Boundaries of groups & spaces

Day 1,2: CAT(0) spaces
Day 3-5: hyperbolic & relatively hyperbolic groups

Reference: Bridson & Haefliger
Metric spaces of non-positive Curvature

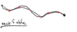
Metric spaces:

- always complete
- usually <u>proper</u>: every closed ball of finite radius is compact
- usually <u>geodesic</u>: if $x, y \in X$ have $d(x,y) = L$, $\exists$ path of length L in X joining x & y. (geodesic)

<u>NOT</u>: uniquely geodesic

X a geodesic metric space, $x, y \in X$. $[x,y]$ is <u>some</u> geodesic joining x & y.

<u>Rectifiable paths:</u>



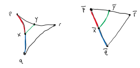
CAT(0) spaces:

X metric space

<u>Def:</u> a triangle $\Delta(p,q,r)$ $p,q,r \in X$ is a union of 3 geodesic segments $[p,q], [q,r], [p,r]$.

$p,q,r \in X$, a <u>Euclidean comparison triangle</u> $\overline{\Delta}(\bar{p}, \bar{q}, \bar{r})$ is a triangle $\mathbb{E}^2$ whose sides have lengths $d(p,q)$, $d(q,r)$, $d(p,r)$.
(unique up to isometry of $\mathbb{E}^2$).
vertices are $\bar{p}, \bar{q}, \bar{r}$.

Given a triangle $\Delta(p,q,r)$ in X, $x \in [p,q]$ <u>a comparison pt</u> for x is $\bar{x} \in [\bar{p}, \bar{q}]$ s.t. $d(\bar{x}, \bar{p}) = d(x, p)$.

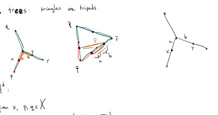


X is CAT(0) if for all $\Delta(p,q,r)$ in X, all $x, y \in \Delta(p,q,r)$,

$$d_X(x,y) \le d_{\mathbb{E}^2}(\bar{x}, \bar{y}).$$

<u>Examples:</u>

- $\mathbb{E}^d$
- trees: triangles are tripods.



- $\mathbb{H}^d$:

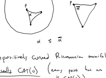
<u>Def:</u> given $x, p, q \in X$

$$\measuredangle_x(p,q) := \limsup_{t,t' \to 0} \measuredangle_{\bar{x}}\left(\overline{c(t)}, \overline{c'(t')}\right)$$

where $c, c' : [0,1] \to X$ are geodesics joining x to p & q respectively.

If X is a Riemannian manifold, this agrees with angle between tangent vectors pointing from x to p and q.

<u>Prop:</u> X is CAT(0) $\iff$ for every $\Delta(p,q,r)$ in X, $\measuredangle_p(q,r) \le \measuredangle_{\bar{p}}(\bar{q}, \bar{r})$.



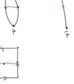
$\alpha \le \bar{\alpha}$

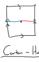
- nonpositively curved Riemannian manifold is <u>locally CAT(0)</u> (every point has an open nbhd which is CAT(0)). (nonpositively curved).
- products of CAT(0) spaces
- identifications of CAT(0) spaces along convex subsets.

<u>Nonexamples:</u>

- Spheres, graphs with nontrivial loops

<u>Prop:</u> CAT(0) spaces have unique geodesics.





<u>Cartan-Hadamard Theorem:</u>

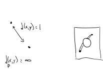
<u>Thm.</u> X is locally CAT(0) (complete, proper, geodesic).
$x_0 \in X$

$$\tilde{X}_{x_0} = \left\{ \begin{matrix} \text{reparametrized local} \\ \text{geodesics } c:[0,1] \to X \end{matrix} : c(0) = x_0 \right\}$$

w/ metric: $d(c, c') = \sup_{t \in [0,1]} d(c(t), c'(t))$.

$\tilde{X}_{x_0} \to X$ is a universal covering
$c \mapsto c(1)$

$\tilde{X}_{x_0}$ w/ <u>path metric</u> coming from d is (globally) CAT(0). ← length of shortest path in $\tilde{X}_{x_0}$ is distance between its endpoints.



$\implies$ CAT(0) spaces are contractible.

<u>Boundaries of CAT(0) spaces</u>

X metric space, $\overline{X}$ a compactification ($X \hookrightarrow \overline{X}$ is open and dense)

$\partial X = \overline{X} - X$ | X is CAT(0).

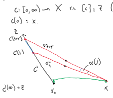
$$\partial X = \{ \text{infinite geodesic rays } c : [0,\infty) \to X \} / \sim$$

$c_1 \sim c_2$ if images have finite Hausdorff distance. $(d_{Haus}(c_1, c_2) < \infty)$.

In $\mathbb{E}^d$: $d_{Haus}(c_1, c_2) < \infty$ iff they're parallel and point in the same direction.

$\partial \mathbb{E}^d$ in bijection with S^{d-1}.

<u>Prop:</u> If $z \in \partial X$, $x \in X$, then $\exists$ unique ray $c : [0,\infty) \to X$ s.t. $[c] = z$ $(c(\infty) = z)$ and $c(0) = x$.



Fix $t' > 0$. $\alpha(t)$ goes to 0 uniformly in t', as $t \to \infty$.

Fix s, take $c(s) = \lim_{t \to \infty} \sigma_t(s)$.

- converges to a geodesic, w/ Hausdorff dist between c and c' is at most $d(x_0, x)$

∂X in bijection with rays based at some fixed basepoint x_0.

<u>Topologizing $X \cup \partial X = \overline{X}$:</u>

X is homeomorphic to <u>eventually constant</u> geodesics: $c : [0,\infty) \to X$ s.t. for some T, for all $t > T$, $c(t) = c(T)$, $c|_{[0,T]}$ is geodesic.

X in bijection w/ eventually constant geodesics starting at a fixed basepoint, ending at $x \in X$.

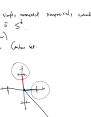
topology: compact-open topology, uniform convergence on compacts.

$\overline{X}$ is $X \cup \partial X$ is a set of geodesic rays $c : [0,\infty) \to X$, topologized w/ compact-open topology.

$\overline{X}$ is compact, X is open and dense.

<u>Examples:</u>

- X is a simply connected nonpositively curved Riemannian mfd, $\partial X \cong S^d$.
- ∂ tree (infinite regular) is Cantor set:



- $\partial(X_1 \times X_2)$ is the <u>join</u> $\partial X_1 * \partial X_2$: