Hyperbolic groups summer mini-course: day 5 exercises
Generalizations of hyperbolicity

August 16, 2019

1. Prove that if γ is any infinite-order element in a hyperbolic group Γ, the map $\mathbb{Z} \to \Gamma$ given by $n \mapsto \gamma^n$ is a quasi-isometric embedding.

   Hint: it will suffice to find a constant K such that that for all R, there is $\rho(R) < KR$ satisfying $|\gamma^{\rho(R)}| > R$. Assume that for every constant K, $|\gamma^n| < R$ for all $n < KR$, and compare geodesics $[1, \gamma^k]$ and $[\gamma^n, \gamma^{n+k}]$, for large k.

2. Show that a map $\sigma : X \times X \to \mathcal{P}(X)$ (satisfying the condition that the endpoints of $\sigma(x, y)$ are x and y) is a combing if and only if the following holds:

   there exist constants $C, D > 0$ such that for any x_1, x_2, y_1, y_2 satisfying $d(x_1, x_2) < D$ and $d(y_1, y_2) < D$, then for all t, we have

   $$d(\sigma_1(t), \sigma_2(t)) < C,$$

   where $\sigma_1 = \sigma(x_1, y_1)$ and $\sigma_2 = \sigma(x_2, y_2)$.

3. A subgroup H of a group Γ with semihyperbolic structure σ is *σ-quasiconvex* if there exists k so that for all $h \in H$, $\sigma(\mathrm{id}, h)$ lies in a k-neighborhood of H.

   Show that σ-quasiconvex subgroups are quasi-isometrically embedded.

4. Let Γ be a group acting by homeomorphisms on a compact metrizable space X. Show that Γ acts as a convergence group if and only if Γ acts properly discontinuously on the space of distinct triples in X.

5. Suppose that Γ has a convergence group action on a compact metrizable space X. Show that every point in X is a conical limit point if and only if Γ acts cocompactly on the space of distinct triples in X.

6. Show that for any finitely generated groups A, B, the free product $A*B$ is hyperbolic relative to the collection of subgroups conjugate to A and B. (Use the coned-off Cayley graph condition.)