Boundaries of Groups & Spaces

6/3/2020

Last Time · CAT(0) groups acts properly discont. & cocompactly on a CAT(0) space X

· Neither X or ∂X is well-def^{n}d (in general)

Sometimes ∂X is well-defined.

Ex. Let $\Gamma = \pi_1(S_g)$ $S_g :=$ closed surface of genus ≥ 2, Γ is CAT(0): S_g has a hyperbolic structure
$\tilde{S}_g = \mathbb{H}^2$, $S_g = \mathbb{H}^2/\Gamma$, Γ acts geometrically on $\mathbb{H}^2$

Prop: every CAT(0) space in which Γ acts geometrically has $\partial \cong$ to $\partial\mathbb{H}^2 = S^1$

WANT property of metric spaces shared by all metric spaces on which a fixed group $\curvearrowright$ geomet.

Def^n: X, Y metric spaces, $K \geq 1$, $A \geq 0$, $f: X \to Y$ is a (K,A)-quasi-isometric embedding if $\forall\, x_1, x_2 \in X$:

$$\tfrac{1}{K} d(f(x_1), f(x_2)) - A \leq d(x_1, x_2) \leq K\, d(f(x_1), f(x_2)) + A$$

f is a Q.I. embedding if its (K,A)-QI embedding for some K, A

↳ NOT necess. injective or continuous

A QI embedding is a QIsometry (QI) if its quasisurjective: $\exists D > 0$ s.t. $\forall y \in Y$, $\exists x \in X$ s.t. $d(f(x), y) < D$

↳ D-nbhd image is all of Y

QI is an equivalence relation.

Examples:
- any bounded metric is QI to a pt
- all norms on $\mathbb{R}^d$ induce QI metrics
- $\mathbb{Z}^2$ is QI to $\mathbb{E}^2$

Thm (Milnor-Schwarz Thm): If $\Gamma \curvearrowright$ geometrically on a proper geodesic metric space X, then:

(i) Γ is fin. generated (ii) any Cayley graph (for a fin. generating set) of Γ is QI to X

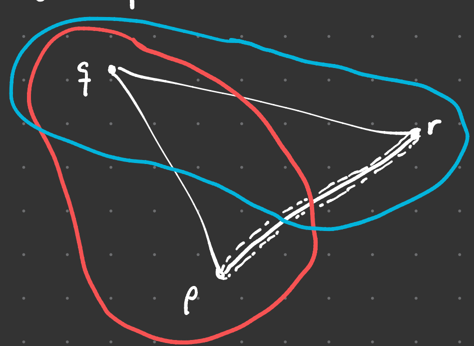
QI invariants of metric spaces are isom. invariants of grps $\curvearrowright$ geometrically

Properties of metric spaces

QI invariant	not QI invariant
· boundedness	· comp
· # of ends	· conn
· δ-hyperbolicity	· k-conn
	· exist of geo
	· isom
	· CAT(0)

Def^n: X a geodesic metric space. A triangle $\Delta(p,q,r)$ in X is δ-slim if for some $\delta \geq 0$

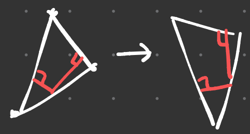
$$N_\delta([p,q]) \cup N_\delta([q,r]) \supset [p,r]$$



Δ

Examples: · Trees  triangles are tripods, 0-slim triangles. every side of a triangle is contained in union of the other sides

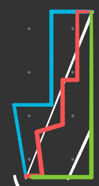
· $\mathbb{H}^2$: (exercise) (do in UHSpace)  "looks like a tripod"

· $\mathbb{E}^2$ is not hyp.: 

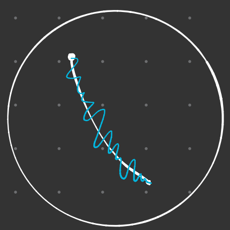
· Any bounded space is hyperbolic

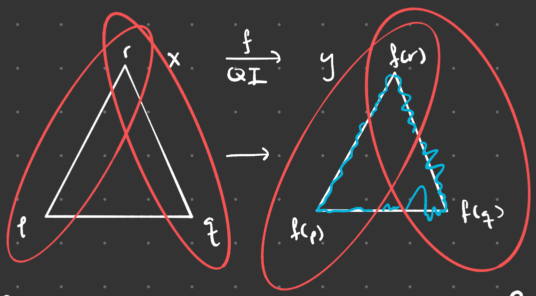
Prop: Hyperbolicity is a QI invariant: If Y is δ-hyp., $f: X \to Y$ (K,A)-QIsom, then X is δ'-hyp. for δ' depending only on K, A

Def^n: A (K,A)-quasigeodesic is a (K,A)-QI embedding $I \to X$, where I interval in $\mathbb{R}$

in $\mathbb{E}^2$ 

Thm (Morse Lemma): X is a δ-hyp metric space, $c: [a,b] \to X$ a (K,A)-quasigeodesic joining $x_1, x_2 \in X$.
$\exists L = L(\delta, K, A)$ s.t. $d_{Haus}(c[a,b], [x_1, x_2]) \leq L$ NOT on $d(x_1, x_2)$





$f^{-1} \circ f$ moves points a bounded amount

Def^n: A hyperbolic group is a group whose Cayley graph is hyperbolic, OR $\curvearrowright$ geom. on a hyp. space (up to QI)

Examples: · Surface groups are QI to $\mathbb{H}^2$ 

· free groups: Cayley graphs are trees

· π_1 of nonpos. curved mflds

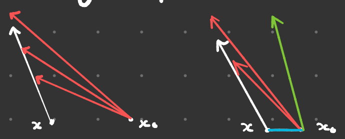
NON Examples: · $\mathbb{Z}^d$, $d \geq 2$, $\tilde{\mathbb{E}}^d$

· Any group containing $\mathbb{Z}^2$

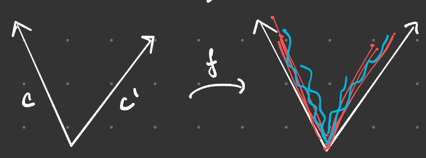
Boundaries of hyp. groups/spaces:

X hyperbolic, $\partial X = \left\{ \begin{array}{c} \text{infinite geodesic rays} \\ c: [0,\infty) \to X \end{array} \right\} / \sim$, $c_1 \sim c_2$ if $d_{Haus}(c_1, c_2) < \infty$

∂X is a bijection w/ classes of rays based at a fixed point $x_0 \in X$



∂X is (on a set) QI invariant:



Topologize: Can do same thing as for CAT(0) spaces: topologize space of rays $c: [0,\infty) \to X$ based at x_0 using compact-open topology. Take quotient by $\sim$

① This QI-invariant: QI's induce homeos ② This agrees with CAT(0) ∂ if space is CAT(0)

Examples: $\partial \mathbb{H}^2$ is S^1
$\partial(\pi_1 S_g)$ is S^1
$\partial(F_d)$ is Cantor Set
↳ free grp on d generators

$\boxed{?}$ Given a hyp. group, can we find a CAT(0) space on which it acts?
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