial y}$:

$$\det\begin{pmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & 0 & 6x-2y\\ 0 & 1 & -2x\end{pmatrix}=(2y-6x)\vec{i}-(8x+2y)\vec{j}+\vec{k}.$$

We also know that $\vec{F}(x,y,f(x,y))=(x-y)\vec{i}+x(3x^2-2xy)\vec{j}-2y(3x^2-2xy)\vec{k}$. So, to compute flux, we evaluate the integral

$$\begin{aligned}&\int_0^2\int_0^1\vec{F}(x,y,f(x))\cdot(\frac{\partial\Phi}{\partial x}\times\frac{\partial\Phi}{\partial y})\ dx\ dy\\ &=\int_0^2\int_0^1(x-y)(2y-6x)-x(3x^2-2xy)(8x+2y)-2y(3x^2-2xy)\ dx\ dy.\end{aligned}$$

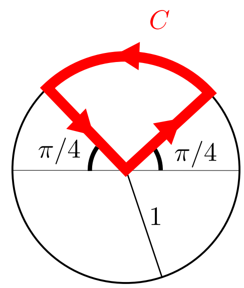
We expand and combine terms:

$$\begin{aligned}&\int_0^2\int_0^1-2y^2-6x^2-4xy+24x^4-10x^3y-4x^2y^2+6x^2y-4xy^2\ dx\ dy\\ &=\int_0^2-2y^2-2-2y+24/5-5y/2-4y^2/3+2y-2y^2\ dy\\ &=\int_0^2 y^2(-4-4/3)+-5y/2+4/5\ dy=\frac{8}{3}(-4-4/3)-5+8/5.\end{aligned}$$

This can be simplified, but there are likely errors in the above computations anyway...

Chapter 8

1. Let C be the curve bounding a wedge-shaped region, pictured in red below:



Use Green's theorem to find the integral

$$\int_C \vec{F} \cdot dr,$$

where $\vec{F}$ is the vector field $\vec{F}(x, y) = x^2 y\vec{\imath} - (2x + y)\vec{\jmath}$.

Solution Green's theorem says that the integral in question is equal to

$$\iint_R (\nabla \times \vec{F})\, dA,$$

where $\nabla \times \vec{F}$ is the *scalar curl*, given by

$$\frac{\partial(2x + y)}{\partial x} - \frac{\partial(x^2 y)}{\partial y} = 2 - x^2,$$

and R is a region with boundary C. We can parameterize R using polar coordinates: it is the region given by $\{(r, \theta) : 0 \le r \le 1, \pi/4 \le \theta \le 3\pi/4\}$. The area element dA is $r\ dr\ d\theta$, so we integrate

$$\int_{\pi/4}^{3\pi/4} \int_0^1 (2 - r^2 \cos^2 \theta) r\ dr\ d\theta.$$

Using the substitution $\cos^2 \theta = \frac{\cos 2\theta + 1}{2}$, this is

$$\int_{\pi/4}^{3\pi/4} 1 - \frac{1}{3}(\cos 2\theta + 1)/2\ d\theta$$

$$= 5\theta/6 - \frac{1}{12} \sin \theta\Big|_{\pi/4}^{3\pi/4}$$

$$= 5\pi/12.$$

2. Let C be the closed, piecewise smooth curve formed by traveling in straight lines between the points $(0, 0, 0)$, $(2, 1, 5), (1, 1, 3)$ in that order. Use Stokes' theorem to evaluate the integral

$$\int_C (xyz)\ dx + (xy)\ dy + (x)\ dz.$$

Solution Stokes' theorem says that the integral of a vector field about C is the integral

$$\iint_S (\nabla \times \vec{F}) \cdot dS,$$

where S is any surface whose boundary is C. There's an easy choice for S: we should just use the triangle whose vertices are the three given points. To parameterize this surface, let $\vec{a} = (2, 1, 5)$ and $\vec{b} = (1, 1, 3)$, and let $\vec{u} = \vec{b} - \vec{a}$. Then, S is just the set of points

$$\{s\vec{a} + t\vec{u} : 0 \leq t \leq s \leq 1\}.$$

The vector area element is then just given by $(\vec{a} \times \vec{u})\, ds\, dt$, which we can compute:

$$(\det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 5 \\ -1 & 0 & -2 \end{pmatrix})\, ds\, dt = (-2\vec{i} - \vec{j} + \vec{k})\, ds\, dt.$$

Now we compute the curl of our vector field $\vec{F}$:

$$\det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & xy & x \end{pmatrix} = (xy - x)\vec{j} + (y - xz)\vec{k}.$$

In coordinates our parameterization $s\vec{a} + t\vec{u}$ gives

$$x(s,t) = 2s - t, \qquad y(s,t) = s, \qquad z(s,t) = 5s - 2t,$$

so at the point (s, t) the curl is

$$[(2s - t)s - (2s - t)]\vec{j} + [s - (2s - t)(5s - 2t)]\vec{k}.$$

So we have

$$\begin{aligned}
\iint_S (\nabla \times \vec{F}) \cdot dS &= \int_0^1 \int_0^s -[(2s - t)s - (2s - t)] + [s - (2s - t)(5s - 2t)]\, dt\, ds \\
&= \int_0^1 \int_0^s -[2s^2 - st - 2s + t] + [s - (10s^2 - 9st + 2t^2)]\, dt\, ds \\
&= \int_0^1 \int_0^s -12s^2 + 10st + 3s - t - 2t^2\, ds\, dt \\
&= \int_0^1 -12s^3 + 5s^3/2 + 3s^2 - s^2/2 - 2s^3/3\, ds \\
&= -3s^4 + 5s^4/8 + s^3 - s^3/6 - s^4/6 \Big|_0^1 \\
&= -3 + 5/8 + 1 - 1/6 - 1/6.
\end{aligned}$$

3. Use the Divergence Theorem to find the flux of the vector field

$$\vec{F}(x, y, z) = (x - y^2)\vec{i} + y\vec{j} + x^3\vec{k}$$

out of the rectangular solid $[0, 1] \times [1, 2] \times [1, 4]$.

Solution The Divergence Theorem says that the flux of the vector field out of the boundary of a region B is given by

$$\iiint_B \nabla \cdot \vec{F}\, dV.$$

So we just need to compute the divergence of this vector field, which is given by

$$\frac{\partial (x - y^2)}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial x^3}{\partial z} = 2.$$

Since the divergence is constant, the integral above is just 2 times the volume of the box, or $2 \times 1 \times 1 \times 3 = 6$.

4. (bonus) Let $\vec{F}(x, y, z)$ be the vector field $\vec{i} + \vec{j} - \vec{k}$. Find the flux of $\vec{F}$ through the surface S given by the part of the graph of $e^{2x-y}\sqrt{1 - x^2 - y^2}$ over the unit disk $x^2 + y^2 \leq 1$, oriented with upward-pointing normal. (Hint: is $\vec{F}$ the curl of a vector field? Why would this help?)

Solution It turns out that $\vec{F}$ *is* the curl of a vector field: any vector field whose *divergence* is zero, and which is defined everywhere on $\mathbb{R}^3$, is a curl. This vector field is a constant, so it does have zero divergence.

Alternatively, you can check that $\vec{F}$ is the curl of the vector field

$$\vec{G} = z\vec{i} - x\vec{j} + y\vec{k}.$$

Stokes' theorem then says that the integral of $\vec{F}$ over the region in question is the same as the *line integral* of $\vec{G}$ over the boundary of S. This line integral is not too bad to compute, but there's an even easier way.

Since $\vec{F}$ is the curl of a vector field, it follows (again by Stokes' theorem) that the flux of $\vec{F}$ through *any* surface whose boundary is ∂S is the same as the flux of $\vec{F}$ through S. So we can pick a surface where this integral is particularly easy: say the unit disk $x^2 + y^2 \leq 1$ in the xy-plane.

We can find this flux by integrating the dot product of $\vec{F}$ with the unit normal vector $\vec{n}$ over the surface. In this case this is just the standard basis vector $\vec{k}$. So we want to integrate the constant function $\vec{F} \cdot \vec{k} = -1$ over the unit disk. Since the unit disk has area π, we just get $-\pi$.