

Warm-up questions

(These warm-up questions are optional, and won't be graded.)

- Let $X = \{a, b, c, d\}$. Let \mathcal{T} be the topology on X

$$\mathcal{T} = \{\emptyset, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}, X\}.$$

- Which elements of X are limits of the constant sequence $x_n = d$? The constant sequence $x_n = a$? The constant sequence $x_n = b$?
 - Give an example of a sequence in X that does not converge.
- Let $X = \{0, 1\}$. Find a topology on X for which the following sequence converges:

$$0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ \dots$$

- Suppose that (X, \mathcal{T}) is a topological space, and that $(a_n)_{n \in \mathbb{N}}$ is a sequence in X that converges to $a_\infty \in X$. Prove that any subsequence of $(a_n)_{n \in \mathbb{N}}$ converges to $a_\infty \in X$.
- Let X be a topological space with the indiscrete topology. Prove that any sequence of points in X converges to every point in X .

Worksheet problems

(Hand these questions in!)

- Worksheet #13 Problems 1, 3.
- Worksheet #14 Problems 2(b), 4.

Assignment questions

(Hand these questions in!)

- Let A be a subset of a topological space A . Show that $\partial(\text{Int}(A))$ is contained in ∂A , but show by example that these sets need not be equal.
 - Let A be a subset of a topological space A . Show that $\partial(\overline{A})$ is contained in ∂A , but show by example that these sets need not be equal.
- Definition (path).** Let X be a topological space, and let $I = [0, 1] \subseteq \mathbb{R}$ be the unit interval with the standard topology. A *path* in X is a continuous function $\gamma : I \rightarrow X$. For points $x, y \in X$, we say that a path γ is a *path from x to y* if $\gamma(0) = x$ and $\gamma(1) = y$.

Note that the path γ does not need to be injective. For example, the constant path $\gamma(t) = x$ is a path from x to x .

For this problem (as always) you may assume any standard results about which functions between Euclidean spaces (with the standard topology) are continuous.

- (a) Given two points $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ in \mathbb{R}^n , construct a path from x to y .
- (b) Let x, y be points in the space \mathbb{R} with the cofinite topology. Construct (with proof) a path from x to y .
- (c) Let $X = \{a, b, c, d\}$ be a topological space with the topology

$$\mathcal{T} = \left\{ \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\} \right\}.$$

Construct (with proof) a path in X from a to d .

3. Let X be a topological space.

- (a) Let $x, y \in X$ and suppose that there exists a path from x to y . Show that there exists a path from y to x .
- (b) Let $x, y, z \in X$. Show that, if there exists a path from x to y , and a path from y to z , then there exists a path from x to z . *Hint:* Homework #10 Problem 2.

Remark: Although we will not formally define this term, we remark that this problem shows that, for a topological space X , the condition “there exists a path from x to y ” defines an *equivalence relation* on the points of X .

4. Let (X, \mathcal{T}_X) be a topological space, and endow the product $X \times X$ with the product topology $\mathcal{T}_{X \times X}$. The set

$$\Delta = \{ (x, x) \mid x \in X \} \subseteq X \times X$$

is called the *diagonal* of $X \times X$. Prove that X is Hausdorff if and only if the diagonal Δ is a closed subset of $X \times X$.

5. Consider the set \mathbb{R} with the **cofinite** topology. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of points in \mathbb{R} .

- (a) Suppose that the sequence has the property that each term is repeated at most finitely many times. More precisely, suppose for each $r \in \mathbb{R}$ that $r = a_n$ for at most finitely many values of $n \in \mathbb{N}$. Which points of \mathbb{R} are limits of the sequence $(a_n)_{n \in \mathbb{N}}$?
- (b) Now suppose the set $\{a_n \mid n \in \mathbb{N}\}$ is finite. Under what conditions will the sequence converge, and what will its limit(s) be?

Remember to justify your solutions!