Quasigeodesics in Gromov Hyperbolic spaces
Exercises

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1. Show that the logarithmic spiral, given in polar coordinates by $r(\theta) = \log(\theta + 1)$, is a quasi-geodesic ray in $\mathbb{E}^2$ that does not satisfy the Morse lemma. This proves again that the Euclidean plane is not hyperbolic.

2. **Proposition.** [Bridson-Haefliger p400] Let c be a continuous rectifiable curve in a δ-hyperbolic geodesic space X. If pq is a geodesic segment connecting the endpoints of c, then for every $x \in \overline{pq}$

$$d(x, \operatorname{im} c) \leq \delta|\log_2 \operatorname{length}(c)| + 1.$$

3. **Lemma.** [Taming Quasi-geodesics (Bridson-Haefliger p403)] Let X be a geodesic space. Given any (L, A) quasi-geodesic $c : [a, b] \to X$, one can find a continuous (L, A') quasi-geodesic $c' : [a, b] \to X$ such that

   (i) $c'(a) = c(a)$ and $c'(b) = c(b)$;

   (ii) $A' = 2(L + A)$;

   (iii) $\operatorname{length}(c'|_{[t,t']}) \leq k_1 d(c'(t), c'(t')) + k_2$ for all $t, t' \in [a, b]$, where $k_1 = L(L + A)$ and $k_2 = (LA' + 3)(L + A)$;

   (iv) the Hausdorff distance between the images of c and c' is less then $(L + A)$.

4. Show that the previous lemma (taming quasi-geodesics) and the version of the Morse lemma we proved (for continuous quasi-geodesics parameterized by arc-length) implies the Morse lemma for quasi-geodesics in general.

5. **Lemma.** [Morse lemma for local quasi-geodesics] Let $c : [a, b] \to X$ be a k-local (L, A) quasi-geodesic in a uniquely geodesic δ-hyperbolic space X and denote by $R = R(\delta, L, A)$ the constant from the Morse lemma. If $k > 2L(2R + 4\delta + A)$ then c lies within $R + 2\delta$ of the geodesic $\overline{c(a)c(b)}$ joining its endpoints.

6. Use the local-to-global principle to produce quasi-isometric embeddings. For example, choose a point $p \in \mathbb{H}^2$ and choose two geodesics l_1, l_2 through p that meet at a right angle. For each pair of translation length $t_1, t_2 > 0$,

there are unique transvections g_1, g_2 along l_1 and l_2 respectively. Use the local-to-global principle to find a constant t_0 such that, if t_1 and t_2 are at least t_0, then the group generated by g_1, g_2 is quasi-isometrically embedded into $\mathbb{H}^2$ under the orbit map (with basepoint p).

7. Let c, c' be geodesic rays in a metric space X. Show that $\sup_t d(c(t), c'(t))$ is finite if and only if the images of c and c' have finite Hausdorff distance. In either case the rays are called *asymptotic*.

8. Use: **Theorem** [Arzelà-Ascoli] If Z is a compact metric space and Y is a seperable metric space, then every sequence of equicontinuous maps $f_n : Y \to Z$ has a subsequence which converges (uniformly on compact subsets) to a continuous map $f : Y \to Z$.

   to prove that a sequence of geodesic rays in a proper geodesic space X with a common basepoint have a subsequence which converges to a geodesic ray.