Hyperbolic groups: day 1 exercises

1. Let Γ be a finitely generated group with generating sets S_1, S_2, and let $\mathrm{Cay}(\Gamma, S_i)$ be the Cayley graph of Γ with respect to the generating set S_i. Show that there is a *bilipschitz equivalence* $\mathrm{Cay}(\Gamma, S_1) \to \mathrm{Cay}(\Gamma, S_2)$.

   That is, there exists $K \geq 1$ such that for all $\gamma_1, \gamma_2 \in \Gamma$,

   $$\frac{1}{K} \cdot d_1(\gamma_1, \gamma_2) \leq d_2(\gamma_1, \gamma_2) \leq K \cdot d_1(\gamma_1, \gamma_2),$$

   where d_i is the metric on $\mathrm{Cay}(\Gamma, S_i)$.

2. Show that quasiisometry is an equivalence relation.

3. Find an example of two finitely generated groups which are quasiisometric but *not* commensurable.

4. Show that all regular trees of valence ≥ 3 are quasiisometric. Conclude that all finitely generated nonabelian free groups are quasiisometric.

5. "Uniqueness" of geodesics in hyperbolic metric spaces: show that in a Gromov-hyperbolic metric space (X, d), there is a constant D satisfying the following:

   If $c_1 : [0, L] \to X$ and $c_2 : [0, L] \to X$ are arclength-parameterized geodesics such that $c_1(0) = c_2(0) = x$, $c_1(L) = c_2(L) = y$, then for all $t \in [0, L]$,

   $$d(c_1(t), c_2(t)) < D.$$

6. Find an example of a metric space which is not Gromov hyperbolic and does not contain a quasi-isometrically embedded copy of the Euclidean plane.