Last time: hyperbolic spaces, defined ∂X as classes of rays in X, topologized w/ cone topology.

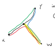
<u>Thm:</u> If X is hyperbolic, ∂X is metrizable. (induces cone topology)

<u>Def:</u> X a metric space, $x, y, w \in X$.

<u>Gromov product</u> is

$$(x \cdot y)_w := \frac{1}{2}\left[d(x,w) + d(y,w) - d(x,y) \right].$$

on a tree: $(x \cdot y)_w = d(w, [x,y])$.



in general: $0 \le (x \cdot y)_w \le d(w, [x,y])$

For any $k \ge 0$:

$a \le_k b$ ("a is coarsely less than b") if $a \le b + k$

$a \simeq_k b$ if $a \le_k b$ and $b \le_k a$

(NOT transitive, but will pretend they are).

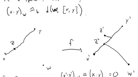
<u>Prop:</u> X is δ-hyperbolic, $S \subset X$, $\#S = n < \infty$.
There exists a map $f: S \to T$, T tree, s.t.

$$d_X(x,y) \simeq_k d_T(f(x), f(y)) \quad \forall x,y \in S.$$

k depends only on δ and n.

<u>Prop:</u> X δ-hyperbolic, $x, y, w \in X$,
$(x \cdot y)_w \simeq_k d(w, [x,y])$.

<u>Pf:</u>



pick $z \in [x,y]$ so that $(x \cdot w)_z = (y \cdot w)_z$.
(goes to 0 as $z \to x, y$ respectively.)

$(x' \cdot y')_{z'} \simeq_k (x \cdot y)_z = 0$
$\Rightarrow z'$ is close to $[x', y']$.
$\Rightarrow z'$ is close to either $[x', w']$ or $[y', w']$.
WLOG: $(x' \cdot w')_{z'} \simeq_k 0$
$(y' \cdot w')_{z'} \simeq_k 0$
$\Rightarrow z'$ is close to z''.
$d(w', z'') = (x' \cdot y')_{w'} = d(w', [x', y'])$
$\simeq_k (x \cdot y)_w$

$d(w, z) \simeq_k (x \cdot y)_w$
$\hookrightarrow \ge d(w, [x,y]) \Rightarrow d(w, [x,y]) \simeq_k (x \cdot y)_w$.

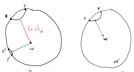
Many statements about distances between finitely many pts and finitely many geodesics in hyperbolic metric spaces are true because they're true on trees.

<u>Def:</u> $x, y \in X \cup \partial X$, $w \in X$:

$$(x \cdot y)_w := \sup \liminf_{i,j \to \infty} (x_i \cdot y_j)_w$$

$\uparrow$ over all sequences $x_i, y_j \to x, y$.
would be well defined (up to additive constants).

This is $\simeq_k d(w, [x,y])$:



For $z_1, z_2 \in \partial X$, define

$$\rho_w(z_1, z_2) = e^{-(z_1 \cdot z_2)_w}.$$

Satisfies positivity: $\rho_w(z_1, z_2) = 0 \iff z_1 = z_2$

Does <u>Not</u> in general satisfy $\Delta \ne$.

<u>Can</u> define metric d_w on ∂X so that:

$$K_1 \rho_w(z_1, z_2) \le d_w(z_1, z_2) \le K_2 \rho_w(z_1, z_2)$$

visual metric

- induces cone topology on ∂X.
- so does $\rho_w(z_1, z_2)$.

This depends on basepoint. (<u>Same</u> notion of equivalence)

Γ a hyperbolic group.

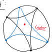
Γ is "coarsely equivalent" to space of ideal triangles in $\partial\Gamma$.

Ideal triangles: $(z_1, z_2, z_3) \in (\partial\Gamma)^3$ s.t. $z_i \ne z_j$.

{Space of ideal triangles} $\subset (\partial\Gamma^3)$ denoted by $U_1\Gamma$.

Γ acts properly discontinuously and cocompactly on $U_1\Gamma$.
(no metric)

<u>$\mathbb{H}^2$:</u>



isometry-equivariant projection
$c: U_1(\partial\mathbb{H}^2) \to \mathbb{H}^2$
which is <u>proper</u>.
(preimages of compacts are compact).

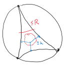
$$\begin{array}{ccc} \pi_1 S_g & & \pi_1 S_g \\ \circlearrowright & & \circlearrowright \\ U_1(\partial\mathbb{H}^2) & \longrightarrow & \mathbb{H}^2 \end{array}$$

<u>properness</u> $\Rightarrow \pi_1 S_g$ on $U_1(\partial\mathbb{H}^2)$ is also cocompact & prop. disc.

Do the same thing in hyperbolic groups:

"Coarse center":

<u>Prop:</u> X δ-hyp. $\exists R, k = R(\delta), k(\delta)$ s.t.
For any $T = \Delta(z_1, z_2, z_3)$ in $U_1 X$, the set
$\{x \in X : d(x, [z_i, z_j]) < k\}$ is nonempty and has diameter $\le R$.
$= c(T)$.

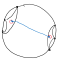


true in a tree.

Set of triangles whose centers intersect is compact in U_1X.

proper equivariant projection $U_1\Gamma \to \Gamma$

<u>Metrize $U_1\Gamma$:</u>



$d(T_1, T_2) = d_{Haus}(c(T_1), c(T_2))$.

Γ acts by isometries!

1. not really a metric.
2. might not be geodesic.

Can fix all these problems and conclude:

Γ is QI to $U_1\Gamma$.

<u>Another way:</u>

Cross-ratios. 4 pts $a_1, b_1, a_2, b_2 \in \partial\Gamma$

$$[a_1, b_1 ; a_2, b_2]_w = \frac{d_w(a_1, b_1) \cdot d_w(a_2, b_2)}{d_w(a_1, a_2) \cdot d_w(b_1, b_2)}$$

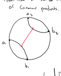
$$\frac{d_w(a_1, a_2) \cdot d_w(b_1, b_2)}{d_w(a_1, b_1) \cdot d_w(a_2, b_2)}$$

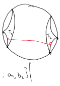
<u>NOT</u> preserved by Γ-action.

$$\log\left([a_1, b_1 ; a_2, b_2]_w\right) \simeq_k d([a_1, b_1], [a_2, b_2]).$$

$\uparrow$ true in a tree.

looks like a sum & difference of Gromov products.





$$\max_{a_i, b_i \in T_i} \log\left|[a_1, b_1 ; a_2, b_2]\right|$$
$$\simeq_k \max d([a_1, b_1], [a_2, b_2])$$
$$\simeq_k d(c(T_1), c(T_2)).$$
(true in a tree).

<u>Thm</u> (Paulin)

Γ_1 and Γ_2 are hyperbolic groups. Suppose $\exists$ homeo $f: \partial\Gamma_1 \to \partial\Gamma_2$ s.t. f and f^{-1} are quasimobius: $\exists$ function $\eta: \mathbb{R}^+ \to \mathbb{R}^+$ s.t.

$$[f(a), f(b); f(c), f(d)] \le \eta([a, b; c, d]).$$

Γ_1 is QI to Γ_2.

<u>Bowditch:</u>

The group action is enough to reconstruct the cross-ratio.

<u>Thm:</u> If Γ acts prop. discontinuously and cocompactly on triples in any space which is

- compact
- metrizable
- perfect (no isolated points)

then Γ is hyperbolic and $\partial\Gamma$ is homeo to space.