

## Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Give an intuitive geometric explanation of each of the 3 properties that define a metric.
2. Let  $X = \{a, b, c\}$ . Which of the following functions define a metric on  $X$ ?

<p>(a) <math>d(a, a) = d(b, b) = d(c, c) = 0</math>  <math>d(a, b) = d(b, a) = 1</math>  <math>d(a, c) = d(c, a) = 2</math>  <math>d(b, c) = d(c, b) = 3</math></p>	<p>(b) <math>d(a, a) = d(b, b) = d(c, c) = 0</math>  <math>d(a, b) = d(b, a) = 1</math>  <math>d(a, c) = d(c, a) = 2</math>  <math>d(b, c) = d(c, b) = 4</math></p>
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3. Which of the following functions  $\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  satisfy the triangle inequality? Which are metrics?

<p>(a) <math>d(x, y) = \frac{ y - x }{2}</math></p>	<p>(c) <math>d(x, y) = \min( y - x , \pi)</math></p>	<p>(e) <math>d(x, y) =  \log(y/x) </math> (for <math>x, y &gt; 0</math>)</p>
<p>(b) <math>d(x, y) =  x - y  + 1</math></p>	<p>(d) <math>d(x, y) = 1</math></p>	

4. Consider the set  $\mathbb{Z}$  with the Euclidean metric (defined by viewing  $\mathbb{Z}$  as a subset of the metric space  $\mathbb{R}$ ). What is the ball  $B_3(1)$  as a subset of  $\mathbb{Z}$ ? What is the ball  $B_{\frac{1}{2}}(1)$ ?
5. Let  $(X, d)$  be a metric space,  $r > 0$ , and  $x \in X$ . Show that  $x \in B_r(x)$ . Conclude in particular that open balls are always non-empty.
6. Let  $(X, d)$  be a metric space, and suppose that  $r, R \in \mathbb{R}$  satisfy  $0 < r \leq R$ . Show the containment of the subsets  $B_r(x) \subseteq B_R(x)$  of  $X$  for any point  $x \in X$ .
7. Let  $X = \mathbb{R}$  with the usual Euclidean metric  $d(x, y) = |x - y|$ .
  - (a) Let  $x$  and  $r > 0$  be real numbers. Show that  $B_r(x)$  is an open interval in  $\mathbb{R}$ . What are its endpoints?
  - (b) Show that every interval of the real line the form  $(a, b)$ ,  $(-\infty, b)$ ,  $(a, \infty)$ , or  $(-\infty, \infty)$  is open, for any  $a < b \in \mathbb{R}$ .
  - (c) Show that the interval  $[0, 1] \subseteq \mathbb{R}$  is closed.
8. Let  $(X, d)$  be a metric space, and let  $U \subseteq X$  be a subset. Does the set  $U$  necessarily need to be either open or closed? Can it be neither? Can it be both?

## Worksheet problems

(Hand these questions in!)

- Worksheet #1 Problem 1(a), 1(b)

## Assignment questions

(Hand these questions in!)

1. Let  $X$  be a set. A function  $f : X \rightarrow \mathbb{R}$  is called *bounded* if there is some number  $M \in \mathbb{R}$  so that  $|f(x)| \leq M$  for all  $x \in X$ . Let  $\mathcal{B}(X, \mathbb{R})$  denote the set of bounded functions from  $X$  to  $\mathbb{R}$ .

(a) Show that the function

$$d_\infty : \mathcal{B}(X, \mathbb{R}) \times \mathcal{B}(X, \mathbb{R}) \longrightarrow \mathbb{R}$$

$$d_\infty(f, g) = \sup_{x \in X} |f(x) - g(x)|$$

is well-defined, that is, the suprema always exist.

- (b) Show that the function  $d_\infty$  defines a metric on  $\mathcal{B}(X, \mathbb{R})$ .  
 (c) Explain why the following metric on  $\mathbb{R}^n$  is a special case of this construction.

$$d_\infty : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

$$d(\bar{x}, \bar{y}) = \max_{1 \leq i \leq n} |x_i - y_i|$$

where  $\bar{x} = (x_1, \dots, x_n)$  and  $\bar{y} = (y_1, \dots, y_n)$ .

2. Let  $(X, d)$  be a metric space. Show that a nonempty subset  $U \subseteq X$  is open if and only if  $U$  can be written as a union of open balls in  $X$ .  
 3. Let  $(X, d)$  be a metric space. Fix  $x_0 \in X$  and  $r > 0$  in  $\mathbb{R}$ . Show that the set  $\{x \mid d(x_0, x) \leq r\}$  is closed.  
 4. (a) Prove *DeMorgan's Laws*: Let  $X$  be a set and let  $\{A_i\}_{i \in I}$  be a collection of subsets of  $X$ .

$$(i) \quad X \setminus \left( \bigcup_{i \in I} A_i \right) = \bigcap_{i \in I} (X \setminus A_i) \qquad (ii) \quad X \setminus \left( \bigcap_{i \in I} A_i \right) = \bigcup_{i \in I} (X \setminus A_i)$$

*Hint:* Remember that a good way to prove two sets  $B$  and  $C$  are equal is to prove that  $B \subseteq C$  and that  $C \subseteq B$ !

- (b) Let  $(X, d)$  be a metric space, and let  $\{C_i\}_{i \in I}$  be a collection of closed sets in  $X$ . Note that  $I$  need not be finite, or countable! Prove that  $\bigcap_{i \in I} C_i$  is a closed subset of  $X$ .  
 (c) Now let  $(X, d)$  be a metric space, and let  $\{C_i\}_{i \in I}$  be a **finite** collection ( $I = \{1, 2, \dots, n\}$ ) of closed sets in  $X$ . Prove that  $\bigcup_{i \in I} C_i$  is a closed subset of  $X$ .  
 5. Let  $(X, d)$  be a metric space, and consider  $Y \subseteq X$  as a metric space under the restriction of the metric to  $Y$ . Show by example that a subset  $U \subseteq Y$  that is open in  $Y$  may or may not be open in  $X$ . For each example you should clearly define the sets  $X, Y, U$ , and the metric being used, but you may state without proof whether the set  $U$  is open in  $Y$  and whether it is open in  $X$ .