Exam 1 Review

1.) * Let P describe $(0,-1,2)$, Q describe $(-1,-1,-2)$ and R describe $(4,1,0)$

* The area can be found by $\dfrac{\|\overrightarrow{QP} \times \overrightarrow{QR}\|}{2}$

$$\overrightarrow{QP} = \langle 0,-1,2\rangle - \langle -1,-1,-2\rangle$$
$$= \langle 1,0,4\rangle$$

$$\overrightarrow{QR} = \langle 4,1,0\rangle - \langle -1,-1,-2\rangle$$
$$= \langle 5,2,2\rangle$$

$$\overrightarrow{QP} \times \overrightarrow{QR} = \begin{vmatrix} i & j & k \\ 1 & 0 & 4 \\ 5 & 2 & 2 \end{vmatrix}$$

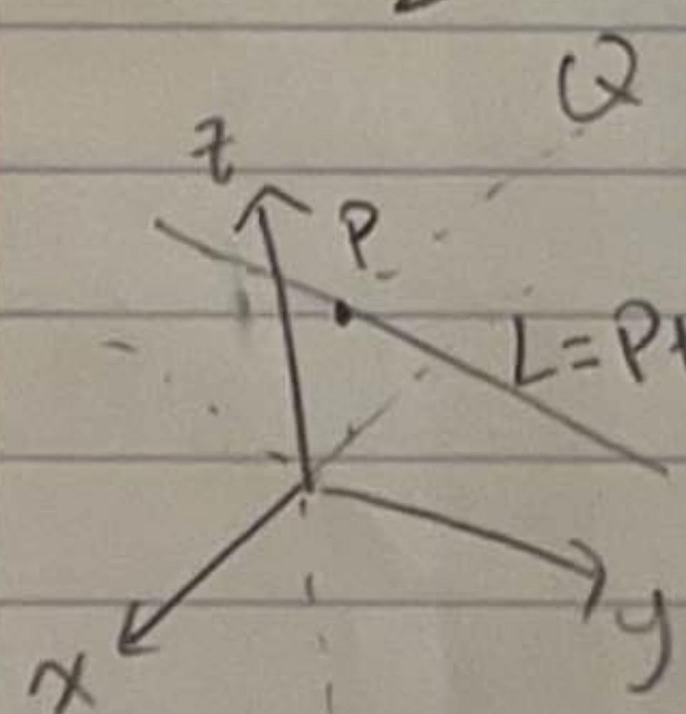
$$= \begin{vmatrix} 0 & 4 \\ 2 & 2 \end{vmatrix} i - \begin{vmatrix} 1 & 4 \\ 5 & 2 \end{vmatrix} j + \begin{vmatrix} 1 & 0 \\ 5 & 2 \end{vmatrix} k$$

$$= (0-8)i - (2-20)j + (2-0)k$$

$$= \langle -8,18,2\rangle$$

$$\frac{\|\overrightarrow{QP} \times \overrightarrow{QR}\|}{2} = \frac{1}{2}\sqrt{(-8)^2+(18)^2+(2)^2} \approx \boxed{9.899}$$

2.)



$$\overrightarrow{PQ} = \langle x+1, y, z-4\rangle$$
$$L = \langle -1+2t, t, 4+t\rangle$$

* the set of points will form a cone around L

$$(\overrightarrow{PQ}) \cdot L = \|\overrightarrow{PQ}\| \, \|L\| \cos\theta$$

$$\|\vec{PQ}\| = \sqrt{(x+1)^2+y^2+(z-4)^2}$$

$$\|L\| = \sqrt{(-1+2t)^2+t^2+(4+t)^2}$$

$$\begin{bmatrix} x+1 & y & z-4 \end{bmatrix}\begin{bmatrix}-1+2t\\ t\\ 4+t\end{bmatrix} = \sqrt{(x+1)^2+y^2+(z-4)^2}\,\sqrt{(-1+2t)^2+t^2+(4+t)^2}\,\cos\frac{\pi}{4}$$

$$(x+1)(2t-1)+ty+(z-4)(4+t) = \frac{\sqrt{2}}{2}\sqrt{(x+1)^2+y^2+(z-4)^2}\,\sqrt{(-1+2t)^2+t^2+(4+t)^2}$$

*Square both sides

$$\boxed{\left[(x+1)(2t-1)+ty+(z-4)(4+t)\right]^2 = \frac{2}{4}\left((x+1)^2+y^2+(z-4)^2\right)\left((-1+2t)^2+t^2+(4+t)^2\right)}$$

↑ this equation represents a cone

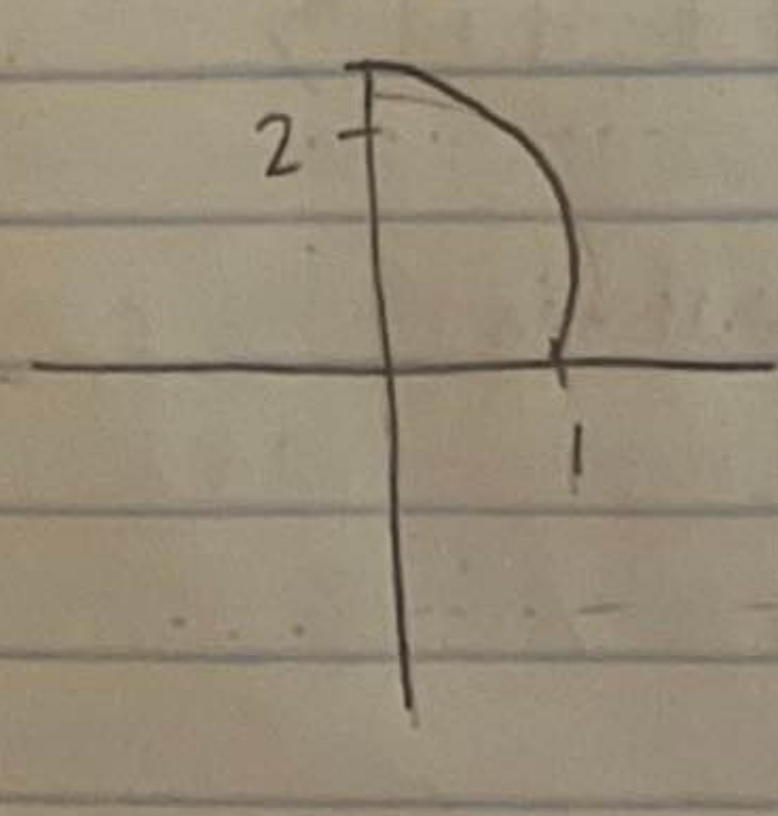
3.) <u>Spherical</u>

$\cos\varphi = \sin\varphi\cos\theta$

<u>Cylindrical</u>

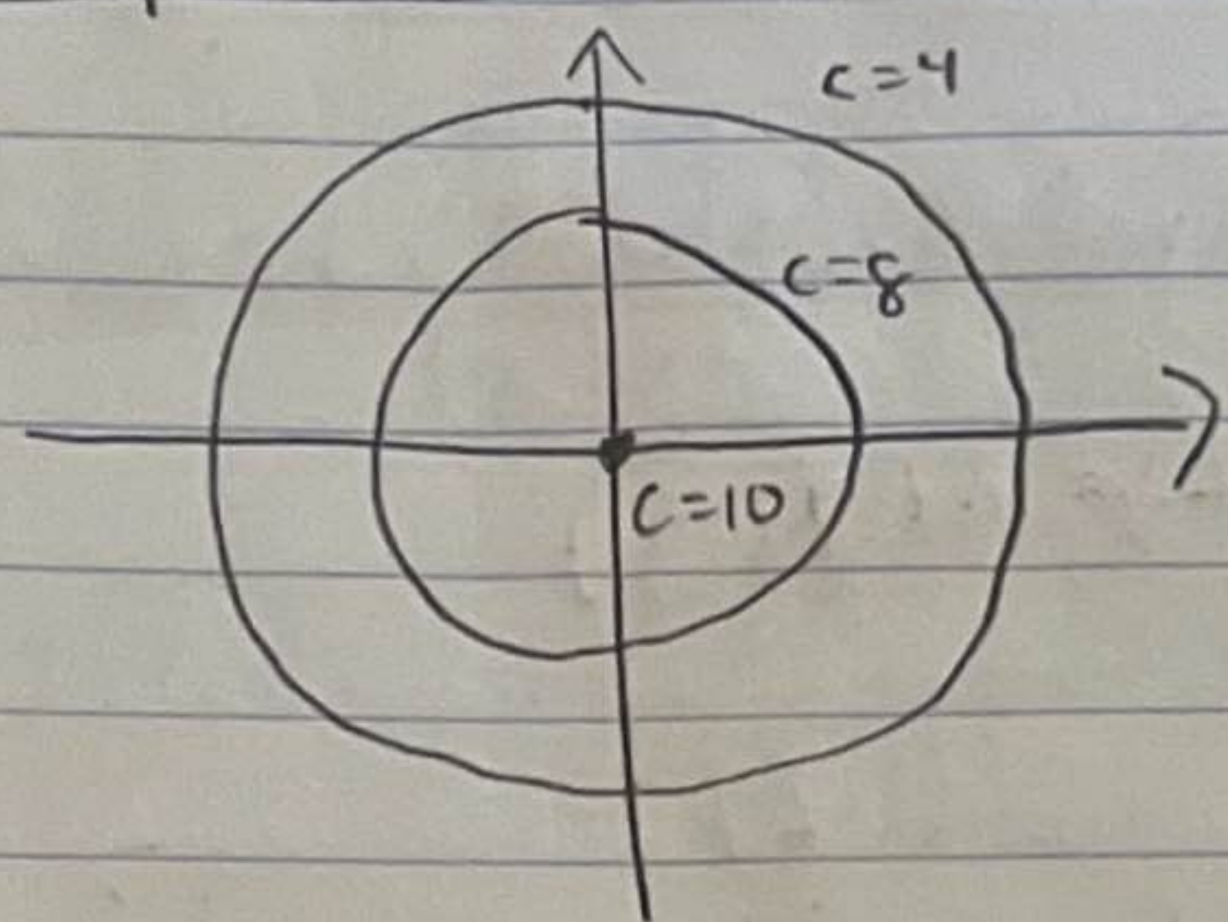
$z = r\cos\theta$

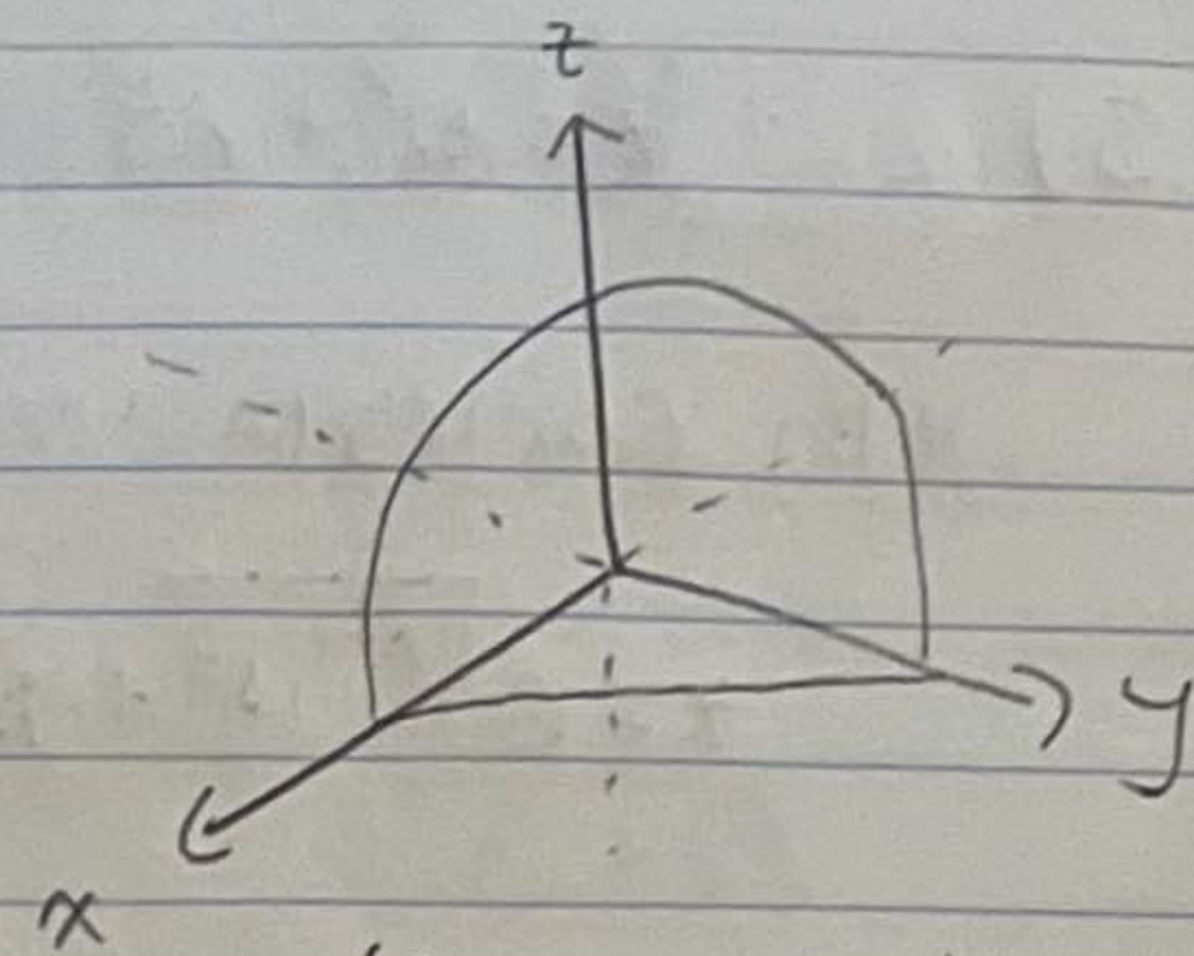
4.)



Chapter 2

1.)





(Sorry, I can't really draw)

2.) * Try $x = 0$

$$\lim_{y \to 0} \frac{-y^2}{y^2} = -1$$

* Try $y = 0$

$$\lim_{x \to 0} \frac{0}{-x^4} = 0 \implies \boxed{\text{DNE}}$$

this is zero over a really small nonzero negative number but zero over any number (except zero) is 0

3.) The function is continuous anywhere except where $x^2 = y^2$

$$\boxed{\{(x,y) : x^2 \neq y^2\}}$$

4.) $f(x,y) = x\cos(y) - \sin(xy)$

$$\boxed{\nabla f(x,y) = \begin{bmatrix} \cos(y) - y\cos(xy) \\ -x\sin(y) - x\cos(xy) \end{bmatrix}}$$

5.) $z = e^r + r^2$

- In cartesian coordinates, this is

$$z = e^{\sqrt{x^2+y^2}} + x^2 + y^2 = f(x,y)$$

- $(r=1, \theta = \pi/6, z = e+1)$ gives

$$(x,y,z) = (r\cos\theta, r\sin\theta, z)$$
$$= (\sqrt{3}/2, 1/2, e+1)$$

$$\vec{n} = \begin{vmatrix} i & j & k \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y} \end{vmatrix}$$

$$= \begin{bmatrix} -\partial f/\partial x \\ -\partial f/\partial y \\ 1 \end{bmatrix}$$

- Equation of a plane is

$$(\vec{x} - \vec{a}) \cdot \vec{n} = 0$$

$$\begin{bmatrix} x - \sqrt{3}/2 \\ y - 1/2 \\ z - (e+1) \end{bmatrix} \cdot \begin{bmatrix} -\partial f/\partial x \\ -\partial f/\partial y \\ 1 \end{bmatrix} = 0$$

$$\Rightarrow -(x - \tfrac{\sqrt{3}}{2})\frac{\partial f}{\partial x} - (y - \tfrac{1}{2})\frac{\partial f}{\partial y} + z - (e+1) = 0$$

- Rearrange

$$\Rightarrow z = e+1+\left(x-\frac{\sqrt{3}}{2}\right)\frac{\partial f}{\partial x}+\left(y-\frac{1}{2}\right)\frac{\partial f}{\partial y}$$

$$\frac{\partial f}{\partial x} = \frac{x e^{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}} + 2x$$

$$\frac{\partial f}{\partial x}\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right) = \frac{\frac{\sqrt{3}}{2} e^{\sqrt{\frac{3}{4}+\frac{1}{4}}}}{\sqrt{\frac{3}{4}+\frac{1}{4}}} + 2\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{2}e+\sqrt{3}$$

$$\frac{\partial f}{\partial y} = \frac{y e^{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}} + 2y$$

$$\frac{\partial f}{\partial y}\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right) = \frac{\frac{1}{2} e^{\sqrt{\frac{3}{4}+\frac{1}{4}}}}{\sqrt{\frac{3}{4}+\frac{1}{4}}} + 2\left(\frac{1}{2}\right) = \frac{1}{2}e+1$$

$$\Rightarrow \boxed{z = e+1+\left(\frac{\sqrt{3}}{2}e+\sqrt{3}\right)\left(x-\frac{\sqrt{3}}{2}\right)+\left(\frac{e}{2}+1\right)\left(y-\frac{1}{2}\right)}$$

6.) * The tangent line at a point p on C is the intersection of plane P ($z=4x-5y$) and the tangent plane of the sphere at point p

* First, find intersection

$$z=4x-5y \Rightarrow z^2=(4x-5y)^2$$
$$=16x^2-40xy+25y^2$$

$$x^2+y^2+z^2=1 \Rightarrow z^2=1-x^2-y^2$$

$$\Rightarrow 16x^2-40xy+25y^2=1-x^2-y^2$$
$$17x^2-40xy+26y^2=1$$

$\underline{x = \frac{1}{2}:}$

$$17\left(\tfrac{1}{2}\right)^2 - 40\left(\tfrac{1}{2}\right)y + 26y^2 = 1$$
$$\tfrac{17}{4} - 20y + 26y^2 = 1$$
$$26y^2 - 20y + \tfrac{13}{4} = 0$$

$$\Rightarrow y = \tfrac{1}{52}\left(20 \pm \sqrt{62}\right)$$

* Take $x = \frac{1}{2}$, $y = \frac{1}{52}(20 - \sqrt{62}) \approx 0.233$ (any point with $x = \frac{1}{2}$ will do)

$$\begin{aligned} z &= 4x - 5y \\ &= 4\left(\tfrac{1}{2}\right) - 5\left(\tfrac{1}{52}\left(20 - \sqrt{62}\right)\right) \\ &= 2 - \tfrac{5}{52}\left(20 - \sqrt{62}\right) \\ &\approx 0.834 \end{aligned}$$

* So let p be $(\frac{1}{2}, 0.233, 0.834)$
  * Note that $\vec{n}_s$ (the normal to a unit sphere) is the position vector described by $\langle 0.5, 0.233, 0.834 \rangle$

* Then the tangent plane to this point on the sphere is

$$(\vec{x} - \vec{a}) \cdot \vec{n} = 0$$

$$\begin{bmatrix} x-0.5 & y-0.233 & z-0.834 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.233 \\ 0.834 \end{bmatrix} = 0$$

$$0.5(x-0.5) + 0.233(y-0.233) + 0.834(z-0.834) = 0$$

(6 continued)

✓ Find the intersection of the two planes

$$z = 4x - 5y$$

$$(z - 0.834) = \frac{1}{0.834}\left[0.5(x-0.5) + 0.233(y-0.233)\right]$$

$$\frac{1}{0.834}\left[0.5(x-0.5) + 0.233(y-0.233)\right] + 0.834 = 4x - 5y$$

$$0.5995x + 0.279y + 0.469 = 4x - 5y$$

$$5.279y = 3.4005x - 0.469$$

$$\boxed{y = 0.644x - 0.089}$$

7.) $f(x,y) = \cos(xy) - xy^2$

$x = 3\cos(t) + \sin(s) - t$

$y = 2s - t$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

$$\boxed{\frac{\partial f}{\partial s} = (-y\sin(xy) - y^2)(\cos(s)) + (-x\sin(xy) - 2xy)(2)}$$

8.) $$\vec{n} = \frac{\frac{\partial S(1,1)}{\partial t} \times \frac{\partial S(1,1)}{\partial s}}{\left\| \frac{\partial S(1,1)}{\partial t} \times \frac{\partial S(1,1)}{\partial s} \right\|}$$

$S = t\vec{v} + s\vec{u} + s^2\vec{w}$

$$\frac{\partial S}{\partial t} = \vec{v} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \qquad \frac{\partial S}{\partial s} = \vec{u} + 2s\vec{w} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 2s\begin{bmatrix} 4 \\ 4 \\ -6 \end{bmatrix}$$

$$\frac{\partial S(1,1)}{\partial s} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 4 \\ -6 \end{bmatrix} = \begin{bmatrix} 8 \\ 9 \\ -11 \end{bmatrix}$$

$$\frac{\partial S}{\partial t} \times \frac{\partial S}{\partial s} = \begin{vmatrix} i & j & k \\ -1 & 2 & 0 \\ 8 & 9 & -11 \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 9 & -11 \end{vmatrix} i - \begin{vmatrix} -1 & 0 \\ 8 & -11 \end{vmatrix} j + \begin{vmatrix} -1 & 2 \\ 8 & 9 \end{vmatrix} k$$

$$= -22i - 11j - 25k = \begin{bmatrix} -22 \\ -11 \\ -25 \end{bmatrix}$$

$$\boxed{\vec{n} = \frac{1}{\sqrt{1230}} \begin{bmatrix} -22 \\ -11 \\ -25 \end{bmatrix}}$$

9.) $D_{\vec{v}}(g \cdot f) = \nabla(g(0,1) \cdot f(0,1)) \cdot \vec{v}$

$$= (f(0,1)\underbrace{\nabla g(0,1)} + \underbrace{g(0,1)}\nabla f(0,1)) \cdot \vec{v}$$

$$\nabla g = \begin{bmatrix} ye^{xy} \\ xe^{xy} \end{bmatrix} \quad g(0,1) = e^0 = 1$$

$$\Rightarrow \nabla g(0,1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \left(-3\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} -1 \\ 4 \end{bmatrix}\right) \cdot \vec{v}$$

$$= \begin{bmatrix} -4 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$= (-4)(-1) + 4(-1)$$

$$= 4 - 4$$

$$= \boxed{0}$$

Chapter 3

1.) $f(x,y) = x - y^2\cos(x) + y^2$

$f_x = 1 + y^2\sin(x)$ $\quad$ $f_y = -2y\cos(x) + 2y$

$f_{xx} = y^2\cos(x)$ $\quad$ $f_{yy} = -2\cos(x) + 2$

$f_{xy} = 2y\sin(x)$ $\quad$ $f_{yx} = 2y\sin(x)$

$$g(x,y) = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$$
$$+ \frac{f_{xx}(1,1)}{2}(x-1)^2 + f_{xy}(1,1)(x-1)(y-1)$$
$$+ \frac{f_{yy}(1,1)}{2}(y-1)^2$$

$$\boxed{\begin{aligned} g(x,y) = &(2-\cos(1)) + (1+\sin(1))(x-1) \\ &+ (2-2\cos(1))(y-1) + \tfrac{1}{2}\cos(1)(x-1)^2 \\ &+ 2\sin(1)(x-1)(y-1) + \tfrac{1}{2}\cdot 2\sin(1)(y-1)^2 \end{aligned}}$$

2.) $f(x,y) = xy + \frac{1}{x} + \frac{1}{y}$

$$\nabla f = \begin{bmatrix} y - \frac{1}{x^2} \\ x - \frac{1}{y^2} \end{bmatrix}$$

$\nabla f = 0 \Rightarrow$

$$y - \frac{1}{x^2} = 0 \qquad\qquad x - \frac{1}{y^2} = 0$$
$$y = \frac{1}{x^2} \qquad\qquad x = \frac{1}{y^2}$$

$$x^2 = \frac{1}{y} \qquad\qquad y^2 = \frac{1}{x}$$
$$\left(\frac{1}{y^2}\right)^2 = \frac{1}{y} \qquad\qquad \left(\frac{1}{x^2}\right)^2 = \frac{1}{x}$$
$$\frac{1}{y^4} = \frac{1}{y} \qquad\qquad \frac{1}{x^4} = \frac{1}{x}$$

$$x^4 = x$$
$$x(x^3-1) = 0$$
$$\Rightarrow x = 0, 1$$

$$y^4 = y$$
$$y^4 - y = 0$$
$$y(y^3-1) = 0$$
$$\Rightarrow y = 0, 1$$

* Compute second derivatives

$f_{xx} = \frac{2}{x^3} \qquad f_{xy} = 1 \qquad f_{yy} = \frac{2}{y^3}$

$\underline{(x,y)=(0,0):}$

$f_{xx}(0,0)\,f_{yy}(0,0)-(f_{xy}(0,0))^2$

$=\infty$

undefined (same for $(x,y)=(0,1)$ and $(x,y)=(1,0)$)

$\underline{(x,y)=(1,1):}$

$f_{xx}(1,1)\,f_{yy}(1,1)-(f_{xy}(0,0))^2$

$=2\cdot 2-(1)^2=3>0$

$f_{xx}=2>0$

$\boxed{\text{so there is a local minimum at } (1,1)}$

3) $z=f(x,y)=(x^2+3y^2)e^{1-x^2-y^2}$

$$\nabla f=\begin{bmatrix} 2xe^{1-x^2-y^2}+(x^2+3y^2)e^{1-x^2-y^2}(-2x) \\ 6ye^{1-x^2-y^2}+(x^2+3y^2)e^{1-x^2-y^2}(-2y) \end{bmatrix}$$

$f_x=2xe^{1-x^2-y^2}(1-x^2-3y^2)=0$

$\Rightarrow x=0 \text{ or } 1-x^2-3y^2=0$

$$f_y = 2ye^{1-x^2-y^2}(3-x^2-3y^2) = 0$$

$$\Rightarrow y = 0 \quad \text{or} \quad 3-x^2-3y^2 = 0$$

* Only one (not both at the same time) of $3-x^2-3y^2=0$ or $1-x^2-3y^2=0$ are true
So do not solve them together

$\underline{x=0}$:

$$3-x^2-3y^2=0$$
$$3-0-3y^2=0$$
$$3y^2=3$$
$$y^2=1$$
$$y=\pm 1$$

$\underline{y=0}$:

$$1-x^2-3y^2=0$$
$$1-x^2=0$$
$$x=\pm 1$$

$$f_{xx} = 2e^{1-x^2-y^2}(1-x^2-3y^2) - 4x^2e^{1-x^2-y^2}(1-x^2-3y^2) + 2xe^{1-x^2-y^2}(-2x)$$

$$f_{xy} = -4xye^{1-x^2-y^2}(1-x^2-3y^2) + 2xe^{1-x^2-y^2}(-6y)$$

$$f_{yy} = 2e^{1-x^2-y^2}(3-x^2-3y^2) - 4y^2e^{1-x^2-y^2}(3-x^2-3y^2) + 2ye^{1-x^2-y^2}(-6y)$$

$\underline{(x,y)=(0,0)}$:

$$f_{xx}f_{yy}-(f_{xy})^2 = (2e)(6e)-(0)$$
$$= 12e^2 > 0$$

$f_{xx} > 0 \Rightarrow \boxed{\text{local min at } (0,0) \text{ with } f(0,0)=0}$

$f(0,0)=0$

$\underline{(x,y) = (0,1):}$

$$f_{xx} f_{yy} - (f_{xy})^2 = (-4)(-12) - (0)^2$$
$$= 48 > 0$$

$f_{xx} = -4 < 0$

$\Rightarrow \boxed{\text{local max at } (0,1) \text{ with } f(0,1) = 3}$

$\underline{(x,y) = (0,-1):}$

$$f_{xx} f_{yy} - (f_{xy})^2 = (-4)(-12) - (0)^2$$
$$= 48 > 0$$

$f_{xx} = -4 < 0$

$\Rightarrow \boxed{\text{local max at } (0,1) \text{ with } f(0,-1) = 3}$

$\underline{(x,y) = (1,0):}$

$$f_{xx} f_{yy} - (f_{xy})^2 = (-4)(4) - (0)^2$$
$$= -16 < 0$$

saddle point at $(1,0)$

$\underline{(x,y) = (-1,0):}$

$$f_{xx} - f_{yy} - (f_{xy})^2 = (-4)(4) - (0)^2$$
$$= -16 < 0$$

saddle point at $(-1,0)$

4.) $2x - y + 2z = 20$

$d^2 = x^2 + y^2 + z^2$ $\}$ maximizing d^2 also maximizes its square root

$g = 2x - y + 2z - 20$

$\nabla d = \lambda \nabla g$

$$\begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix} = \lambda \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

$2x = 2\lambda \Rightarrow x = \lambda$

$2y = -\lambda \Rightarrow y = -\frac{\lambda}{2}$

$2z = 2 \Rightarrow z = \lambda$

$2x - y + 2z = 20$

$$2\lambda + \frac{\lambda}{2} + 2\lambda = 20$$
$$4\lambda + \lambda + 4\lambda = 40$$
$$9\lambda = 40$$
$$\lambda = 40/9$$

$\Rightarrow x = \lambda = 40/9$

$y = -\lambda/2 = -20/9$

$z = \lambda = 40/9$

$$\boxed{(x, y, z) = (40/9, -20/9, 40/9)}$$

5.) $f(x,y) = 4x + 2y$

$2x^2 + 3y^2 = 21$

$\Rightarrow g(x,y) = 2x^2 + 3y^2 - 21$

$\nabla f = \lambda \nabla g$

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \lambda \begin{bmatrix} 4x \\ 6y \end{bmatrix} \Rightarrow \begin{matrix} 4 = 4\lambda x \Rightarrow \lambda x = 1 \Rightarrow x = 1/\lambda \\ 2 = 6\lambda y \Rightarrow 3\lambda y = 1 \Rightarrow y = \frac{1}{3\lambda} \end{matrix}$$

$$2(1/\lambda)^2 + 3(1/3\lambda)^2 = 21$$

$$\frac{2}{\lambda^2} + \frac{3}{9\lambda^2} = 21$$

$$2 + \frac{1}{3} = 21\lambda^2$$

$$\frac{6}{3} + \frac{1}{3} = 21\lambda^2$$

$$\frac{7}{3} = 21\lambda^2$$

$$\Rightarrow \lambda^2 = \frac{1}{9}$$

$$\Rightarrow \lambda = \pm\frac{1}{3} \Rightarrow \begin{cases} x = \pm 3 \\ y = \pm 1 \end{cases}$$

$\underline{(x,y) = (3,1):}$

$\boxed{f(3,1) = 14}$ max

$\underline{(x,y) = (-3,-1):}$

$\boxed{f(-3,-1) = -14}$ min

6.) $g(x,y,z) = xyz$
$h(x,y,z) = x^2+y^2+z^2-1$

- Check for $x^2+y^2+z^2<1$

$$\nabla g = \begin{bmatrix} yz \\ xz \\ xy \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} yz=0 \\ xz=0 \\ xy=0 \end{matrix}$$

$$\Rightarrow (x,y,z) = (c_1,0,0)$$
$$\text{or } = (0,c_2,0) \quad -1<c_1,c_2,c_3<1$$
$$\text{or } = (0,0,c_3)$$

$$g(c_1,0,0) = g(0,c_2,0) = g(0,0,c_3) = \boxed{0}$$ is an extrema value (potentially)

- Check for $x^2+y^2+z^2=1$

$$\nabla g = \lambda \nabla h \Rightarrow \begin{bmatrix} yz \\ xz \\ xy \end{bmatrix} = \lambda \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix}$$

$$\left.\begin{matrix} yz = \lambda 2x \\ xz = \lambda 2y \\ xy = \lambda 2z \end{matrix}\right\} \begin{matrix} \text{mult by } x \\ \text{mult by } y \\ \text{mult by } z \end{matrix} \Rightarrow \begin{matrix} xyz = 2\lambda x^2 \\ xyz = 2\lambda y^2 \\ xyz = 2\lambda z^2 \end{matrix}$$

$$\begin{aligned} & xyz = 2\lambda x^2 \\ -&(xyz = 2\lambda y^2) \\ \hline & 0 = 2\lambda x^2 - 2\lambda y^2 \Rightarrow x^2 = y^2 \end{aligned}$$

- we end up with $x^2=y^2=z^2$ which we can sub into $x^2+y^2+z^2=1$

$$x^2+x^2+x^2=1 \Rightarrow 3x^2=1 \Rightarrow x = \pm\frac{1}{\sqrt{3}}$$

* Then we get

$$(x,y,z) = \left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right)$$

$$g\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = g\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) = g\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$$
$$= g\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \boxed{\frac{1}{3\sqrt{3}} \text{ is a max value}}$$

$$g\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = g\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = g\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$$
$$= g\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) = \boxed{\frac{-1}{3\sqrt{3}} \text{ is a min value}}$$

7) * The formula for the distance between a plane described by $Ax + By + Cz + D = 0$ and a point (x_0, y_0, z_0) is

$$d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

* We want

$$\min_{(x,y,z)} d \quad \text{such that}$$
$$(x-1)^2 + (y-2)^2 + (z-3)^2 = 1$$
$$\Rightarrow g(x,y,z) = (x-1)^2 + (y-2)^2 + (z-3)^2 - 1$$

* Here,

$$d = \frac{|x + y + z|}{\sqrt{3}}$$

* Now we use Lagrange multipliers

$$\nabla d = \lambda \nabla g$$

$$\Rightarrow \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} = \lambda \begin{bmatrix} 2(x-1) \\ 2(y-2) \\ 2(z-3) \end{bmatrix}$$

$$\left.\begin{aligned} \frac{1}{\sqrt{3}} &= 2\lambda(x-1) \Rightarrow (x-1) = \frac{1}{2\lambda\sqrt{3}} \\ \frac{1}{\sqrt{3}} &= 2\lambda(y-2) \Rightarrow (y-2) = \frac{1}{2\lambda\sqrt{3}} \\ \frac{1}{\sqrt{3}} &= 2\lambda(z-3) \Rightarrow (z-3) = \frac{1}{2\lambda\sqrt{3}} \end{aligned}\right\} (\star)$$

$$(x-1)^2 + (y-2)^2 + (z-3)^2 = 1$$

* Sub $\star$ into ↗

$$\left(\frac{1}{2\lambda\sqrt{3}}\right)^2 + \left(\frac{1}{2\lambda\sqrt{3}}\right)^2 + \left(\frac{1}{2\lambda\sqrt{3}}\right)^2 = 1$$

$$3\left(\frac{1}{12\lambda^2}\right) = 1$$

$$\Rightarrow \lambda^2 = \frac{1}{4}$$

$$\lambda = \pm\frac{1}{2}$$

$\underline{\lambda = -\frac{1}{2}}$:

$$x = 1 + \frac{1}{2\lambda\sqrt{3}} = 1 - \frac{1}{\sqrt{3}}$$

$$y = 2 + \frac{1}{2\lambda\sqrt{3}} = 2 - \frac{1}{\sqrt{3}}$$

$$z = 3 + \frac{1}{2\lambda\sqrt{3}} = 3 - \frac{1}{\sqrt{3}}$$

$$d = \frac{\left|1 - \frac{1}{\sqrt{3}} + 2 - \frac{1}{\sqrt{3}} + 3 - \frac{1}{\sqrt{3}}\right|}{\sqrt{3}} = \frac{\left|6 - \frac{3}{\sqrt{3}}\right|}{\sqrt{3}}$$

$$= \frac{6-\sqrt{3}}{\sqrt{3}}$$

$\underline{\lambda = \frac{1}{2}}$:

$$\left.\begin{aligned} x &= 1 + \frac{1}{\sqrt{3}} \\ y &= 2 + \frac{1}{\sqrt{3}} \\ z &= 3 + \frac{1}{\sqrt{3}} \end{aligned}\right\} \Rightarrow d = \frac{6+\sqrt{3}}{\sqrt{3}}$$

* Thus, the distance is $\boxed{\frac{6-\sqrt{3}}{\sqrt{3}}}$