Boundaries of groups and spaces: day 3 exercises

June 3, 2020

1. Use the Milnor-Schwarz theorem to show: if G is a finitely generated group, and H is a finite-index subgroup, then H is quasi-isometric to G.

2. (a) Find an example of a hyperbolic metric space where geodesics are not unique.

   (b) Find an example of a hyperbolic group Γ such that: for some $z \in \partial\Gamma$, there are geodesic rays $c_1, c_2 : [0, \infty) \to \Gamma$ with $c_i(\infty) = z$, $c_1(0) = c_2(0)$, and c_1 and c_2 only intersect at their basepoint.

3. Prove that the Morse lemma is true on a tree.

4. Prove that $\mathbb{H}^2$ is δ-hyperbolic, and find the optimal δ.

5. Prove that if the boundary of a hyperbolic group Γ has exactly two points in it, then Γ is quasi-isometric to $\mathbb{Z}$.

   (In fact, it's true that in this case Γ is virtually isomorphic to $\mathbb{Z}$, although this is harder to prove.)

6. Prove that if X is CAT(0) and hyperbolic, it is a visibility space.

   Remark: if in addition X is cocompact (e.g. if Γ is a hyperbolic group acting geometrically on X) then the converse is also true (although it is somewhat more difficult to prove). Ultimately one can conclude that a CAT(0) group is hyperbolic if and only if it does not contain a quasi-isometrically embedded copy of $\mathbb{E}^2$.