- Last time: CAT(0) groups: Γ acts prop. discontinuously & cocompactly on a CAT(0) space X
- Unlike X or ∂X is well-defined (in general).

Sometimes: ∂X is well-defined.

$\Gamma = \pi_1 S_g$, S_g = closed surface of genus $g \geq 2$.

Γ is CAT(0): S_g has a hyperbolic structure, $\tilde{S}_g \cong \mathbb{H}^2$. $S_g = \mathbb{H}^2/\Gamma$, Γ acts geometrically on $\mathbb{H}^2$.

Prop: Every CAT(0) space on which Γ acts geometrically has ∂ ≅ to $\partial\mathbb{H}^2 \cong S^1$.

What property of metric spaces shared by all metric spaces on which a fixed group acts geometrically.

Def: X, Y metric spaces, $K \geq 1$, $A \geq 0$.

$f: X \to Y$ is a (K,A)-quasi-isometric embedding (QI embedding) if for all $x_1, x_2 \in X$,

$$\frac{1}{K} d(f(x_1), f(x_2)) - A \leq d(x_1, x_2) \leq K\, d(f(x_1), f(x_2)) + A$$

f is a QI embedding if it is (K,A)-QI embedding for some K, A.

Not necessarily injective, or continuous.

A QI embedding is a quasi-isometry (QI) if it is quasi-surjective: $\exists D > 0$ s.t. $\forall y \in Y$ there is $x \in X$

$$d(f(x), y) < D.$$

D-nbhd of image is all of Y.

QI is an equivalence relation.

Examples:
- any bounded metric space is QI to a point.
- all norms on $\mathbb{R}^n$ induce QI metrics.
- $\mathbb{Z}^2$ is QI to $\mathbb{E}^2$.

Thm (Milnor–Schwarz Theorem): (Fundamental observation of GGT):

If Γ acts geometrically on a proper geodesic metric space X, then
(1) Γ is finitely generated
(2) Any Cayley graph of Γ (from a finite generating set) is QI to X.

Quasi-isometry invariants of metric spaces are isomorphism invariants of groups acting on them geometrically.

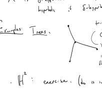
Properties of metric spaces

QI invariant	not QI invariant
boundedness	completeness
# of ends	connectedness
hyperbolicity	k-connectedness
	existence of geodesics
	isometry group
	CAT(0)

Def: X a geodesic metric space. A triangle Δ(p,q,r) in X is δ-slim if for δ ≥ 0,

$$N_\delta([p,q]) \cup N_\delta([q,r]) \supset [p,r].$$

$N_\delta(A)$ = δ-nbhd of A

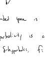


X is δ-hyperbolic if every triangle is δ-slim.
hyperbolic if δ-hyperbolic for some δ.

Examples: Trees.

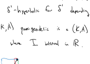
triangles are tripods
0-slim triangles.
every side of a triangle is contained in union of the other sides.

- $\mathbb{H}^2$: exercise. (do it in upper-half-plane).



"looks kind of like a tripod".

$\mathbb{E}^2$ is not hyperbolic:

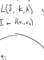


- Any bounded space is hyperbolic.

Prop: hyperbolicity is a QI invariant:
If Y is δ-hyperbolic, $f: X \to Y$ (K,A)-quasi-isometry, then X is δ'-hyperbolic for δ' depending only on K, A.

Def: A (K,A) quasigeodesic is a (K,A)-QI embedding $I \to X$, where I interval in $\mathbb{R}$.

in $\mathbb{E}^2$

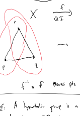


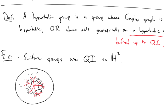
← are quasigeodesics w/ $K = \sqrt{2}$, $A = 0$.

Thm (Morse lemma):
X is a δ-hyperbolic metric space. $c: [a,b] \to X$ a (K,A)-quasigeodesic, joining x_1 and $x_2 \in X$.

$$\exists L = L(\delta, K, A) \text{ s.t. } d_{Haus}\big(c([a,b]), [x_1, x_2]\big) \leq L$$

NOT on $d(x_1, x_2)$.





$f^{-1} \circ f$ moves pts a bounded amount.

Def: A hyperbolic group is a group whose Cayley graph is hyperbolic, OR which acts geometrically on a hyperbolic space. defined up to QI.

Ex:
- Surface groups are QI to $\mathbb{H}^2$.

- free groups: Cayley graphs are trees.
- fundamental groups of negatively curved compact manifolds.

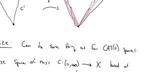
Nonexamples:
- $\mathbb{Z}^d$, $d \geq 2$ ($\simeq \mathbb{E}^d$)
- Any group containing $\mathbb{Z}^2$.

Boundaries of hyperbolic groups/spaces:

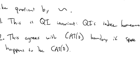
X hyperbolic.

$$\partial X = \left\{ \begin{matrix} \text{infinite geodesic rays} \\ c: [0,\infty) \to X \end{matrix} \right\} \Big/ \sim \qquad c_1 \sim c_2 \text{ if } d_{Haus}(c_1, c_2) < \infty$$

- ∂X is in bijection w/ classes of rays based at a fixed point $x_0 \in X$.



∂X is (as a set) QI invariant:



Topologize: Can do same thing as for CAT(0) spaces:
Topologize space of rays $c: [0,\infty) \to X$ based at x_0 using compact-open topology (uniform convergence on cpcts).
Take quotient by ∼.
1. This is QI invariant: QI's induce homeomorphisms
2. This agrees with CAT(0) boundary if space happens to be CAT(0).

Ex: $\partial\mathbb{H}^2$ is S^1

$\partial(\pi_1 S_g)$ is S^1

$\partial(F_2)$ = Cantor set

free gp on 2 generators

- Given a hyperbolic group, can we find a CAT(0) space on which it acts? Open.
- Metric (and other) structures on the boundary.