## Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Consider $\mathbb{R}$ with the Euclidean metric. Which of the following maps $f : \mathbb{R} \to \mathbb{R}$ are homeomorphisms (Assignment Question 2)?

   (a) $f(x) = ax + b$ (b) $f(x) = x^2$ (c) $f(x) = x^3$ (d) $f(x) = \sin(x)$

2. Let $f : X \to Y$ be an **invertible** function of sets, and let $S \subseteq Y$. The notation $f^{-1}(S)$ could denote either the preimage of $S$ under $f$, or the image of $S$ under the inverse function $f^{-1}$. Show that these two sets are equal, so there is no ambiguity in using the notation $f^{-1}(S)$.

3. Let $f : X \to Y$ be an **invertible** function of sets.
   (a) Show that, for subsets $B \subseteq Y$, there is equality $f(f^{-1}(B)) = B$.
   (b) Show that, for subsets $A \subseteq X$, there is equality $f^{-1}(f(A)) = A$.

4. See the definition of bounded in Assignment Question 3.
   (a) Negate the definition of *bounded* to state what it means for a subset $S$ of a metric space to be *unbounded*.
   (b) Is $\varnothing$ a bounded set?
   (c) Show that any **finite** subset of a metric space is bounded.

5. Give examples of subsets of $\mathbb{R}$ (with the Euclidean metric) that satsify the following.

   (a) open, and bounded
   (b) closed, and bounded
   (c) open, and unbounded
   (d) closed, and unbounded

6. Consider $\mathbb{R}$ with the Euclidean metric. For each of the following sets $A$, find $\text{Int}(A)$, $\overline{A}$, $\partial A$, $\text{Int}(\mathbb{R} \setminus A)$, $\overline{\mathbb{R} \setminus A}$, and $\partial(\mathbb{R} \setminus A)$. See Assignment Question 4 for the definition of a boundary $\partial$.

   (a) $\mathbb{R}$
   (b) $[0, 1]$
   (c) $(0, 1)$
   (d) $(0, 1]$
   (e) $\{0, 1\}$
   (f) $\{\frac{1}{n} \mid n \in \mathbb{N}\}$

7. Consider the real numbers $\mathbb{R}$ with the Euclidean metric. Find examples of subsets $A$ of $\mathbb{R}$ with the following properties.
   (a) $\partial(A) = \varnothing$
   (b) $A$ has a nonempty boundary, and $A$ contains its boundary $\partial A$.
   (c) $A$ has a nonempty boundary, and $A$ contains no points in its boundary
   (d) $A$ has a nonempty boundary, and $A$ contains some but not all of the points in its boundary.
   (e) $A$ has a nonempty boundary, and $A = \partial A$.
   (f) $A$ is a **proper** subset of $\partial A$.

8. Let $X$ be a nonempty set with the discrete metric. Let $A \subseteq X$. Show that $A = \text{Int}(A) = \overline{A}$. Conclude that $\partial A = \varnothing$.

## Worksheet problems

(Hand these questions in!)

- Worksheet #5 Problems 1(a), 2, 3.

## Assignment questions

(Hand these questions in!)

1. Let $f : X \to Y$ be a function of sets $X$ and $Y$. Let $A, B \subseteq X$. For each of the following, determine whether you can replace the symbol $\square$ with $\subseteq, \supseteq, =$, or none of the above. Justify your answer by giving a proof of any set-containment or set-equality you claim. If set-equality does not hold in general, give a counterexample.

   (a) $f(A \cap B) \quad \square \quad f(A) \cap f(B)$

   (b) $f(A \cup B) \quad \square \quad f(A) \cup f(B)$

   (c) For $A \subseteq B$, $\quad f(B \setminus A) \quad \square \quad f(B) \setminus f(A)$

2. Consider the following definition.

   **Definition (Homeomorphism.)** Let $X$ and $Y$ be metric spaces. Then a map $f : X \to Y$ is a *homeomorphism* if

   - $f$ is continuous;
   - $f$ has an inverse $f^{-1}$;
   - $f^{-1}$ is continuous.

   The metric space $X$ is called *homeomorphic* to $Y$ if there exists a homeomorphism $f : X \to Y$.

   (a) Show that, if $f : X \to Y$ is a homeomorphism, then $f^{-1} : Y \to X$ is a homeomorphism. Conclude that $X$ is homeomorphic to $Y$ if and only if $Y$ is homeomorphic to $X$. (We simply call the two spaces *homeomorphic*.)

   (b) Let $f : X \to Y$ be a homeomorphism of metric spaces. Show that

   $$\{U \subseteq Y \mid U \text{ is open}\} \longrightarrow \{V \subseteq X \mid V \text{ is open}\}$$
   $$U \longmapsto f^{-1}(U)$$

   defines a bijection between the collection of all open subsets of $Y$, and the collection of all open subsets of $X$.
   *Hint:* To show it is a bijection, you could check that the assignment $V \mapsto f(V)$ is its inverse.

   (c) Consider the function

   $$f : (\mathbb{R}, \text{ discrete metric}) \to (\mathbb{R}, \text{ Euclidean metric})$$
   $$f(x) = x$$

   Prove that $f$ is continuous and invertible, but its inverse $f^{-1}$ is not continuous.

*Remark:* Note the contrast to other mathematical fields, such as linear algebra: if a linear map has an inverse, then the inverse is automatically linear. This exercise shows that this is not true for continuous maps!

3. Consider the following definition.

   **Definition (Bounded subset.)** Let $(X, d)$ be a metric space. A subset $S \subseteq X$ is called *bounded* if there is some $x_0 \in X$ and some $R \in \mathbb{R}$ with $R > 0$ such that $S \subseteq B_R(x_0)$.

   (a) Let $(X, d)$ be a metric space, and $A \subseteq X$ a subset. Show that $A$ is bounded if and only if, for every $x \in A$, there is some $R_x > 0$ such that $A \subseteq B_{R_x}(x)$.

   (b) Let $(X, d)$ be a metric space. Suppose that $(a_n)_{n \in \mathbb{N}}$ is a sequence in $X$ converging to an element $a_\infty$. Show that the set $\{ a_n \mid n \in \mathbb{N} \}$ is a bounded subset of $X$.

4. Consider the following definition.

   **Definition (Boundary of a set $A$.)** Let $(X, d)$ be a metric space, and let $A \subseteq X$. Then the *boundary* of $A$, denoted $\partial A$, is the set $\overline{A} \setminus \text{Int}(A)$.

   Let $(X, d)$ be a metric space, and let $A \subseteq X$.

   (a) Prove that $\text{Int}(A) = \overline{A} \setminus \partial A$.

   (b) Prove that $\partial A = \overline{A} \cap \overline{(X \setminus A)}$.

   (c) Conclude from part (b) that $\partial A$ is closed.

   (d) Additionally conclude from part (b) that $\partial A = \partial(X \setminus A)$.

   (e) Prove the following characteriziation of points in the boundary:

   **Theorem (An equivalent definition of $\partial A$).** Let $(X, d)$ be a metric space, and let $A \subseteq X$. Then $x \in \partial A$ if and only if every ball $B_r(x)$ about $x$ contains at least one point of $A$, and at least one point of $X \setminus A$.

   (f) Deduce that we can classify every point of $X$ in one of three mutually exclusive categories:

   (i) interior points of $A$;

   (ii) interior points of $X \setminus A$;

   (iii) points in the (common) boundary of $A$ and $X \setminus A$.