

Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Give an example of a metric space (X, d) and a continuous function $f : X \rightarrow \mathbb{R}$ such that f has a finite supremum on X , but f does not achieve its supremum at any point $x \in X$.
2. Let (X, d) be a metric space and let $S \subseteq X$ be a bounded set. Show that any subset of S is bounded.
3. Let (X, d_X) and (Y, d_Y) be bounded metric spaces. Show that $(X \times Y, d_{X \times Y})$ is bounded. What if we only assumed X is bounded?
4. Let (X, d_X) and (Y, d_Y) be metric spaces, and consider their product metric $(X \times Y, d_{X \times Y})$. Prove that a subset $U \subseteq X \times Y$ is open if and only if U can be written as the union of subsets of the form $U_x \times U_y$, with U_x an open subset of X and U_y an open subset of Y .
5. Let (X, d_X) and (Y, d_Y) be metric spaces. Let $A \subseteq X$ and $B \subseteq Y$. Then A and B inherit metric space structures from the metrics on X and Y . Consider the Cartesian product $A \times B$. There are two ways we can define a metric on $A \times B$: (1) using the product metric defined by the metric spaces A and B , and (2) as a subspace of the metric space $X \times Y$ with the product metric $d_{X \times Y}$. Verify that these two metrics agree, so there is no ambiguity in how to metricize the product $A \times B$.

Worksheet Problems

(Hand these questions in!)

- Worksheet #7 Problems 1, 2, 3.

Assignment questions

(Hand these questions in!)

1. Let $f : X \rightarrow Y$ be a continuous function of metric spaces. For each of the following statements, either prove the statement, or state a counterexample. You may state the counterexample without proof.
 - (a) If $U \subseteq X$ is open, then $f(U) \subseteq Y$ is open.
 - (b) If $C \subseteq X$ is closed, then $f(C) \subseteq Y$ is closed.
 - (c) If $C \subseteq X$ is sequentially compact, then $f(C) \subseteq Y$ is sequentially compact.
 - (d) If $C \subseteq Y$ is sequentially compact, then $f^{-1}(C)$ is sequentially compact.
 - (e) If $B \subseteq X$ is bounded, then $f(B) \subseteq Y$ is bounded.
 - (f) If $B \subseteq Y$ is bounded, then $f^{-1}(B) \subseteq X$ is bounded.
 - (g) If X is complete, then $f(X)$ (with metric space structure inherited from Y) is complete.

Hint: Below are some continuous functions that are useful sources of counterexamples. Subsets of \mathbb{R} have the Euclidean metric unless otherwise noted.

$$\begin{array}{ll} (\mathbb{R}, \text{discrete}) \longrightarrow (\mathbb{R}, \text{Euclidean}) & \mathbb{R} \longrightarrow (-1, 1) \\ x \longmapsto x & x \longmapsto \frac{\arctan(x)}{\pi/2} \end{array}$$

$$\begin{array}{ll} (0, \infty) \longrightarrow (0, \infty) & \mathbb{R} \longrightarrow \mathbb{R} \\ x \longmapsto \frac{1}{x} & x \longmapsto 0 \end{array}$$

- Let (X, d_X) and (Y, d_Y) be metric spaces. Suppose that $C \subseteq X$ and $D \subseteq Y$ are closed subsets. Prove or find a counterexample: the subset $C \times D \subseteq X \times Y$ is closed with respect to the product metric.
- Let (X, d_X) , (Y, d_Y) , and (Z, d_Z) be a metric spaces, and suppose that $f : Z \rightarrow X$ and $g : Z \rightarrow Y$ are continuous functions. Prove that the function

$$\begin{aligned} (f \times g) : Z &\longrightarrow X \times Y \\ (f \times g)(z) &= (f(z), g(z)) \end{aligned}$$

is continuous with respect to the product metric $d_{X \times Y}$ on $X \times Y$.

- Let (X, d_X) and (Y, d_Y) be metric spaces.
 - Let $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ be sequences of points in X and Y , respectively. Show that the sequence of points $((x_n, y_n))_{n \in \mathbb{N}}$ in $(X \times Y, d_{X \times Y})$ converges if and only if both $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ converge.
Hint: Worksheet # 7 Problems 2(b) and 3(a).
 - Show that $X \times Y$ is sequentially compact if and only if both X and Y are.
- Suppose that B is a closed and bounded subset of \mathbb{R} . Explain why the supremum of B exists, and show that B contains its supremum.
 - Prove the following theorem. This theorem is one of the important reasons we care about sequential compactness!

Theorem (Extreme value theorem for metric spaces). Let (X, d) be a metric space and C a sequentially compact subset of X . Let $f : X \rightarrow \mathbb{R}$ be a continuous function. Then there is a point $c \in C$ so that

$$f(c) = \sup_{x \in C} f(x).$$

In other words, prove that f achieves its supremum on C .