# Boundaries of Groups & Spaces

6/4/2020

**Thm**: If $X$ is hyperbolic, $\partial X$ is metrizable. (induces cone topology)

**Def$^n$**: $X$ a metric space, $x, y, w \in X$. The Gromov product is

$$(x \cdot y)_w := \tfrac{1}{2}\left[ d(x,w) + d(y,w) - d(x,y) \right]$$

$\boxed{\text{Ex.}}$ on a tree $(x\cdot y)_w = d(w, [x,y])$

↳ generally, $0 \le (x\cdot y)_w \le d(w,[x,y])$


For any $k \ge 0$: $a \le_k b$ ("a is coarsely less than b") if $a \le b + k$

$a \simeq_k b$ if $a \le_k b$ and $b \le_k a$ (not transitive, but we'll pretend)

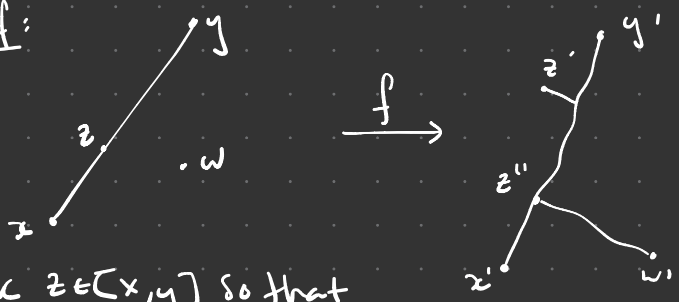
**Prop**: $X$ is $\delta$ hyperbolic, $S \subset X$, $\# S = n < \infty$. $\exists$ a map $f: S \to T$, $T$ a tree, s.t.

$d_X(x,y) \simeq_k d_T(f(x), f(y)) \; \forall x, y \in S$

$k$ only depends on $\delta$ and $n$

**Prop**: $X$ $\delta$-hyp., $x, y, w \in X$, $(x\cdot y)_w \simeq_k d(w, [x,y])$

**Pf**:




$(x'\cdot y')_{z'} \simeq (x\cdot y)_z = 0$

$\Rightarrow z'$ is close to $[x', y']$

$\Rightarrow z'$ is close to either $[x', w']$ or $[y', w']$

WLOG $(x'\cdot w')_{z'} \simeq_k 0$

$(y' \cdot w')_{z'} \simeq_k 0$

$z'$ is close to $z''$

$d(w', z'') = (x'\cdot y')_{w'}$

$= d(w', [x', \quad])$

$\simeq_k (x\cdot y)_w$

pick $z \in [x,y]$ so that $(x\cdot w)_z = (y\cdot w)_z$ ⟶ goes to 0 as $z \to x, y$ resp.

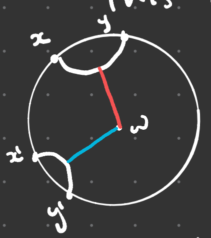
$d(w, z) \simeq_k (x\cdot y)_w \to \ge d(w, [x,y]) \Rightarrow d(w,[x,y]) \simeq_k (x, y)_w$

**REM** Many statements about distances between finitely many points and finitely many geodesics in hyperbolic spaces are true because they're true on trees

**Def$^n$**: $x, y \in X \cup \partial X$, $w \in X$, $(x\cdot y)_w := \sup\left(\liminf_{i,j\to\infty} (x_i \cdot x_j)_w\right)$ (over all sequences $x_i, y_j \to x, y$)

would be well-.

This is $\simeq_k d(w, [x,y])$




For $z_1, z_2 \in \partial X$, def$^n$ $\rho_w(z_1, z_2) = e^{-(z_1 \cdot z_2)_w}$

↳ nearly a metric, but does not satisfy the $\Delta$-inequality (but up to add. const. it does!)

We can def$^n$ a metric $d_w$ on $\partial X$ so that: $k_1 \rho_w(z_1, z_2) \le d_w(z_1, z_2) \le k_2 \rho_w(z_1, z_2)$ visual metric

- induces cone topology on $\partial X$
- so does $\rho_w(z_1, z_2)$

↳ But, this depends on basepoint (some notion of equivalence)

$\Gamma$ is a hyperbolic group. $\Gamma$ is "coarsely equivalent" to space of ideal triangles in $\partial\Gamma$

$\triangleright$ Ideal triangles: $(z_1, z_2, z_3) \in (\partial\Gamma)^3$ s.t. $z_i \neq z_j$; {Space of ideal $\Delta$'s} $\subset (\partial\Gamma^3)$, denoted by $U_1\Gamma$

$\Gamma \curvearrowright U_1\Gamma$ properly discont. & cocompactly (but since no metric, no isometries $\Rightarrow$ can't be geometric)

$\mathbb{H}^2$:


claim: $\Delta$'s with same center are rotations

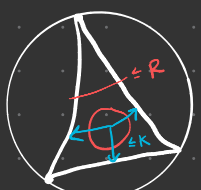
isometry-equiv projection $c: U_1(\partial\mathbb{H}^2) \longrightarrow \mathbb{H}^2$ which is proper

$\pi_1 S_g \curvearrowright U_1(\partial\mathbb{H}^2)$ $\qquad$ $\pi_1 S_g \curvearrowright \mathbb{H}^2$

Properness $\Rightarrow \pi_1 S_g$ on $U_1(\partial\mathbb{H}^2)$ is also cocompact & prop. discont. $\triangleleft$

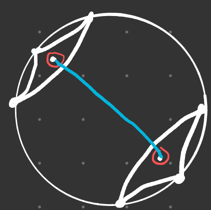
$\hookrightarrow$ We can do the same thing in hyperbolic groups: (we consider a "coarse center")

Prop: $X$ $\delta$-hyp $\exists R, K = R(\delta), K(\delta)$ s.t. for any $T = \Delta(z_1, z_2, z_3)$ in $U_1X$, the set $\{x \in X : d(x, [z_i, z_j]) < K\}$ is nonempty and has diam $\leq R$ $= C(T)$ (coarse center) (this is true in a tree)




Set of triangles whose centers intersect is compact in $U_1X$
proper equivalent projection: $U_1\Gamma \to \Gamma$

$\triangleright$ Metrize $U_1\Gamma$



$d(T_1, T_2) = d_{Haus}(c(T_1), c(T_2))$

$\Gamma$ acts by isom! But:
1. not really a metric
2. might not be geodesic $\triangleleft$

$\hookrightarrow$ Can fix all three problems & conclude: $\Gamma$ is QI to $U_1\Gamma$

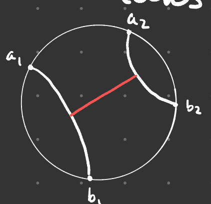
Another way to metrize this space: CROSS RATIO

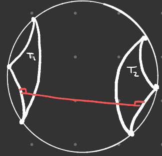
Def$^n$: The cross ratio of 4 pts $a_1, b_1, a_2, b_2 \in \partial\gamma$: $[a_1, b_1; a_2, b_2]_w = \dfrac{d_w(a_1, b_1) \cdot d_w(a_2, b_2)}{d_w(a_1, a_2) \cdot d_w(b_1, b_2)}$

$\hookrightarrow$ not preserved by $\Gamma$ action.

$\log([a_1, b_1; a_2, b_2]_w) \simeq_K d([a_1, b_1], [a_2, b_2])$ (true in a tree)

$\hookrightarrow$ looks like a sum & diff of Gromov products




$\max_{a_i, b_i \in T_i} \log|[a_1, b_1; a_2, b_2]| \simeq_K \max d([a_1, b_1], [a_2, b_2]) \simeq_K d(c(T_1), c(T_2))$ (true in a tree)

Thm (Paulin): $\Gamma_1$ and $\Gamma_2$ are hyperbolic groups. Suppose $\exists$ homeo $f: \partial\Gamma_1 \to \partial\Gamma_2$ s.t. $f$ and $f^{-1}$ are quasimöbius: $\exists \eta: \mathbb{R}^+ \to \mathbb{R}^+$ s.t.

(Teddy's Favorite™) $\quad [f(a), f(b); f(c), f(d)] \leq \eta([a, b; c, d])$; $\Gamma_1$ is QI to $\Gamma_2$

$\hookrightarrow$ Thm (Bowditch): If $\Gamma$ acts prop. discont. and cocompact on triples in a space which is compact metrizable, perfect (no isolated pts) then $\Gamma$ is hyp. and $\partial\Gamma$ is homeo to space

Punchline: Group action is enough to reconstruct the cross-ratio