

Warm-up questions

(These warm-up questions are optional, and won't be graded.)

- Let X be a topological space with the cofinite topology, and let $S \subseteq X$. Show that the subspace topology on S is the cofinite topology.
- Let X be a set, and $A \subseteq X$ a proper subset. What are the interior and closure (Assignment Problems 1 and 2) of A if X is given
 - the discrete topology?
 - the indiscrete topology?
- Let $X = \{a, b, c, d\}$. Let \mathcal{T} be the topology on X

$$\mathcal{T} = \{\emptyset, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}, X\}.$$

Find the interior, closures, boundaries, and accumulation points of the subsets

- (a) $\{a, b, c\}$ (b) $\{a, c, d\}$ (c) $\{a, b, d\}$ (d) $\{b\}$ (e) $\{d\}$ (f) $\{b, d\}$

- Consider the following subsets of \mathbb{R}

- | | | | |
|----------------|-----------------------------|------------------|----------------------------------|
| • \mathbb{R} | • $[0, \infty)$ | • $(-\infty, 0)$ | • $\{-n \mid n \in \mathbb{N}\}$ |
| • \emptyset | • $(1, 2)$ | • $(-\infty, 0]$ | |
| • $\{0, 1\}$ | • $[1, 2] \cup [3, \infty)$ | • \mathbb{N} | |

Find the interior, closures, boundaries, and accumulation points of these subsets ...

- ... when \mathbb{R} has the discrete topology.
- ... when \mathbb{R} has the indiscrete topology.
- ... when \mathbb{R} has the Euclidean topology.
- ... when \mathbb{R} has the topology $\mathcal{T} = \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{\emptyset\} \cup \{\mathbb{R}\}$.
- ... when \mathbb{R} has the cofinite topology.
- ... when \mathbb{R} has the topology $\mathcal{T} = \{\emptyset\} \cup \{U \subseteq \mathbb{R} \mid 0 \in U\}$.
- ... when \mathbb{R} has the topology $\mathcal{T} = \{\mathbb{R}\} \cup \{U \subseteq \mathbb{R} \mid 0 \notin U\}$.

Worksheet problems

(Hand these questions in!)

- Worksheet #10 Problems 1, 4.

Assignment questions

(Hand these questions in!)

In this assignment, we will generalize the notions of interior, closure, boundary, and accumulation point to general topological spaces—and verify that many of the results we proved for metric spaces hold in general. This is an opportunity to revisit their proofs!

1. **Definition (Interior of a set in a topological space).** Let (X, \mathcal{T}) be a topological space, and let $A \subseteq X$. A point $a \in A$ is an *interior point* of A if there exists an open subset $U \subseteq X$ such that $a \in U \subseteq A$. Define the *interior* of A to be the set of interior points of A . Concretely,

$$\text{Int}(A) = \{ a \in A \mid \text{there is some open neighbourhood } U \text{ of } a \text{ such that } U \subseteq A \}.$$

Observe that, by definition, $\text{Int}(A) \subseteq A$. Prove the following.

- (a) A is open if and only if $A = \text{Int}(A)$.
 - (b) $\text{Int}(A)$ is an open set.
 - (c) $\text{Int}(\text{Int}(A)) = \text{Int}(A)$.
 - (d) Suppose that $A \subseteq X$ is any subset, and $U \subseteq A$ is an open subset of X . Prove that $U \subseteq \text{Int}(A)$.
 - (e) $\text{Int}(A) = \bigcup_{\substack{U \subseteq A, \\ U \text{ open in } X}} U$.
2. **Definition (Closure of a set in a topological space).** Let (X, \mathcal{T}) be a topological space, and let $A \subseteq X$. Define the *closure* of A to be the set

$$\bar{A} = \{ x \in X \mid \text{any neighbourhood } U \text{ of } x \text{ contains a point of } A \}.$$

Prove the following.

- (a) $A \subseteq \bar{A}$.
- (b) A is closed if and only if $A = \bar{A}$.
- (c) $\overline{\bar{A}} = \bar{A}$.
- (d) \bar{A} is a closed set.
- (e) Suppose that $A \subseteq X$ is any subset, and C is a closed set containing A . Then $\bar{A} \subseteq C$.
- (f) $\bar{A} = \bigcap_{\substack{C \text{ closed in } X, \\ A \subseteq C}} C$.
- (g) **Definition (Accumulation points of a set).** Let (X, d) be a metric space, and let $S \subseteq X$ be a set. A point $x \in X$ is called an *accumulation point* of S if every open neighbourhood U of x also contains a point in S distinct from x .

By Homework #2 Problem 4(b), when the topology on X is induced by a metric, this agrees with our metric space definition of an accumulation point.

Let A' be the set of accumulation points of A . Then $\bar{A} = A \cup A'$.

