Hyperbolic groups: day 3 exercises
Boundaries of hyperbolic groups

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1. Find a quasi-isometry of a non-hyperbolic metric space which does not take geodesics to curves lying finite Hausdorff distance from geodesics.

   Note: the logarithmic spiral gives an example of a quasi-isometric *embedding* which does not take geodesics to neighborhoods of geodesics; here we are looking for a *quasi-isometry*.

2. For a hyperbolic metric space X, show that there is a bijection (commuting with isometries) between the sets

$$\{\text{infinite geodesic rays based at } x_0\}/\sim,$$
$$\{\text{infinite geodesic rays based at } x_1\}/\sim$$

   for any x_0, x_1 in X (here $\sim$ is the usual equivalence relation: $c_1 \sim c_2$ when c_1 and c_2 lie within bounded Hausdorff distance of each other).

   To make this question interesting, do *not* assume that there is an isometry of X taking x_0 to x_1.

3. Visibility of hyperbolic spaces: given any two points η_1, η_2 in the boundary of a hyperbolic space X, show that there exists a bi-infinite geodesic $c : \mathbb{R} \to X$ with "endpoints" η_1, η_2. That is, the equivalence classes of the two infinite rays $c\big|_{[0,\infty)}$ and $c\big|_{(-\infty,0]}$ are η_1 and η_2.

   (Hint: use Arzelà-Ascoli.)

4. Show that the boundary of a hyperbolic group cannot contain exactly one point.

5. Show that if the boundary of a hyperbolic group contains exactly two points, then it is quasi-isometric to $\mathbb{Z}$.

   Harder: show that if the boundary of a hyperbolic group contains exactly two points, then it is virtually isomorphic to $\mathbb{Z}$.

6. Use the convergence group property to show that if the boundary of a hyperbolic group Γ contains at least 3 points, then it is *perfect* (i.e. every point is an accumulation point).