Boundaries of groups and spaces: day 4 exercises

June 4, 2020

1. Find an homeomorphism between the space of *ordered* ideal triangles in $\partial\mathbb{H}^2$ and the unit tangent bundle of $\mathbb{H}^2$, equivariant under isometries of $\mathbb{H}^2$.

   This is the justification for the notation $U_1\Gamma$.

2. If X is a hyperbolic space and $x \in X$, show that the collection of triangles in ∂X whose centers contain x is compact in U_1X.

3. (a) Let T be a tree, and fix a basepoint $x \in T$. For any four points a_1, a_2, a_3, a_4 in T, write down a formula for $d([a_1, a_2], [a_3, a_4])$ in terms of Gromov products $(a_i \cdot a_j)_x$. (You will need to take a max/min to deal with all of the cases.)

   (b) Show that if a_1, a_2, a_3, a_4 are points in a δ-hyperbolic metric space, then there is a tree T and a map $f : \{a_i\} \to T$ so that

   $$d([a_1, a_2], [a_3, a_4]) \simeq_k d_T([f(a_1), f(a_2)], [f(a_3), f(a_4)]),$$

   where k only depends on δ. The fact that this is not totally trivial may convince you that my "true in a tree" handwaves really were handwaves and not proofs, if you weren't already convinced.

   (c) Use (2) to see that in any δ-hyperbolic metric space, there exists a constant k such that for any 4 points in a hyperbolic metric space, at least two of the following hold:

   $$d([a_1, a_2], [a_3, a_4]) \simeq_k 0,$$
   $$d([a_1, a_3], [a_2, a_4]) \simeq_k 0,$$
   $$d([a_1, a_4], [a_2, a_3]) \simeq_k 0.$$

4. A function $f : X \to Y$ between metric spaces is called *quasisymmetric* if there is a function $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ so that for every $x_1, x_2, x_3 \in X$,

   $$\frac{d(f(x_1), f(x_2))}{d(f(x_1), f(x_3))} \leq \eta\left(\frac{d(x_1, x_2)}{d(x_1, x_3)}\right).$$

   Prove that for finitely generated free group F_d, changing the visual metric on ∂F_d by changing the basepoint is quasisymmetric.