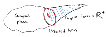
Relatively hyperbolic groups

Hyperbolic groups: model to understand QI invariant properties of "negatively curved" groups: $\pi_1 M$ for M compact negatively curved manifolds.

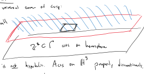
Coarse geometry of universal cover to understand group.

compact ⟹ finite volume.

Ex: Cusped hyperbolic 3-manifold M, $\pi_1 M = \Gamma$

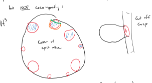


in $\mathbb{H}^3$: universal cover of cusp:



horoball

Γ is not hyperbolic. Acts on $\mathbb{H}^3$ properly discontinuously, but NOT cocompactly!



Γ acts cocompactly & prop. disc. on $\mathbb{H}^3$ − (horoballs).

not convex as subset of $\mathbb{H}^3$ w/ metric.

Take path metric on $\mathbb{H}^3$ − (horoballs).

Apply Milnor–Schwarz: Γ is QI to $\mathbb{H}^3$ − (horoballs) w/ path metric.

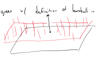
horospheres are isometrically embedded, iso. to $\mathbb{E}^2$.

Def. X hyperbolic metric space, $c: [0,\infty) \to X$

A Busemann function $b_c(x) = \lim_{t\to\infty} [d(x, c(t)) - t]$.

A horoball at $z = c(\infty)$ in ∂X is any sublevel set $b_c^{-1}(-\infty, R)$, $R \in \mathbb{R}$.

Agrees w/ definition of horoball in $\mathbb{H}^d$.



Def: A group Γ is hyperbolic relative to a collection $\mathcal{P}$ of subgroups (peripheral subgroups) if: Γ acts prop. discontinuously on a hyperbolic space X, s.t. X/Γ = (compact piece) ∪ (cusps).

cusp = image of $(H/\text{Stab}_\Gamma(H))$, H horoball in X,

$\text{Stab}_\Gamma(H)$ acts cocompactly on ∂H.

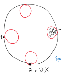
$\mathcal{P} = \{\text{Stab}_\Gamma(H) : H \text{ lift of a cusp}\}$.

"relatively hyperbolic in the sense of Gromov"

geometrically finite action of Γ on X

if X convex subset of $\mathbb{H}^d$: this is exactly geometrically finite Kleinian group.

X is not well-defined, even up to QI.



many ways to fill in the cusp respecting group action, give your finite action, NOT QI to each other.

Specialize to case of ONE cusp.

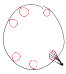
$z \in \partial X$ (cusp).

$\text{Stab}_\Gamma(z) = P = \text{Stab}_\Gamma(P)$

$\text{Stab}_\Gamma(\gamma \cdot z) = \gamma P \gamma^{-1} = \text{Stab}_\Gamma(\gamma P)$

⟹ horospheres correspond to cosets of peripheral subgroups.

Farb: Relative hyperbolicity characterized by fact non-hyperbolicity is "trapped" in these cosets.



Cone off the coset w/ extra vertex.

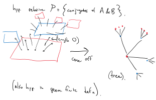
$\hat{\Gamma}$ = coned-off Cayley graph

$\hat{\Gamma}$ not proper.

- Γ is not QI to $\hat{\Gamma}$.
- QI class of $\hat{\Gamma}$ is well-defined. (in terms of Γ, $\mathcal{P}$)
- if Γ rel. hyp. (in sense of Gromov): $\hat{\Gamma}$ is hyperbolic.

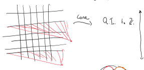
Def: Γ is weakly hyperbolic relative to $\mathcal{P}$ if coned-off Cayley graph is hyperbolic.

Ex: A, B f.g. groups. $A * B$ is (weakly) hyp. relative $\mathcal{P} = \{\text{conjugates of } A \,\&\, B\}$.

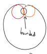


(also hyp. in geom. finite def^n).

Ex: $\mathbb{Z} \times \mathbb{Z}$, relative to a $\mathbb{Z}$ factor:



Not geometrically finite. "Cosets are too close".



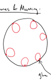
Solution:

weak relative hyperbolicity + EITHER { "bounded coset penetration" / "fineness" }

⟺ geometrically finite definition.

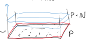
Can build a model X w/ geometrically finite action for Γ (noncanonical, but we can work with it).

Groves & Manning:



define a way to fill in cusp: "combinatorial horoball":

glue



Ensure metric on $P \times \{k\}$ shrinks exponentially in k.

Cusped space: $\Gamma \cup$ (combinatorial horoballs)

Γ is hyp. relative $\mathcal{P}$ ⟺ cusped space is hyperbolic.

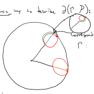
Γ acts geom. finitely on cusped space.

Thm: X not well-defined, but ∂X is.

$= \partial(\Gamma, \mathcal{P})$.

Bowditch boundary of Γ.

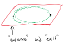
Intrinsic way to describe $\partial(\Gamma, \mathcal{P})$:



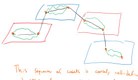
Corresponding geodesic in Γ:

Rough correspondence:

{ infinite geodesics in cusped space } ⟷ { quasi-geodesic paths w/ detours in Γ } ∪ { cosets of peripheral subgp }



"entrance" and "exit" points are well-defined.



This sequence of cosets is coarsely well-defined, and entrance & exit points are coarsely well-defined.

(Relative Morse lemma).

Given X proper geodesic metric space:

find two hyp. metric spaces w/ one pt. in their boundary which can be glued to a horosphere in a hyp. metric space Y, QI for X.