Boundaries of groups and spaces: day 1 exercises

June 1, 2020

1. Prove X is CAT(0) if and only if: for every triangle $\Delta(p,q,r)$ and every point $x \in [p,q]$
$$d(x,r) \leq d(\bar{x},\bar{r}).$$

2. Prove that if X, Y are CAT(0), and $Z \subset X$, $Z' \subset Y$ are convex subspaces with Z isometric to Z', then the space obtained by gluing X and Y along an isometric identification Z, Z' is CAT(0).

3. Prove that if X is CAT(0) and $Y \subseteq X$ is convex, then ∂Y embeds into ∂X.

4. Prove that in a CAT(0) space, the distance function satisfies a *strict convexity* property: for any geodesic $c : [0,1] \to X$, and any point $x \in X$, the function
$$f(t) = d(x, c(t))^2$$
satisfies
$$f(t) < (1-t)f(0) + tf(1)$$
for all $t \in (0,1)$.

   (Note the ^{2} in the definition of f, and the strict inequality!)

   Use this to prove that: for any closed convex set $Y \subseteq X$, and any $z \in X$, there is a unique point $y \in Y$ such that
$$d(z,y) = d(z,Y).$$

5. Find an example of a CAT(0) space whose boundary has exactly n connected components.

6. Building an explicit basis for the topology on $\bar{X}$: Fix a basepoint $x_0 \in X$, so that each point in $\bar{X}$ is identified with a geodesic ray or segment starting at x_0.

   For an infinite geodesic ray c, and $r, \varepsilon > 0$, define the set $U(c,r,\varepsilon)$ by
$$U(c,r,\varepsilon) := \{x \in \bar{X} : d(x,x_0) > r, d_{\text{Haus}}(x \cap B(x_0,r), c \cap B(x_0,r)) < \varepsilon\}.$$
   (if x is an infinite geodesic ray, then $d(x,x_0) = \infty$).

   Show that the set of open balls in X together with the sets $U(c,r,\varepsilon)$, ranging over all $[c] \in \partial X$, $r, \varepsilon > 0$, give a basis for $\bar{X}$.