Boundaries of groups and spaces: day 2 exercises

June 2, 2020

1. Give an example of a CAT(0) space where the boundary with the topology induced by the *angle metric* is connected, but not a sphere.

2. Give an example of a complete and proper CAT(0) space which is not a visibility space, but nevertheless does not contain an isometric copy of $\mathbb{E}^2$. (Which hypothesis of the theorem are we leaving out?)

3. (Exercise II.2.10 in Bridson and Haefliger) Let $\Delta(p,q,r)$ be a triangle in a CAT(0) space, and let $\Delta(\bar{p},\bar{q},\bar{r})$ be a Euclidean comparison triangle. Use the flat triangle lemma to show that if there is some $\bar{x}$ in the interor of $[\bar{p},\bar{q}]$ and $\bar{y}$ in $[\bar{p},\bar{r}]$ such that $d(\bar{x},\bar{y}) = d(x,y)$, then $\Delta(p,q,r)$ is flat.

4. Prove that if X is a complete, proper, and cocompact metric space containing isometric copies of arbitrarily large flat disks, then X contains an isometric copy of $\mathbb{E}^2$. (Hint: pick a countable dense subset of $\mathbb{E}^2$ and argue that there is an isometric embedding of this subset into X).

5. Prove that if Γ_1, Γ_2 are CAT(0) groups, then the free product $\Gamma_1 * \Gamma_2$ is also CAT(0).

6. Prove that if M is a manifold, and Γ is a group acting freely and properly discontinuously on M, then $M \to M/\Gamma$ is a covering map (and so M/Γ is a manifold).