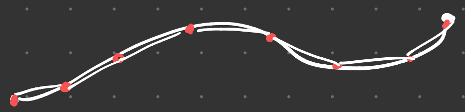
Boundaries of Groups & Spaces

6/1/2020

Metric Spaces:
- always complete
- usually proper: every closed ball of fin. radius is compact
- usually geodesic: if $x,y \in X$ have $d(x,y) = L$, $\exists$ a path of length L in X joining x,y
  - ↳ NOT: uniquely geodesic

[?] How do we measure the length of a path in a metric space?

↳ [A]: Consider rectifiable paths: take the sum of the lengths of these path components, → take the sup over all such sums

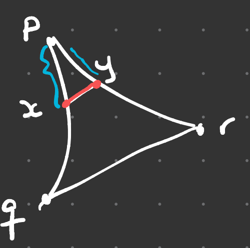


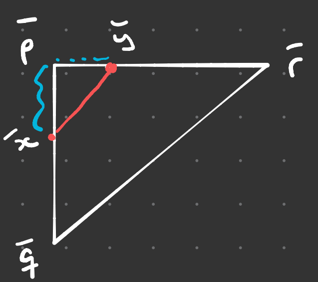
CAT(0) Spaces: X, a metric space

Def^n: a triangle $\Delta(p,q,r)$, $p,q,r \in X$, is a union of 3 geodesic segments, $[p,q]$, $[q,r]$, $[p,r]$

Def^n: a Euclidean comparison triangle $\bar{\Delta}(\bar{p},\bar{q},\bar{r})$, $p,q,r \in X$ is a triangle in $\mathbb{E}^2$ whose sides have lengths $d(p,q)$, $d(q,r)$, $d(p,r)$ (unique up to isom$(\mathbb{E}^2)$)
↳ vertices are $\bar{p},\bar{q},\bar{r}$

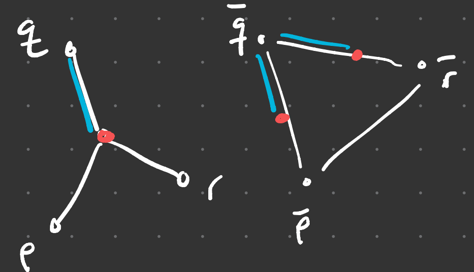
Def^n: Given a triangle $\Delta(p,q,r)$ in X, $x \in [p,q]$, a comparison pt for x is $\bar{x} \in [\bar{p},\bar{q}]$ s.t. $d(\bar{x},\bar{p}) = d(x,p)$

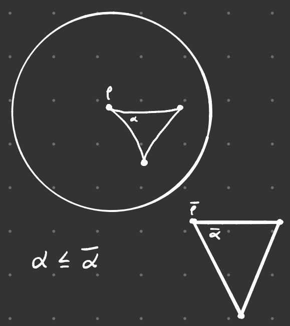




Def^n: X is CAT(0) if for all $\Delta(p,q,r)$ in X, all $x,y \in \Delta(p,q,r)$
↳ $d_X(x,y) \le d_{\mathbb{E}^2}(\bar{x},\bar{y})$

Examples:
- $\mathbb{E}^d$ $\left(d_{\mathbb{E}^d}(x,y) = d_{\mathbb{E}^2}(x,y)\right)$
- trees: triangles are tripods →





$\alpha \le \bar{\alpha}$

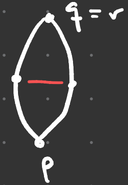
- $\mathbb{H}^d$
  ↳ given $x, p, q \in X$, $\measuredangle_x(p,q) := \limsup_{t,t' \to 0} \measuredangle_{\bar{x}}(\overline{c(t)}, \overline{c'(t')})$, where $c, c' : [0,1] \to X$ are geodesics joining x to p & x to q resp.
  If X is a Riemannian mfld, this agrees with $\measuredangle$ between tangent vectors pointing from x to p and q

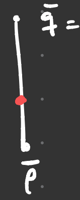
Prop: X is CAT(0) $\Leftrightarrow$ for every $\Delta(p,q,r)$ in X, $\measuredangle_p(q,r) \le \measuredangle_{\bar{p}}(\bar{q},\bar{r})$

REM: a nonpositively curved Riemannian mfld is locally CAT(0) [every pt has an open nbhd which is (nonpos. curved)-CAT(0)

- products of CAT(0) spaces are CAT(0)]

NON Examples: · spheres, graphs with nontrivial loops





Prop: CAT(0) spaces have unique geodesics

Cartan-Hadamard Thm: X is locally CAT(0) (complete, proper, geodesic), $x_0 \in X$

$$\tilde{X}_{x_0} = \{\text{reparametrized local geodesics } c:[0,1]\to X : c(0)=x_0\}$$

w/ metric $d(c,c') := \sup_{t\in[0,1]} d(c(t), c'(t))$

length of shortest path in $\tilde{X}_{x_0}$ is d between endpoints

$\tilde{X}_{x_0} \to X$ is a universal covering ; $\tilde{X}_{x_0}$ w/ path metric coming from d is (globally) CAT(0)
$c \longmapsto c(1)$ $\Rightarrow$ CAT(0) spaces are contractible

[?] when is the path metric not the same as the one from the space?

[A]: ① (silly ex)

$d(x,y)=1$

$d_p(x,y)=\infty$

②

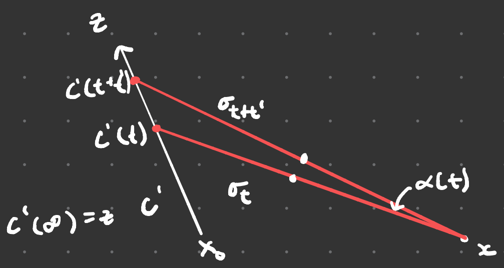
Boundaries of CAT(0) spaces

X metric space, $\bar{X}$ a compactification $\left(\begin{array}{l} X \hookrightarrow \bar{X} \\ \text{is open \& dense}\end{array}\right)$, $\partial X = \bar{X} - X$

$\partial X = \{\text{infinite geodesic rays } c:[0,\infty)\to X\}/\sim$, $c_1 \sim c_2$ if images have finite Hausdorff distance ($d_{Haus}(c_1,c_2) < \infty$)

In $\mathbb{E}^d$: $d_{Haus}(c_1,c_2)<\infty$ iff they're parallel and point is same direction
$\partial\mathbb{E}^d$ in bijection with S^{d-1}

Prop: If $z \in \partial X$, $x \in X$, then $\exists$ unique ray $c:[0,\infty)\to X$ s.t. $[c]=z$ ($c(\infty)=z$) and $c(0)=x$



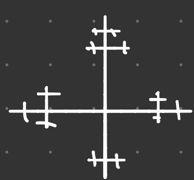
Fix $t'>0$. $\alpha(t)$ goes to 0 uniformly in t' as $t\to\infty$.
Fix s, take $c(s) = \lim_{t\to\infty} \sigma_t(s)$; converges to a geodesic, w/ Haus. dist. btwn c and c' is at most $d(x_0,x)$

∂X in bijection w/ rays based at same fixed basepoint x_0

X is homeomorphic to eventually constant geodesics : $c:[0,\infty)\hookrightarrow X$ s.t. for some T, $\forall t>T$, $c(t)=c(T)$, $c|_{[0,T]}$ is geodesic

X in bijection w/ eventually constant geodesics starting at a fixed basepoint, ending at $x\in X$
$\hookrightarrow$ topology: compact-open topology, uniform conv. on compact sets

$\bar{X}$ is $X \cup \partial X$ is a set of geodesic rays $c:[0,\infty)\longmapsto X$, topologized w/ compact open; $\bar{X}$ compact, X open & dense



Examples: · X is a simply connected, nonpos. curved Riem. mfld $\partial X \cong S^d$
· ∂tree (infinite) is a Cantor set · $\partial(X_1 \times X_2)$ = join of $\partial X_1 * \partial X_2$