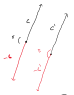
$X$ a CAT(0) space

$\partial X$: $\{$infinite geodesic rays in $X\}/\sim$, topologized via uniform convergence.

- isometries of $X$ induce homeomorphisms of $\partial X$.

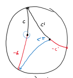
$\mathbb{E}^2$ and $\mathbb{H}^2$ have homeomorphic boundaries ($S^1$).

$\mathbb{E}^2$:




Transitive subgroup of $\mathrm{Isom}(\mathbb{E}^2)$ acts trivially on $\partial\mathbb{E}^2$

$\mathbb{H}^2$:




$\mathrm{Isom}(\mathbb{H}^2)$ has nontrivial action on $\partial\mathbb{H}^2$

## Angle metric on $\partial X$

Def: $p \in X$, $c_1, c_2 : [0,1] \to X$, $c_i(0) = p$

$$\measuredangle_p(c_1, c_2) := \limsup_{t,t' \to 0} \measuredangle_{\bar p}\left(\overline{c_1(t)}, \overline{c_2(t')}\right) = \lim_{t \to 0} \measuredangle_{\bar p}\left(\overline{c_1(t)}, \overline{c_2(t)}\right)$$

$z_1, z_2 \in \partial X$

$$\measuredangle_p(z_1, z_2) := \measuredangle_p(c_1, c_2) \text{ where } c_i(\infty) = z_i,\ c_i(0) = p.$$

$$\measuredangle(z_1, z_2) := \sup_{p \in X} \measuredangle_p(z_1, z_2)$$

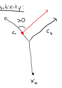
Prop: This is an isom-invariant metric on $\partial X$.

Pf: Triangle inequality for based angles:

$$\measuredangle_p(c_1, c_2) \le \measuredangle_p(c_1, c_3) + \measuredangle_p(c_2, c_3).$$

(Law of cosines in Euclidean space)

Positivity:



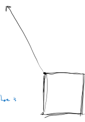

- When is the sup in definition a max?

Sufficient: $\mathrm{Isom}(X)$ acts cocompactly on $X$.

$\measuredangle(z_1, z_2) = \measuredangle_p(z_1, z_2)$ for some $p \in X$.

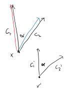
Cocompact action: $\exists$ compact $K \subset X$ s.t. ~~for~~ every $x \in X$, there is an isometry of $X$ $\varphi$ s.t. $\varphi(x) \in K$.

(universal covers of compact nonpositively curved spaces).

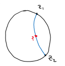


Does NOT induce $\underbrace{\text{the topology}}_{\text{cone topology}}$ from yesterday.

$\mathbb{E}^2$:




$\mathbb{H}^2$:




$\measuredangle(z_1, z_2) = \pi$

get discrete topology.

## Visibility Spaces:

Def. A CAT(0) space $X$ is a visibility space if $\forall z_1, z_2$ $z_1 \ne z_2$ $\in \partial X$ $\exists$ $c : (-\infty, \infty) \to X$ geodesic s.t. $c(\infty) = z_1$, $c(-\infty) = z_2$.

- $\mathbb{E}^2$ is <u>not</u> a visibility space.
- $\mathbb{H}^2$ is a visibility space.
- Visibility spaces ~ negative curvature
- trees are visibility spaces

Thm: $X$ is a proper, complete, cocompact CAT(0) space.

Isom(X) acts cocompactly

$X$ is a visibility space $\iff$ $X$ does not contain an isometrically embedded copy of $\mathbb{E}^2$.

Main ingredient: Flat triangles in CAT(0) spaces.

$\Delta(p,q,r)$ in $X$, $\alpha = \measuredangle_p(q,r)$.

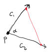
$\bar\Delta(\bar p, \bar q, \bar r)$ comparison triangle, $\bar\alpha = \measuredangle_{\bar p}(\bar q, \bar r)$.

CAT(0): $\alpha \le \bar\alpha$.

Flat triangle lemma: if $\alpha = \bar\alpha$, then convex hull of $\Delta(p,q,r)$ is isometric to convex hull of $\bar\Delta(\bar p, \bar q, \bar r)$.

$\Rightarrow$ of visibility space statement is clear: can't draw geodesics between pts in boundary of embedded $\mathbb{E}^2$.

$\Leftarrow$:




$\measuredangle_p(c_1, c_2) = \measuredangle(c_1, c_2) < \pi$.

$\measuredangle_p(c_1(t), c_2(t))$ as $t \to \infty$ is equal to $\measuredangle_p(c_1, c_2)$.

Concluding: $X$ contains large flat triangles w/ fixed angle.

$\Rightarrow$ $X$ contains arbitrarily large flat disks.

(exercise) $\Rightarrow$ $X$ contains a copy of $\mathbb{E}^2$.

## CAT(0) groups:

- Groups can be made into metric spaces (Cayley graph) after choosing presentation.
- might want: group whose Cayley graph is CAT(0).

A graph is CAT(0) $\iff$ it's a tree.

Def: A group $\Gamma$ is CAT(0) if it acts $\underbrace{\text{properly discontinuously and cocompactly by isometries}}_{\text{geometric action}}$ on a proper CAT(0) space $X$.

Proper discontinuity: $\forall K \subset X$ compact, the set

$$\{\gamma \in \Gamma : \gamma \cdot K \cap K \ne \emptyset\}$$

is finite.

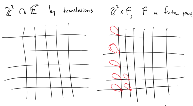
(point stabilizers are finite).

$\Gamma$ acts freely on a manifold $M$,

$\Gamma$ acts properly discontinuously $\iff$ $M \to M/\Gamma$ is a covering map. ($M/\Gamma$ is a manifold).

geometric action: Cayley graph "coarsely embeds" into the space $X$, via orbit map $\gamma \mapsto \gamma \cdot x$, $x \in X$ fixed.

$\mathbb{Z}^2 \curvearrowright \mathbb{E}^2$ by translations. $\mathbb{Z}^2 \times F$, $F$ a finite group:



$\mathbb{Z}^2 \times F$ acts on $\mathbb{E}^2$ w/ $F$ acting trivially:

$\mathbb{Z}^2 \rtimes \mathbb{Z}/2 \curvearrowright \mathbb{E}^2$

$(x,y) \mapsto (x,-y)$

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- CAT(0) groups have solvable word problem. (in quadratic time).

## Boundaries of CAT(0) groups:

- these do not make sense.

CAT(0) space $X$ is <u>not</u> well-defined.

$\mathbb{Z}^2 \curvearrowright \mathbb{E}^2 \times [0,1]$

Thm: (Croke and Kleiner):

exists a CAT(0) group acting geometrically on two CAT(0) spaces w/ nonhomeomorphic boundaries.