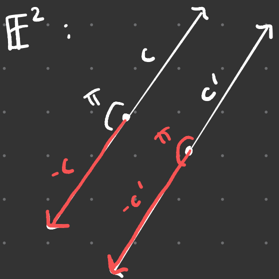
Boundaries of Groups & Spaces

6/2/2020

- X a CAT(0) space, $\partial X = \{\text{infinite geodesic rays in } X\}/\sim$, topologized via unif. convergence
- isometries of X induce homeomorphisms of ∂X

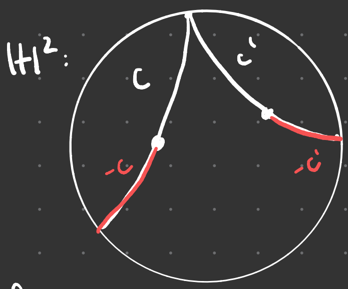
$\boxed{\text{REM}}$ $\mathbb{E}^2$ and $\mathbb{H}^2$ have homeomorphic boundaries (S^1) but are NOT isometric

$\mathbb{E}^2$:



Transitive subgrp of $\mathrm{Isom}(\mathbb{E}^2)$ acts trivially on $\partial\mathbb{E}^2$

$\mathbb{H}^2$:



$\mathrm{Isom}(\mathbb{H}^2)$ has nontrivial action on $\partial\mathbb{H}^2$

Def^n: The angle metric on ∂X is def^{n}d as follows: $p \in X$, $c_1, c_2 : [0,1] \to X$, $c_i(0) = p$

$$\angle_p(c_1,c_2) := \limsup_{t,t' \to 0} \angle_{\bar p}\left(\overline{c_1(t)}, \overline{c_2(t')}\right) = \lim_{t\to 0} \angle_{\bar p}\left(\overline{c_1(t)}, \overline{c_2(t)}\right)$$

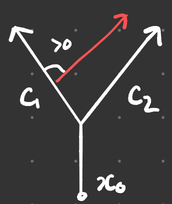
$z_1, z_2 \in \partial X$, $\angle_p(z_1,z_2) := \angle_p(c_1,c_2)$ where $c_i(\infty) = z_i$, $c_i(0) = p$

$$\angle(z_1,z_2) := \sup_{p\in X} \angle_p(z_1,z_2)$$

Prop: This is an isom-invariant metric

Pf: Δ-ineq for based angles: $\angle_p(c_1,c_3) \le \angle_p(c_1,c_2) + \angle_p(c_2,c_3)$
(Law of Cosines in Euc. Space)

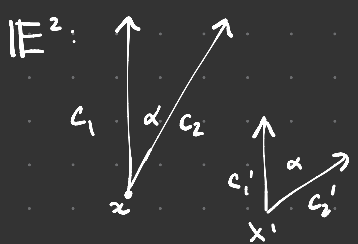
Positivity:



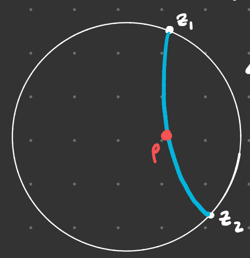
When is the sup in the def^n a max?
Sufficient: $\mathrm{Isom}(X)$ acts cocompactly on X
$\angle(z_1,z_2) = \angle_p(z_1,z_2)$ for some $p \in X$

↳ NOTE: This does not induce the same topology (cone topology) from yesterday

$\mathbb{E}^2$:



$\mathbb{H}^2$:



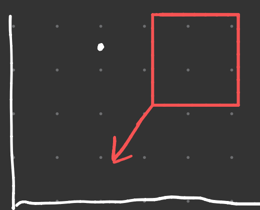
$\angle(z_1,z_2) = \pi$
↳ get discrete topology

$\boxed{?}$ What does cocompact mean?

$\boxed{A}$: A cocompact action: $\exists K \subset X$ s.t. for every $x \in X$, $\exists$ an isometry of X, φ, s.t. $\varphi(x) \in K$

A cocompact space is a space with a cocompact action via isometries

Ex. · Universal covers of nonpos. curved spaces



Def^n: A CAT(0) space X is a visibility space if $\forall z_1, z_2$, $z_1 \ne z_2 \in \partial X$
$\exists\, c : (-\infty,\infty) \to X$ geodesic s.t. $c(\infty) = z_1$, $c(-\infty) = z_2$

↳ $\boxed{\text{Ex}}$ · $\mathbb{E}^2$ is not a visibility space, but $\mathbb{H}^2$ is

- visibility spaces ⇝ negative curvature
  ↳ so trees are visibility spaces

Thm: X is a proper, complete, cocompact CAT(0) space. (cocompact := Isom(X) acts cocompactly)

↳ X is a visibility space ⟺ X does not contain an isometrically embedded copy of $\mathbb{E}^2$

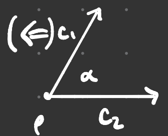
(Main ingredient of Pf): Flat triangles in CAT(0) spaces. $\Delta(p,q,r)$ in X, $\alpha = \measuredangle_p(q,r)$
$\bar\Delta(p,q,r)$ comp. triangle, $\bar\alpha = \measuredangle_p(\bar q, \bar r)$

CAT(0) $\alpha \leq \bar\alpha$

Flat triangle lemma: if $\alpha = \bar\alpha$, then convex hull of $\Delta(p,q,r)$ is isometric to convex hull of $\bar\Delta(\bar p, \bar q, \bar r)$

($\Rightarrow$) of visibility space statement is clear: can't draw geodesics between pts in ∂ of embedded $\mathbb{E}^2$

($\Leftarrow$)



$\measuredangle_p(c_1,c_2) = \measuredangle(c_1,c_2) < \pi$
WTS $\measuredangle_p(c_1(t), c_2(t))$ as $t \to \infty$ is equal to $\measuredangle_p(c_1,c_2)$

Concluding: X contains large flat triangles w/ fixed angle
$\Rightarrow$ X contains arbitrarily large flat disks
$\Rightarrow$ X contains a copy of $\mathbb{E}^2$ (exercise)

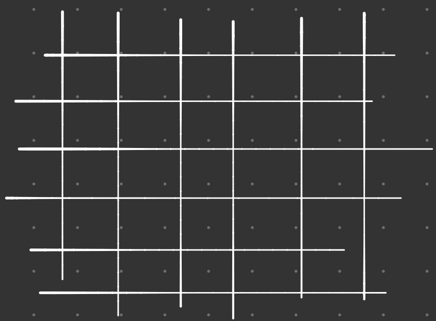
CAT(0) Groups: · Groups can be made into metric spaces (Cayley graph) after choosing presentation
· might want: groups where Cayley graph is CAT(0)
↳ BUT: a graph G is CAT(0) iff its a tree $\Rightarrow$ only yields free groups

Def^n: A group Γ is CAT(0) if it acts properly discont. and cocompactly by isometries (geometric action) on a proper CAT(0) space X

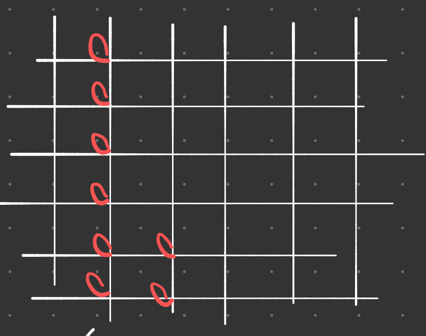
Proper discontinuity: $\forall K \subset X$ compact, the set $\{\gamma \in \Gamma : \gamma . K \cap K \neq \emptyset\}$ is finite (pt stabilizers are finite)

↳ Γ acts freely on a mfld M, Γ acts properly discont. ⟺ M / Γ is a covering map (M/Γ is a mfld)

Geometric action: Cayley graph "coarsely embeds" into the space X, via orbit map $\gamma \mapsto \gamma . x$, $x \in X$ fixed



$\mathbb{Z}^2 \curvearrowright \mathbb{E}^2$ by trans.



$\mathbb{Z}^2 \times F$, F a finite group

$\mathbb{Z}^2 \times F$ acts on $\mathbb{E}^2$ w/ F acting trivially #; $\mathbb{Z}^2 \rtimes \mathbb{Z}/2 \curvearrowright \mathbb{E}^2$ $(x,y) \mapsto (x,-y)$

$\boxed{\text{FUN FACT}}$: CAT(0) groups have solvable word problems (in quadratic time)

Boundaries of CAT(0) Groups: These do not make sense
↳ CAT(0) space X is not well-defined

(Croke+Kleiner) $\boxed{\text{Ex}}$ $\mathbb{Z}^2 \curvearrowright \mathbb{E}^2$ but also $\mathbb{Z}^2 \curvearrowright \mathbb{E}^2 \times [0,1]$

↳ **Thm**: $\exists$ a CAT(0) group $\curvearrowright$ geometrically on 2 CAT(0) spaces with non-homeomorphic ∂'s.

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